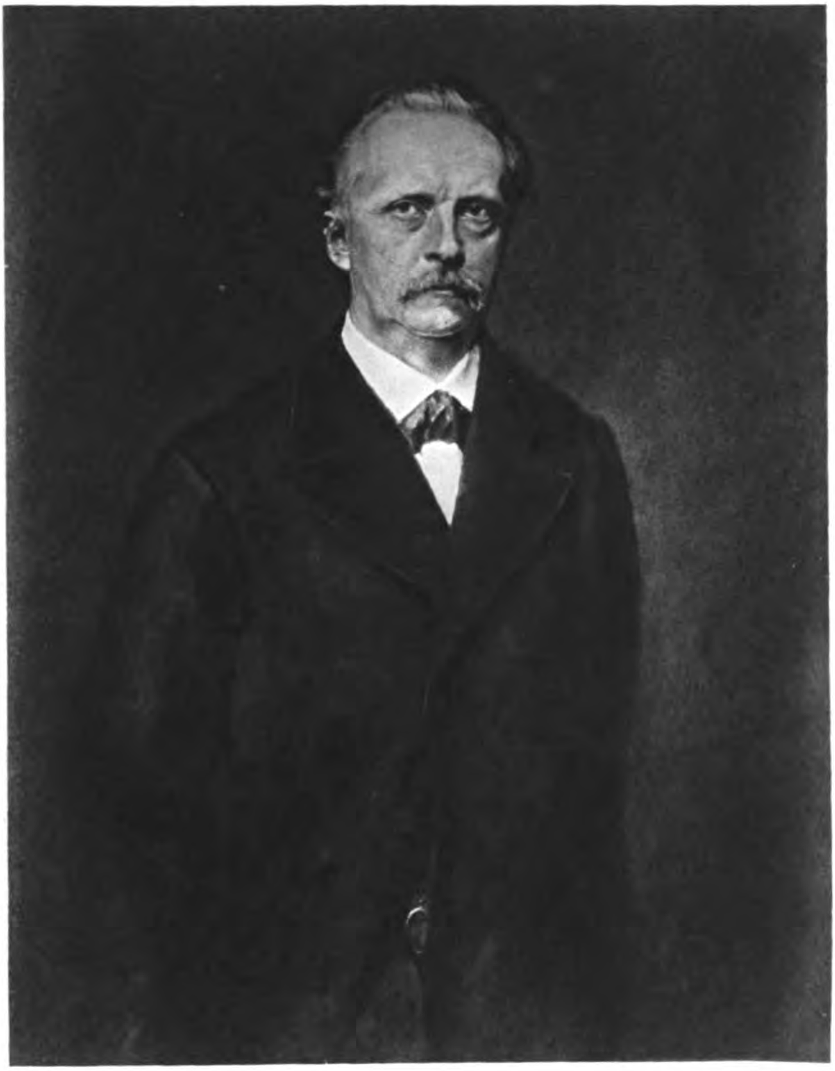
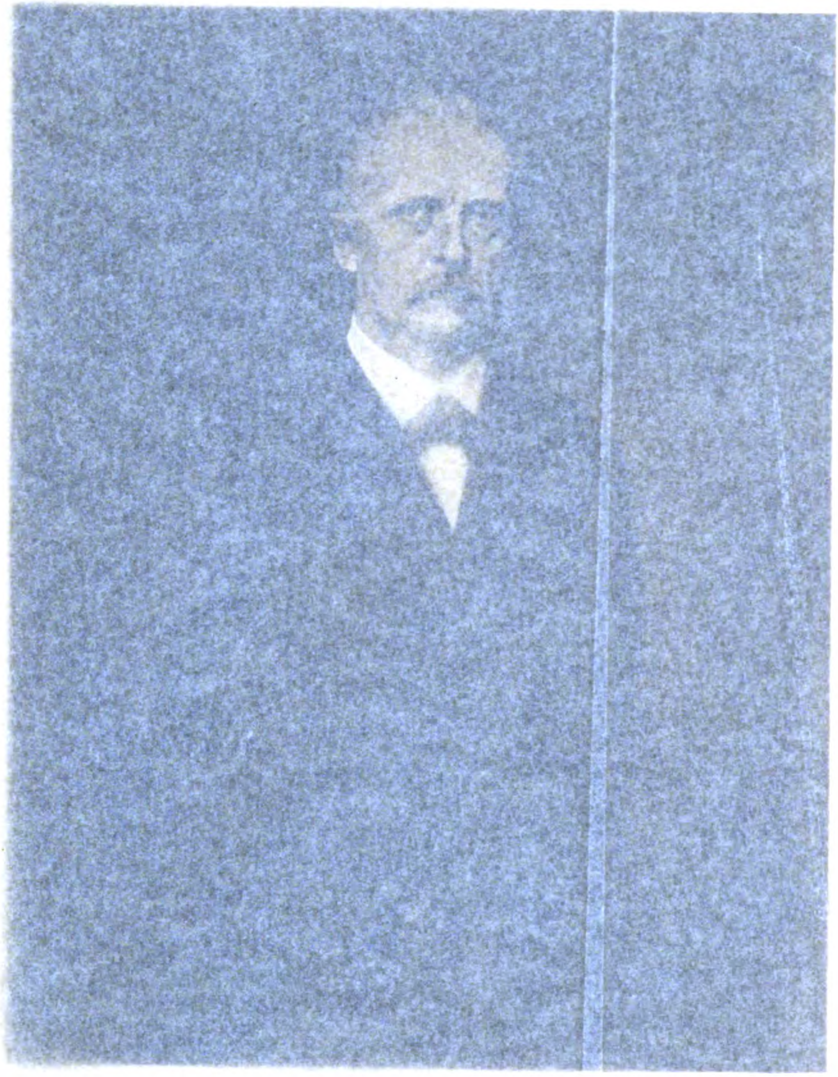


H. v. Helmholtz



Helmholtz's Treatise

on

Physiological Optics

Translated from the Third German Edition

Edited by
James P. C. Southall
Professor of Physics in Columbia University

Volume I

Published by
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1924

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Handbuch

der

Physiologischen Optik

von

H. von Helmholtz.

Dritte Auflage

ergänzt und herausgegeben in Gemeinschaft mit

Prof. Dr. A. Gullstrand und **Prof. Dr. J. von Kries**
Upsala Freiburg

von

Professor Dr. W. Nagel
Rostock

Erster Band

Mit 146 Abbildungen im Text

Einleitung herausgegeben von Prof. Dr. **W. Nagel**

Die Dioptrik des Auges herausgegeben von Prof. Dr. **A. Gullstrand**

Hamburg und Leipzig
Verlag von Leopold Voss
1909.

PREFACE TO THE ENGLISH TRANSLATION

HELMHOLTZ's "Popular Scientific Lectures" have spread his fame far and wide among educated people everywhere. His great work "On the Sensations of Tone as a Physiological Basis for the Theory of Music" has long been accessible to English readers (3rd ed., 1895). While he was professor at Heidelberg and still a comparatively young man, nearly three-score years ago he composed the preface to the first edition of his monumental work on light and vision, in all their intricate and manifold relations to each other; and already considerably more than a decade has passed since the publication of the posthumous third edition of the *Physiologische Optik* which was brought up to date and greatly enlarged under the collaboration of NAGEL, GULLSTRAND and v. KRIES. Yet in all these years there has been no English translation of this great classical treatise; and unfortunately no similar work in English of any kind. It is interesting to note that both YOUNG and HELMHOLTZ, the two great pioneers in Physiological Optics, started on their careers in the medical profession, and each of them afterwards gained his greatest renown in Physics. Apart from its own intrinsic value, the treatise on Physiological Optics is a model of scientific method and logical procedure that has hardly ever been excelled in these respects.

The meeting of the Optical Society of America in 1921 in Rochester, N. Y., was made notable by the celebration of the hundredth anniversary of the birth of HELMHOLTZ. On that occasion it was proposed to commemorate this event in a more useful and substantial way by bringing out the long-delayed English translation of the Physiological Optics, and accordingly a special committee was appointed to have charge of this business. After due deliberation the committee decided that under the circumstances the third German edition was beyond question the one to be reproduced in English, not simply because it was the last and, so to speak definitive edition, but because, besides containing the original HELMHOLTZ text absolutely unchanged, each of the three principal divisions of the subject, namely, let us say, the physical, the physiological and the psychological portions, had been supplemented and enriched by recent additions, all of the highest value and importance. This plan involved, however, a much bigger undertaking and much greater expense than had perhaps been contemplated at first. But luckily for the success of the project, there was one member of the committee who was determined that no obstacles should stand in the way; and it is literally true that without the continual advice and

encouragement of Mr. ADOLPH LOMB, the achievement would never have been accomplished. He was resolved at all costs that this great thesaurus of physiological optics should henceforth be at the disposal of ophthalmologists and scientific investigators in England and America in their own tongue. And certainly the very existence of this book in English should lead to new treatises and new text-books which are sorely needed at present.

The editor has received also valuable assistance from other sources without which he could hardly have accomplished his own task. Herewith appended is a list of those collaborators who have lent their aid in the arduous work of translation:

R. P. ANGIER, Yale University (Volume III, v. KRIES's Appendix, I., Nos. 6, 7 and 8).

M. DRESBACH, Albany Medical College (Volume II, §§22, 23, 24 and 25).

HARRY S. GRADLE, Chicago, Ill. (Volume III, §§27, 28).

D. H. HOOKER, University of Pittsburgh (Volume I, §§1, 2, 3, 4, 5, 6, 7, 12, and GULLSTRAND's Appendix IV).

WILLIAM KUNERTH, Iowa State College (Volume III, §§26, 29).

JAKOB KUNZ, University of Illinois (Volume III, v. KRIES's Note on §31, and §§32 and 33, and v. KRIES's Appendix I., Nos. 1, 2, 3, 4, 5).

HENRY LAURENS, Yale University (Volume II, §§17, 18, 18A, and Appendices of W. NAGEL and v. KRIES)

A. LOMB and H. C. LOMB, New York City (new matter contributed by v. KRIES to Volumes II and III).

G. W. MOFFITT, Frankford Arsenal (Volume III, v. KRIES's Appendix II).

L. T. TROLAND, Harvard University, and E. J. WALL, Wollaston, Massachusetts (Volume II, §§19, 20, 21).

L. D. WELD, Coe College (Volume I §§8, 9, 10, 11, 13, 14, and part of 15, Volume III, §3 except v. KRIES's Note on this section).

W. WENIGER, Oregon State Agricultural College (Volume III, §30).

The single object that has been kept steadily in mind throughout was a faithful rendition of the original, at the same time without being too literal and awkward. And while I believe every word has been carefully scrutinized, I must still hope that the indulgent critic will be disposed to overlook many shortcomings. At the same time I wish it to be clearly understood that the editor, and the editor alone, assumes all responsibility for the version as it stands. He has had to exercise the right of revising the preliminary manuscripts without returning them to the various collaborators for their approval.

In no sense is this work a new edition of the Physiological Optics. Nevertheless, it does contain some new and original material of distinct value, notably as follows: 1. A chapter on Ophthalmoscopy by Professor GULLSTRAND, taken by his permission from his book entitled *Einführung in die Methoden der Dioptrik des Auges des Menschen* (Leipzig, 1911), which is inserted at the end of the first volume; 2. Several special contributions prepared by Professor v. KRIES, which will appear at various places in the second and third volumes; and finally, 3. An article by Dr. CHRISTINE LADD-FRANKLIN on her colour theory and related matters, which will be found at the end of the second volume. Here and there throughout the entire work the editor and his associates have ventured to insert a few explanatory footnotes and occasional, more or less haphazard, references to more recent literature. Anything like a complete bibliography in the ever-widening domain of this vast subject would have to be a separate task in itself and was manifestly out of the question here. This is something that I hope will be systematically undertaken perhaps also by the Optical Society of America; and the sooner, the better, if it is ever to be done at all. The additional footnotes are indicated by the sign ¶ prefixed to them; in each instance the initials of the responsible writer being appended in parentheses at the end.

With a few necessary modifications now and then, the plates and illustrations have been reproduced from the original copper plates purchased directly from the German publishers. The latter, by the way, together with Professor GULLSTRAND and Professor v. KRIES, have coöperated in the most friendly and helpful way to make the English edition in every way worthy of its handsome prototype. This is also true of the American printers, who, as will be seen, have spared no pains to make the book useful and suitable from every point of view.

To my friend and colleague Mr. C. L. TRELEAVEN I am under many obligations for assisting me in reading the proof-sheets. My wife has likewise aided me in this labour. And, last, but not least, let me inscribe here, *magna cum laude*, the name of Miss RUTH TOWNSEND, who has diligently copied the entire manuscript thus far, besides taking time to perform some of HELMHOLTZ's experiments as she went along!

As these lines are being written, the first volume is about ready to be issued from the press, and the manuscript of the second volume is now in the hand of the printers. I trust the entire treatise in three volumes will be completed before the end of another year.

JAMES P. C. SOUTHALL

Department of Physics,
Columbia University,
New York, N. Y.
June 1 1924

PREFACE TO THE FIRST EDITION

The first section of this treatise appeared in 1856, the second in 1860, and the third partly at the beginning and partly toward the close of 1866. The long interval that elapsed before the publication of the final instalment was due in part to outside influences (two changes of residence and occupation involving other scientific undertakings) and partly also to internal causes. Of late years the theory of visual perceptions has been the subject of much investigation and has just begun to develop its rich contents and the absorbing interest which it possesses. It might be a fair question to ask whether it is now indeed quite feasible to carry out the general plan of this book and of the encyclopedia to which it belongs and to bring even to a preliminary conclusion a work which treats of this youthful and at the same time effervescing field of knowledge. However, owing to the peculiar nature of this domain of science, it is hardly to be expected that a final answer can be very soon given to some of the questions that are still debatable. The whole region is closely entangled with physiological problems of the utmost difficulty, and moreover the investigators who can make advances are necessarily limited, because they must have long practice in the observation of subjective phenomena before they are qualified to do more than see what others have seen before them. Unless a person is duly cautious about this kind of experimental work, he may have to pay for it afterwards by some impairment of his eyesight. The result is that just here where psychical processes intervene there appears to be much greater room for individual peculiarities than in other regions of physiology.

Finally, however, an effort had to be made to introduce law and order in this region and to rid it of the curious contradictions which have heretofore impeded progress. I have proceeded on the conviction that law and order even if they are not fundamentally sound are better than contradictions and lawlessness. Accordingly, I have taken as my guide the principle of the empiristic theory as expounded in §§26 and 33 concerning which I am persuaded more and more, the longer I ponder over it, that it is the only safe guide through the labyrinth of the facts known at present. Along this route other pioneers have already preceded me whose labours, perhaps in consequence of a certain predilection for direct mechanical explanations that are characteristic of the materialistic tendency of the day, have on the whole not won such favour as they probably deserved. The reason

for this may be due to the fact that my predecessors have always been busy with single chapters in the theory of the visual perceptions, and their opinions, to be of weight, should be placed in the right perspective with respect to the whole subject. I have been at much pains to develop this connection completely.

The inconveniences resulting from not waiting to publish the entire work before the appearance of the two earlier portions I have endeavoured to remedy by collecting the recent literature on the subject in a supplement and calling attention at least briefly to the most important facts that have been ascertained since the publication of the first parts of this book. Fortunately, none of these new observations have necessitated any essential modification of the previous conclusions and opinions.

As far as possible and with the means at my command, I have tried to follow the literary plan of the encyclopedia for which these volumes were intended. The more recent literature will be found to be fairly complete; but the earlier writings I have frequently had to compile from secondary sources and cannot guarantee their accuracy. The execution of a really trustworthy history of physiological optics would be of itself an undertaking that would demand the time and strength of an investigator over a long period of years, and such a work would hardly be worth the labour until the science itself was in a much maturer state than it is at present.

In the preparation of this treatise the chief aim which I have had in view has been to verify all the fairly important facts by the evidence of my own eyes and by my own experience. Those methods of observation which seemed to me the most reliable have been selected always for description, and if they are sometimes different from those of the original investigator, I hope no one will suppose that these variations are introduced without purpose and merely for the sake of novelty.

I trust that competent judges will bear in mind the difficulty and intricacy of the problem to be solved when they are disposed to find fault with the book which is here submitted to them.

H. Helmholtz

Heidelberg, December, 1866

PREFACE TO THE THIRD EDITION

The first edition of HERMAN VON HELMHOLTZ's Treatise on Physiological Optics, which was published in 1866, has long been out of print, and even the second edition published in 1885 is no longer to be found in the bookshops. The demand for the book has not ceased and will not cease for a long time to come, for no new treatise has superseded HELMHOLTZ's work. Containing in its scope such a wealth of material presented in the simplest and clearest fashion, the "Physiological Optics" bears the stamp of a genuinely classical treatise which will always retain its value, even if new investigations lead to some modifications here and there of the points of view which HELMHOLTZ himself entertained.

To preserve a work of this character for the scientific world and for the book-trade is not simply a pious duty of mere historical value but is also a practical service in a very real sense. And so when the publishers announced that a new edition was needed and appealed to me to undertake it, I was glad to coöperate with them, although I was aware of the enormous difficulties which were necessarily involved.

It was obvious from the outset that the mere reproduction either of the first or of the second edition without any changes whatever would not be satisfactory at all. Undoubtedly, for a long time to come a new impression of the original text would be found to be valuable and useful. And yet in view of the wealth and significance of much of the new material of research, it seemed as if it might be a matter of almost universal regret if there were no reference to it whatever, whereas by taking some account of recent progress in this domain of science the value of the entire work might be essentially enhanced. If a new edition along these lines appeared practicable instead of simply a new impression of the original text, evidently this was the best solution. Of course, I had to admit that an adequate revision of the entire work was probably beyond the power of any single individual and certainly beyond my power. But this difficulty was quickly disposed of in the most satisfactory way when Professor GULLSTRAND and Professor VON KRIES consented to join me in the enterprise and to edit the parts on the Dioptrics of the Eye and the Visual Perceptions, respectively; thus leaving me free to devote my labours exclusively (except for certain essentially technical problems) to revising the second part of the book, which is concerned with the subject of the Visual Sensations.

The principal difficulty thereafter was to decide on the nature and plan of the revision. It was immediately apparent that we could not hope to satisfy every desire in this respect. However, this problem also was soon clarified and settled by consultations among the editors themselves and also with the publishers and Madam Ellen v. Siemens, *neé* v. Helmholtz, who represented the author's heirs. All were unanimously of the opinion that a revision in the sense of producing an entirely new work in which the text of the original was freely employed was out of the question. Undoubtedly, such a procedure might have resulted in a uniform, well rounded work which would be useful for reference. But this is exactly what Helmholtz meant when he spoke of an editor who had become so completely merged and, as it were, melted in his work as to have lost all identity of his own. A compilation of this sort may indeed be desirable one of these days, but that is for future times to determine. At present (we were all agreed on that point) the time was not ripe for this. The significance of Helmholtz's own views is still too great to suffer what he wrote and his mode of presenting these ideas to be submerged and more or less obliterated by being restated and re-edited. The facts and problems of physiological optics as they appeared to Helmholtz should be still accessible for general information on this subject.

Under the circumstances there was nothing else to be done but to preserve the text of the original work intact, and at the expense of a certain unity and uniformity in the work as a whole to limit the revision to supplementary chapters. The unavoidable disadvantage of this method will be less evident wherever such additions are substantially in harmony with the author's own views and opinions and appear therefore as a sort of super-structure on the basis of his ideas. However, the circumstances of the case are found to be quite different in the separate parts of the treatise. In the theoretical and for the most part solid territory belonging to the physiology of both the sensations and the perceptions of vision, the editors considered the conditions so far favourable for their plan that even where it was necessary to record a notable advance of scientific investigation since the date of the appearance of the first edition of the Physiological Optics, a natural outcome of this kind should not involve at all an adverse attitude towards the theories of Helmholtz. It is precisely because the editors believe that in such questions as these it is necessary to start from the fundamental conceptions as Helmholtz represented them that they decided to undertake the revision of the work. The plan of arrangements of the separate divisions will be explained further on.

The decision to reproduce the original text with supplements having been reached, the next question was whether to use the text of the

first or second edition. Undoubtedly, the natural thing was to select the later version for this purpose. How could an editor choose to disregard any alterations or additions which had been made by the author himself? If, therefore, in spite of so obvious an objection, the editors have preferred the text of the first edition, there must have been special reasons that were responsible for this decision. Foremost of these was the consideration that the whole imperishable significance of HELMHOLTZ's achievements in the domain of the physiology of vision, the elegant physical methods which he adopted and improved, his painstaking observations of the sensations themselves and the allied psychical phenomena, the mathematical analysis and philosophical and critical discussion, all these characteristic features are essentially interwoven with the first edition of the Physiological Optics. It is the classical work that marked the dawn of a new era in the science of the physiology of the senses.

However, there was another reason also for preferring the text of the original edition. Curiously enough the changes and additions which were introduced in the second edition seem to be called in question and discarded at present to a far greater extent than the contents of the first edition.* The explanation of this is hardly to be attributed to the fact that at the time when HELMHOLTZ was at work on the second edition, he was diverted from the task by other absorbing labours and consequently could not give to physiological optics the same undivided strength and interest as formerly, and so had not been able to keep pace with the more recent developments in this domain of science. A sufficient answer to that is that at this very time, under his influence and with his collaboration, A. KÖNIG's important researches were begun and had aroused his keenest interest. Through these investigations he had become acquainted with part of a newly explored region. Subsequent study of these phenomena and better knowledge of them placed the results in a somewhat different light and suggested the probability of a different interpretation. It is easy for us looking backwards to see now how, on account of the unfinished state of these researches, the beginning of the year 1890 happened to be a singularly unfortunate time for a new edition of the Physiological Optics. Thus at the present time it is not only easier but wiser to undertake the revision on the safe basis of the first edition instead of attempting to reconcile the tentative investigations in the second edition. This criticism applies, of course, to only one part of physiological optics; but it is the very part in which the changes in the second edition are most numerous.

* It will be of interest to read the review of the second edition written by Professor J. McKEEN CATTELL and published in *Science*, N.S. Vol. VIII, pp. 794-796. Dec. 2, 1898. (J. P. C. S.)

Accordingly, it was decided to reproduce word for word the text of the first edition, including the supplements which HELMHOLTZ had added himself. With regard to the contributions of the editors, the best plan seemed to be to allow as much latitude as possible to each of them to present the subject in his own way. Occasional brief comments and corrections are to be found in footnotes; whereas the more extensive notes are placed at the conclusions of the various single sections. In each of the three main divisions of the work certain subjects required to be revised with perfect freedom and in more detail. Parts of a new chapter arising in this fashion may be inserted in between the sections of the original text and the other parts appended at the end of the main division. In these supplementary portions certain regions of physiological optics which HELMHOLTZ merely touched on or ignored entirely and which modern investigations have opened up for the first time are examined, whereas in other chapters recent theoretical views are presented.

A revision of the introduction on the anatomy of the eye appeared to be unnecessary. HELMHOLTZ's treatment of this subject is a model of terseness and clearness and enables an ordinary reader to obtain a sufficient grasp of the matter; whereas a complete revision going more in detail would exceed the scope of the work.

Of the three main divisions of the entire book the first one of the "Dioptrics of the Eye" required special consideration owing to the very considerable progress which has been made recently in the study of the actual image-process in optical systems as compared with the ordinary more or less imperfect methods of Dioptrics that have long been in vogue. Consequently, mere minor alterations and additions in the original text were not adequate here for the purpose in hand, and therefore it seemed better to present this whole subject from these new standpoints.

While it was not to be gainsaid that a complete new treatment of this division was advisable for the reasons mentioned above, on the other hand it began to be evident very soon that this plan necessarily involved such a disproportionate augmentation of this part of the book that it had to be abandoned for a compromise. And so the editor of this subject has endeavoured to limit himself mainly to the essential facts in the region where notable advances have taken place and to present these modern ideas in as simple a form as possible for a reader who was not versed in higher mathematical analysis. An outline of the new theory as a whole is given here for the first time, including also the hitherto unknown laws of optical imagery in media of variable index of refraction which could not be omitted because, first of

all, they are essential for finding the data of a schematic eye that will agree with the facts as they are now known, and also because these considerations are of importance in connection with the effort to substitute a new theory of accommodation in place of HELMHOLTZ's theory, whereas undoubtedly this attack as well as others of a similar kind are to be regarded as affording new supports to the author's views on this subject.

In the region of the "Sensations of Vision" the main question to be decided first of all was whether HELMHOLTZ's conception of the structure and action of the mechanism of colour-perception could still be considered as an adequate explanation of all the new observations that have been made in the last four decades; and if not, whether these ideas should be discarded perhaps altogether, or, finally, whether it would be really profitable to introduce here additional supplementary hypotheses. The editor's position on this question is that there is no reason whatever to abandon the fundamental ideas of the colour-theory which HELMHOLTZ espoused; although the assumption of the organization of the mechanism of colour-perception in three components is no longer sufficient to give an entirely satisfactory account of all the known facts of colour-vision. However, in the theory of the separate functions of the rods and cones of the retina an opportunity is afforded of making certain phenomena intelligible that are ignored in HELMHOLTZ's original statement of the retinal functions. Since this theory of the twofold function of the retina is fundamentally and essentially bound up with the differences of vision in bright and feeble illumination and with the so-called adaptation of the eye, a special chapter had, first of all, to be devoted to this subject in order to explain how the light-sense and colour-sense are related to the state of adaptation of the eye. The purpose of another chapter is to describe and estimate comparatively recent advances in laboratory methods of measurement of the colour-sensitivity of the eye (spectrophotometry, colorimetry, peripheral and flicker reactions, etc.). And, finally, a third chapter here treats of the abnormalities of the colour-sense as to their bearing on the theory of colour-vision.

With respect to the last of the three main divisions of the treatise, the comprehension and explanation of the "Perceptions of Vision" will probably depend for a long time to come on personal opinions that are hardly open to discussion, because here we are concerned with a whole set of debatable propositions that cannot be submitted to experiment or direct observation but are determined by considerations of a philosophical and psychological nature. The corner-stone in HELMHOLTZ's theory of the visual perceptions, the doctrine of "empiricism," as he called it, has to be described by a word which in our opinion

begs the question, and indeed is utterly opposed to it, that is, a conception, which, while it may still be justifiable and is fundamentally so, being supported by the same identical facts, is nevertheless subject to the same doubts and difficulties now as it was forty years ago. And inasmuch as it was HELMHOLTZ's intention in 1894 to republish this part of the book for the second edition without any considerable alterations, it can perhaps be positively affirmed that the facts which had come to life meantime, supposing he was aware of them then, would certainly not have altered at all his main convictions on this subject. Under such circumstances it might therefore seem permissible to limit the revision of this portion of the work merely to certain simple statements of fact without attempting any explanations of the theoretical and fundamental questions which they involved.

And yet a desire of presenting the opposite aspect of the matter in relation to a whole set of items, together with numerous other considerations which will be duly mentioned as they arise, appeared to make such a restriction impractical, so that at last the editor felt obliged to attempt a general and independent review of the fundamental questions that are aroused by the battle-cry of "empiricism" and "nativism" (or intuitionalism). To this third main division of the work another chapter also has been added dealing with binocular optical instruments, which is a subject entirely within the scope of the work and which indeed was discussed at some length in the original edition, but which of late years has received such extensive and important practical developments in the construction of new apparatus that it was deemed advisable to present the whole matter in a more thorough and systematic way.

In outward appearance and mechanical execution the publishers have spared neither pains nor expense. The page is larger in size and the letter-press more agreeable to the eye. The paper likewise is better than it was in the former editions, and since the actual text has been augmented by additions considerably beyond that of the first and even of the second edition, it has been found necessary to issue the work in three volumes, as a single volume would have been entirely too unwieldy. Following the author's example in the second edition, the illustrations have been distributed in these volumes in the places in the text where reference to them occurs instead of assembling them in plates as in case of the first edition.

The bibliography which was prepared by ARTHUR KÖNIG for the second edition has not been included in the present edition. It would certainly have needed to be brought up to date, and in that case it would have required a whole volume by itself. Besides, the expenditure of time and labour would have been out of proportion to its value,

since nowadays good journals are available containing periodical lists and abstracts of the literature of the subject, notably, for example, the "Zeitschrift für Psychologie und Physiologie der Sinnesorgane" which was started by A. König himself and is specially devoted to the physiology of the senses.

The bibliographies of the first edition (together with those contained in the supplements) have been retained as containing references to early investigations in physiological optics which otherwise might be difficult to trace. The citations in the new parts of the work will be found in footnotes.

For convenience of reference and comparison with the two preceding editions, the corresponding page numbers of the first edition are given at the top of each page, as was done in the second edition. In place of these numbers, the contributions of the various editors, Gullstrand, v. Kries and Nagel, are indicated at the top of the page by the initial letter of the author's name, G., K. or N. The editors' comments in the footnotes are indicated in the same way.

A portrait of H. v. Helmholtz will appear as a frontispiece of the next volume that is issued.

W. Nagel

Rostock, September, 1909

The first volume of the third edition of the "Handbuch der physiologischen Optik" appeared in 1909. The next volume to be issued was the third dated 1910; which was followed by the second volume in 1911.

PREFACE TO THE SECOND VOLUME

With the appearance of the second volume the new edition of the Physiological Optics reaches its completion, later and differently from what was anticipated when the work was begun. Professor W. A. NAGEL to whom was entrusted the revision of the second volume suffered an accident late in March, 1910, which developed into a severe illness resulting in his death. When his work was interrupted he had completed the revision of the original text together with the minor additions thereto. Of the new chapters which were to be included in this volume the first one on Adaptation, Twilight Vision and Duplicity Theory had likewise been written and printed and the first proof of it corrected. When it became necessary to entrust the completion of the work to other hands, the publisher begged me to undertake the task; and I have complied with this wish not without many misgivings but influenced finally by the fact that my personal relations with NAGEL made me acquainted with his views in the main, and since my opinions were not materially divergent from his at any point I was perhaps in the best position to continue what he had already begun and to bring it to a conclusion to some extent as he had intended. The task was indeed all the more difficult for me because NAGEL had already planned the remaining chapters in detailed fashion (as indicated by numerous preliminary memoranda) but except for a very summary outline had not prepared any formal manuscript on the subject. Here therefore I was obliged in the main to follow my own judgment. On the other hand with respect to the first of the new chapters, apart from the fact that it had already been put in type as stated above, it was due to the author and his memory not to alter that materially. It may therefore very well be that the disadvantages, to some extent unavoidable in the whole work, a certain lack of unity and perhaps also of completeness, will be more apparent in this volume than in the first and third volumes. Should this be the case, the special difficulties incident to the change of editorship may be alleged by way of explanation and excuse.

The alphabetical index of subjects for all three volumes will be found at the end of this volume; and likewise a few corrections.

V. KRIES

Freiburg, April, 1911

Table of Contents of Volume I

Anatomical Description of the Eye

Physiological Optics

Part I

The Dioptrics of the Eye

Appendices by A. Gullstrand

Helmholtz's Treatise

on

Physiological Optics

Anatomical Description of the Eye

§1. General Structure of the Organ of Vision

The eyes of animals may distinguish only light from darkness or may perceive form as well.

1. *Distinguishing only light and darkness.* This is probably the case in the "eyespots" of the lowest forms of animal life (annelids, intestinal worms, starfish, sea-urchins, jelly-fish, infusoria, etc.). The only essential for this purpose is a nerve sensitive to light, the peripheral end of which lies exposed to the exterior under a transparent covering. In most cases, the peripheral end of this nerve is surrounded by pigment of one colour or another and is thus rendered visible. However, it has not been ascertained whether all pigmented "eyespots" in the lower animal forms actually serve for the perception of light. On the other hand, the reactions to light of some lower animals without "eyespots" force us to the conclusion that nerves, sensitive to light but unaccompanied by pigment, may be present in transparent animals, though an investigator has no way of recognizing them.

2. *Distinguishing not only light and darkness, but form as well.* The ability to do this requires an apparatus of separate nerve fibres in order to perceive light coming from separate luminous points. It is no longer necessary for each nerve fibre to receive light rays from all parts of the environment, but only from one small portion of it. Thus to each individual nerve fibre corresponds a certain field of vision, and consequently it is possible to distinguish which elementary fields of vision of the entire area perceived contain luminous bodies and which do not. The smaller the size of each single field of vision and the larger their total number, the more minute will be the portions of surrounding bodies which may be perceived. In the highest development of the organ of vision, the size of these separate elements becomes imperceptibly small in proportion to that of the total field. The requirement for clear vision in such an organ might be expressed by saying, light coming from a single illuminated point in the environment must fall only upon a single point of the nervous substance or retina that is sensitive to light.

The subdivision of the light coming from different parts of the environment is brought about either (1) by means of funnel-shaped, opaque septa (the composite eyes of invertebrates), or (2) by the refraction of light at curved refracting surfaces (the simple eyes of invertebrates and the eyes of vertebrates).

There is no sharp line of demarcation between eyes which perceive only light and darkness and those which also perceive form. Even in the lowest forms of animal life, the layers of pigment surrounding the light-sensitive nerves are so arranged that light can fall only on the exposed sides at the ends of the fibres. By moving its body, an animal with such eyespots would be able to ascertain from which direction most light comes, in very much the same way as a human being determines the direction of radiant heat by means of his cutaneous sensation or as a patient with an entirely opaque crystalline lens is able to make out the position of a window in a room. The pigment layers of the eyespots are thus seen to have a very important function. In forms such as the leeches and planaria, in which a transparent, spherical or conical body lies in front of the nervous substance, different parts of the retina may be impressed in different degrees by light coming from different directions. From this type there occurs a gradual differentiation of structure through the simple eyes of crustacea, arachnids and insects (which usually have something on the order of a lens and vitreous humor beyond the cornea) to those of the molluscs and especially of the cephalopods, whose eyes are quite like those of vertebrates. That clearness of vision in such eyes is, in general, directly proportional to their linear dimensions, may be inferred because the miscroscopical elements of animal tissues, especially those of the nervous system, are more or less of the same size in all classes of animals, and also because clearness of vision is essentially dependent upon the number of individual receptive elements present, which must be approximately proportional to the extent of the posterior surface of the vitreous humor of the simple eye.

Compound eyes occur in crustacea, where they often appear as an aggregate of conically elongated simple eyes. They are best developed among insects. Their outer surface is somewhat spherical and often composes more than one-half or even as much as two-thirds of the surface of a sphere. At the centre of the sphere lies a club-shaped swelling of the optic nerve, from which fibres run out radially in all directions towards conically shaped and radially arranged vitreous bodies. The bases of these latter are turned towards the cornea, which generally presents on its outer surface a rather flat, six-sided or four-sided facet for each cone, whereas on the inner surface it often has lens-like swellings. The individual transparent cones are separated from one another by the funnel-like pigment layers which surround them. The accompanying illustration (Fig. 1), taken

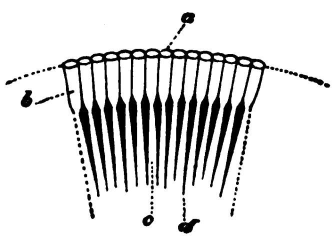

Fig. 1.

from JOH. MÜLLER,[1] represents a number of such cones from the eye of a nocturnal butterfly. The facets of the cornea are indicated by the letter *a*, the transparent cones by *b*, the fibres of the optic nerve by *c*, and the pigment between them by *d*.

If each cone were provided with only one nerve fibre, the field of vision would be divided only into as many parts as there were cones. Recently, however, GOTTSCHE[2] has demonstrated that an optical image of objects in front of the eye is thrown on the inner ends of the

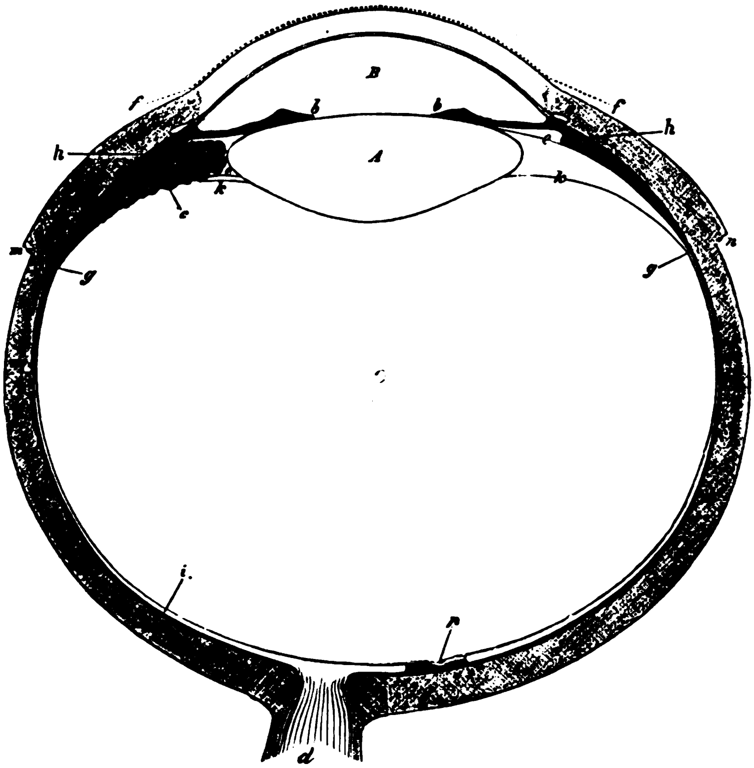

Fig. 2.

cones. Consequently, if a number of perceptive nerve elements were present, there might be still another subdivision of individual impressions in each cone. If there were only one nerve element for each cone, the refraction of the light would still have a functional significance, because light falling parallel to the axis of the cone is concentrated on the end of the nerve fibre, and that from other adjacent points in the field of vision is prevented from reaching it more effectively by this means than by septa.

[1] *Zur vergleichenden Physiologie des Gesichtssinnes.* Leipzig 1826. S. 349. Taf. VII. Fig. 5.

[2] J. MÜLLERS *Archiv. für Anat. u. Physiol.* 1852. S. 483.

Fig. 2 represents a horizontal section of the human eye magnified five times. The eyes of vertebrates are similar to the human eye in all important features. These eyes contain the following transparent elements:

1. The aqueous humor in the anterior chamber of the eye *B*.
2. The crystalline lens *A*.
3. The vitreous humor *C*.

These are enclosed by three concentrically situated systems of membranes or tunics.

1. The system of the *retina* (*i*) and *Zonula Zinnii* (*e*) which directly surrounds the vitreous humor and is attached in front to the lens *A*.

2. The system of the *uvea*, which consists of the *choroid* (*chorioidea*) (*g*) (indicated by the heavy black line), the ciliary body (*h*) and the *iris* (*b*). It surrounds the preceding systems together with the lens, and has but one opening, the *pupil*, situated in front of the lens.

3. The hard capsule of the eyeball, which, over the greater part of its posterior surface, consists of the opaque white *sclerotica*, and over the smaller anterior part, of the transparent cartilaginous *cornea*. In the living eye, one sees between the eyelids the anterior portion of the *sclerotica* (the white of the eye) and, behind the transparent convex cornea, the brown or blue ring-shaped iris, in the middle of which is the pupil.

The line passing through the middle of the cornea and the centre of the whole eye is known as the *axis of the eye*, because around this line the eye is practically symmetrical. A plane perpendicular to the axis through the widest part of the eyeball is called the *equatorial plane*.

In the following paragraphs a description will be given of the separate parts of the eye, which however will include only such details as are requisite for explaining the functions of the eye.[1]

The principal reference works on the comparative anatomy and physiology of the eye are:

J. Müller, *Zur Physiologie des Gesichtssinnes.* Leipzig 1826. S. 315.
R. Wagner, *Lehrbuch der vergleichenden Anatomie.* 1835.
J. Müller, *Handbuch der Physiologie des Menschen.* Coblenz 1840. Bd. II, S. 305.
R. Wagner, *Lehrbuch der speziellen Physiologie.* 1843. S. 383.
v. Siebold und Stannius, *Lehrbuch der vergleichenden Anatomie.* Berlin 1848.
Bergmann und Leuckart, *Anatomisch-physiologische Übersicht des Tierreichs.* Stuttgart 1852.

[1] The description of the anatomy of the eye given in the next six sections is not only brief, as the author implies, but also in many respects inaccurate according to our present knowledge, as will be occasionally pointed out in footnotes. (D. H.)

The following give good descriptions of the structure of the human eye:

TH. SÖMMERING, *Abbildungen des menschlichen Auges.* Frankfurt a. M. 1801.—In Latin also.

C. F. TH. KRAUSE, *Handbuch der menschlichen Anatomie.* Hannover 1842. Bd. I, T. II. S. 511—551.—Contains also earlier literature on the anatomy of the eye. S. 733-745.

E. BRÜCKE, *Anatomische Beschreibung des menschlichen Augapfels.* Berlin 1847.

W. BOWMAN, *Lectures on the parts concerned in the operations on the eye and on the structure of the retina and the vitreous humour.* London 1849.

A. KÖLLIKER, *Mikroskopische Anatomie oder Gewebelehre des Menschen.* Leipzig 1854. Bd II, S. 605.—More recent literature also, S. 734-736.

DUJARDIN, Remarques sur certaines dispositions de l'appareil de la vision chez les insectes. *C. R.* XLII, 941. Inst. 1856, 194.

§2. Sclerotica and Cornea

The sclerotica of the eye (*σκληρὸν*, *tunica albuginea*, *sclerotica dura*, tough membrane) encloses the greater part of the eyeball, controls its form and protects it from external injury. Its outer form is distinctly different from that of a sphere; the posterior side being quite flattened, while along the equator it is a little indented above and below and on the right and left sides by the pressure of the rectus muscles of the eye. In between these four places it bulges considerably. In most individuals the greatest diameter of the eyeball passes from a point on the upper nasal side to one on the lower temporal side. In front the sclerotica passes over into the very convex cornea; at the back a little toward the nasal side, it is perforated for the passage of the optic nerve (*nervus opticus*), (Fig. 2, *d*), and is here continuous with the fibrous covering of the latter. The sclerotica is thicker both in front and behind than it is at the equator of the eye, as shown in the figure. The anterior thickening is caused by the attachment of the tendons of the motor muscles of the eye to the sclerotica and their fusion with it. The place of insertion of the internal rectus is at *m*, and that of the external rectus is at *n*.

The sclerotica is composed of fibrous tissue. It is white, only slightly translucent, flexible and quite inelastic. It may be classed with the collagenous substances on the basis of its chemical composition. Microscopically, it consists of a very dense felt-work of connective tissue fibres, which in the main run parallel to the surface, thereby making it possible to split the tunic in imperfect layers. Between these, as in other tendons, lies a network of very delicate elastic fibres which show thickenings containing nuclear remnants at those places where their cells of origin were once located.

The cornea is set into the anterior margin of the sclerotica and has the general shape of a very convex watch-glass. Its anterior surface closely approximates the form of a segment of a prolate spheroid generated by the revolution of an ellipse around its axis major. The end of this axis lies at the centre of the cornea. The shape of the

posterior surface is not definitely known. In adults the cornea is somewhat thinner in the middle than at the periphery.

The cornea consists of the following layers from the outside inwards:

1. An *epithelium*,[1] composed of stratified, flat cells of horny consistency (pavement epithelium), indicated in the figure by the broken line *ff*. It is continuous with the conjunctiva of the eyelids. The anterior surface is kept smooth and moist by a continual flow of tear fluid.

2. The fibrous layer of the cornea (*Substantia propria corneae*)[2] is the thickest of them all and is left white in the figure. Its chemical composition places it among the cartilages, as on boiling it gives chondrin. It consists of a felt-work of fibres similar to those of the sclerotica except that the fibres are united in flat bundles lying parallel to the surface of the cornea; and hence the cornea also may be divided in imperfect layers. In the adult the cornea contains no blood vessels. However, between the bundles of fibres, there is a system of branched nucleated cells resembling undeveloped elastic tissue as found in many organs rich in connective tissue. It is possible that these cells may carry on the necessary exchange of fluids for the nutrition of the substance of the cornea. The substance of the cornea appears to be entirely transparent by ordinary illumination. However, if considerable light is concentrated by a convex lens on a point of the cornea, it appears cloudy, as the amount of light reflected by the surfaces of its microscopical components becomes sufficient to be visible under these conditions.

3. The posterior homogeneous membrane (*Membrana Descemeti, Membrana Demoursii*, etc.) is a structureless, transparent, fragile membrane, 0.007 to 0.015 mm in thickness. It rolls up when separated from the cornea. It resembles elastic tissue[3] in its resistance to the action of boiling water, acids and hydroxides. On the surface next the aqueous humor, it bears a layer of large polygonal epithelial cells,[4] indicated by the dotted line on the inner surface of the cornea (Fig. 2).

The plane of union between cornea and sclerotica is not perpendicular to the surface of the eyeball, but passes inwards and backwards,

[1] ¶The corneal epithelium (*conjunctiva*) is composed of about five rows of cells; one row of cylindrical cells, the basal cells; two rows of polyhedral cells in the middle layer; and the external layer of two rows of flattened cells. (D. H.)

[2] ¶The anterior homogeneous membrane or membrane of DESCEMET is now regarded as the second layer of the cornea, the *substantia propria* being the third. (D. H.)

[3] ¶SASSE (Zur Chemie der Descemet'schen Membran, *Unters. d. physiol. Inst. Univ. Heidelberg*, Bd. 2, 1879) has demonstrated that the tissue of the anterior and posterior homogeneous membranes is not identical with elastic tissue. (D. H.)

[4] ¶This layer is usually considered as a separate layer, the endothelium of the cornea. (D. H.)

the sclerotica overlapping on the outer surface, the cornea on the inner. On the inner surface the margin of the cornea forms a fairly regular circle. On the outer surface, on the contrary, it appears to be a horizontal oval, as the sclerotica overlaps it above and below more than on the sides. At this plane of union, the fibres of the cornea are directly continuous with those of the sclerotica.

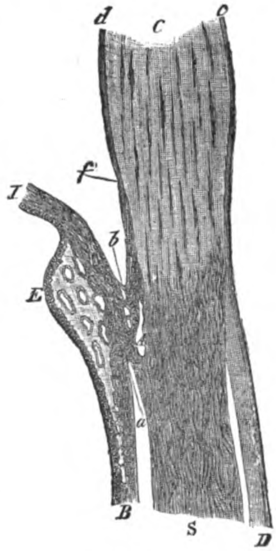

Fig. 3.

The membrane of DESCEMET, however, acts quite differently at the border of the cornea. A cross-section of this region is shown in Fig. 3, where the sclerotica is marked *S*, the cornea *C*, its external epithelium *c*, passing over into the conjunctiva *D*, and the membrane of DESCEMET *d*. From *f* a network of elastic fibres arises between the membrane of DESCEMET and the substance of the cornea, while the former appears to end abruptly at this point. By the separation of this layer of elastic fibres from the sclerotica and its fusion with the lamella *a*, the canal of SCHLEMM is formed further back. This is a ring-shaped canal situated at the boundary between cornea and sclerotica. Laterally it is bounded by the sclerotica, but its medial wall consists of elastic tissue in front and of fibrous tissue behind. To this inner wall the muscular portions of the uvea are attached. The canal of SCHLEMM appears to carry blood.

Although measurements of the dimensions of the eye are of the greatest importance in physiological optics, many difficulties are usually involved in executing them, in the first place because the form of the entire eyeball and of its individual parts varies greatly in different eyes, and, secondly, because after death the eye undergoes many changes. The individual variations are so great that the averages of observations on different eyes should be used only with great caution. For the determination of exact and accurate results, all important measurements must be made on the same eye.

The external form of the eyeball is determined by the pressure of the fluids which it contains. Immediately after death a majority of its blood vessels empty themselves, which naturally produces a diminution in the pressure. The volume of the fluids inside the eye gradually becomes reduced still further by endosmotic paths, so that the eyeball becomes flaccid and the tunics, especially the cornea, become wrinkled. Either measurements on the form of the eyeball must be made on very fresh eyes, or the pressure must be artificially restored, as BRÜCKE[1] did, by passing a canula through the optic nerve and connecting it with a vertical tube containing a column of water about 0.4 m in height. These methods suffice for the measurement of the different diameters of the eyeball, but in order to measure one of the most

[1] E. BRÜCKE, *Anat. Beschreibung des menschl. Augapfels.* Berlin 1847. S. 4.

important optical elements of the eye, the convexity of the cornea, it is not sufficient merely to restore the pressure approximately. The radius of curvature at the vertex of the cornea becomes greater in direct proportion to the pressure, as the writer has found by a method of measurement described below. The reason of this is probably to be found in the fact that a membranous covering which contains fluid must approach more and more the form of a sphere in proportion as the internal pressure of the fluid increases, because for a given superficial area a sphere has the greatest volume. When this happens in the eye, the constriction between the cornea and the sclerotica must be pushed out and as a result the cornea becomes less convex.

Under these circumstances it is evidently an essential requirement to determine all the more important dimensions of the eyeball as far as possible by measurements of the eyes of living persons.

The earlier measurements of the eye were usually made with compasses. C. KRAUSE, who carried out a very extensive system of measurements, measured the external dimensions of the eye with this instrument. Then he cut the eyes in half along a line which had been previously marked out, using a razor to section the cornea, iris and lens, and cutting the sclerotica with scissors. Next he laid the halves in a bowl full of egg-albumen solution with the cut surfaces just below the surface of the fluid. Thus he measured the dimensions of the cross sections partly with compasses, partly with a barred glass micrometer in the ocular of a microscope of low magnification, and partly with a glass ruled in squares which he laid upon the surface of the fluid. He had many opportunities of using very fresh eyes. The external measurements of the sclerotica obtained from these may be considered as thoroughly reliable, but the convexity of the cornea, which depends upon the pressure of the fluids, must have been greatly altered in the sectioned eyes.

KRAUSE's table for the form of eight eyeballs is appended below. No. I is that of a drowned man 30 years old; No. II is the right eye of a man 60 years old who died from having his throat cut; Nos. III and IV are the left and right eyes of a man 40 years old who was hanged; Nos. V and VI and Nos. VII and VIII are the left and right eyes of two men, 20 and 21 years old, respectively, both of whom were executed by the sword. The measurements are given in Paris lines.[1]

No.	Axis of the eye		Diameter					
			trans-verse	perpendicular		diagonal		
						larger		smaller
	outer	inner		outer	inner	outer	inner	
I.	10.9	9.85	10.9	10.8	9.9	11.25	10.3	
II.	11 05	10.0		10.3	9.4	11.1	10.2	11.05
III.	10.7	9.8	10.7	10.5	9.6	11	10.2	10.6
IV.	10.5	9.5	10.6	10.3	9.5	10.9	10.1	10.7
V.	10.8	9.55	10.9	10.55	9.6	11.3	10.35	11
VI.	10.8	9.55	11	10.6	9.45	11.3	10.2	11.1
VII.	10.65	9.4	10.75	10.3	9.45	10.75	9.6	10.75
VIII.	10.65	9.45	10.75	10.3	9.15	10.9	9.75	10.7

BRÜCKE has made measurements on eyes which were distended by a water pressure of 4 decimetres. He maintains that the axis of the eyeball varies between 23 and 26 mm, the greatest horizontal diameter between 22.8 and 26 mm, and the greatest vertical diameter between 21.5 and 25 mm.

[1] 1 Paris line = 2.2558 mm. N.

C. Krause compares the inner convexity of the sclerotica with the surface of an ellipsoid of revolution. The axes which he has calculated and his results concerning the thickness of the cornea and sclerotica at different points are given in the following table.

No.	Thickness of the Sclerotica along the optic axis	Thickness of the Sclerotica at the equator	Thickness of the Sclerotica at the corneal border	Semi-axes of the ellipsoid of the posterior corneal surface major	Semi-axes of the ellipsoid of the posterior corneal surface minor	Thickness of the cornea centre	Thickness of the cornea edge
I.	0.55	0.45	0.35	5.12	4.45	0.4	0.5
II.	0.5	0.35		5.05	4.15	0.35	0.5
III.	0.45	0.4	0.35	5.12	4.23	0.4	0.5
IV.	0.5	0.4	0.3	5.07	4.41	0.4	0.45
V.	0.65	0.4	0.3	5.14	4.58	0.5	0.55
VI.	0.65	0.5	0.3	5.05	4.43	0.48	0.55
VII.	0.55	0.5	0.4	5.05	4.41	0.53	0.63
VIII.	0.6	0.5	0.4	4.93	4 19	0.5	0.62

C. Krause's measurements of the form of the cornea have been omitted, because his method does not appear to the writer to be sufficiently trustworthy in such an important matter. It may be observed that he pronounced the anterior surface of the cornea to be spheroidal and the posterior surface to be that of a paraboloid of revolution near its vertex. In the case of several corneae examined by the writer the thickness in the middle half of the cross section was found to be nearly constant and increased rapidly only near the periphery, so that in the middle of the vault the two surfaces appeared to be almost concentric.

Kohlrausch attempted to measure the radius of curvature of the cornea in the living eye by determining the size of the reflex image in it. For purposes of examination, the patient sat on a very masssive stool with a high back. His head was held by a special apparatus so that he could be comfortable and passive at the same time. He fixed his eyes on a little white spot in the centre of the objective of a Kepler telescope placed two to three feet away. The telescope was directed towards the eye in such a manner that the white spot lay in the same horizontal plane as the vertex of the cornea. Two fine threads were stretched parallel to each other in the focal plane of the ocular. These two parallel lines could be moved nearer together or farther apart by means of a screw. On either side, and in the same horizontal plane, a light was placed whose rays fell upon the eye through a round opening in a small screen and were reflected from the cornea in such a manner that two small images of the luminous points were seen through the telescope. When the threads were exactly aligned on these images, a finely divided rule was placed in front of the eye and the separation of the images in the cornea read off on it. The radius of the cornea may be approximately calculated from the following data: (1) the amount of separation of the images, (2) the distances of the eye from the openings in the screen and from the centre of the objective, and (3) the distance between the two points last mentioned.

Kohlrausch obtained an average of 3.495 Paris lines (7.87mm) for measurements on twelve eyes, the smallest value being 3.35, the highest 3.62. The probable error of the individual observations was calculated as 0.02 Paris lines.

By a similar, but not accurately described method, Senff determined not only the radius of curvature, but also the ellipticity of the cornea, his results being given in the following table.

	Radius of curvature at vertex	Square of the eccentricity	Axis major	Axis minor	α
Right eye. Vertical	7.796	0.1753	9.452	8.583	3.6°
Right eye. Horizontal	7.794	0.2531	10.435	9.019	3.9°
Left eye. Vertical	7.746	0.4492	11.243	8.344	1.6°

SENFF calls the angle between the vertex of the ellipse and the end point of the axis of the eye the angle α. In the vertical section the vertex of the ellipse lies below the end point of the axis of the eye, and in the horizontal section it is situated more outwards. Apparently what SENFF means here by the "axis of the eye" is the line which is hereafter defined as the "visual axis" (*Gesichtslinie*) of the eye.

In making these measurements the greatest difficulty is to keep the patient's eye and head steady. In any method in which images are measured and in which it is necessary to read the mark on a scale which corresponds to one edge of the corneal image and then to read that which coincides with the other edge, even the slightest movement of the head from side to side between the two readings will add to or subtract from the size of the image as measured. Accordingly, the author has constructed an instrument whereby it is possible to make these and other measurements of the eye accurately and quite independently of small movements of the head, and which is therefore called an *ophthalmometer*, although it may be advantageously used also for making many other measurements besides, especially measurements of optical images.

If an object is observed through a glass plate with plane parallel faces, which is held obliquely to the line of vision, it will be seen in its natural size, but shifted slightly to one side. This displacement increases as the angle is diminished between the direction of the rays of light and the surface of the plate. The ophthalmometer is essentially a telescope, adapted to vision at short distances, having two glass plates placed close together, in front of its objective. They are so adjusted that over one half of the objective, the observer sees through one plate, and over the other half through the other plate. If the two plates are in a plane perpendicular to the axis of the telescope, only a single image of the object in question will be seen. If, however, both plates are rotated a little in opposite directions, the single image will be divided into two images, whose distance apart will increase with the angle through which the plates are turned. This separation of the twin images may be calculated from the angles which the plates make with the axis of the telescope. If the two images of a line which is to be measured are so adjusted that they just touch at their ends, the length of the object-line is equal to the distance between a pair of corresponding points of the two images and may be calculated from it.

The instrument itself is shown in Fig. 4 in vertical section and in Fig. 5 in horizontal section, one-half actual size. The rectangular box $B_1B_1B_2B_2$ containing the adjustable glass plates is attached to the end of the telescope A next the object-glass. In Fig. 4, the side wall of the box has been removed and all parts in the lower half are drawn as though sectioned in the middle vertical plane. The base of the box is formed by a strong rectangular frame, shown in Fig. 4, running around the box; and to it thin brass plates are fastened to form the walls, as may be seen in Fig. 5. Conical holes are drilled in the middle of the horizontal portion of the frame, in which the axles CC of the two plates rotate. Outside the box each axle has a cylindrical disc d which is graduated in degrees on the curved surface. At a there is a vernier scale reading

to tenths of a degree. Inside the box, each axle carries a toothed wheel *ee* and a metal frame *g* in which the glass plate *f* is fastened. The frame of each plate has only three sides, the side turned towards the other glass plate being missing. The two glass plates constituted originally a single plane parallel

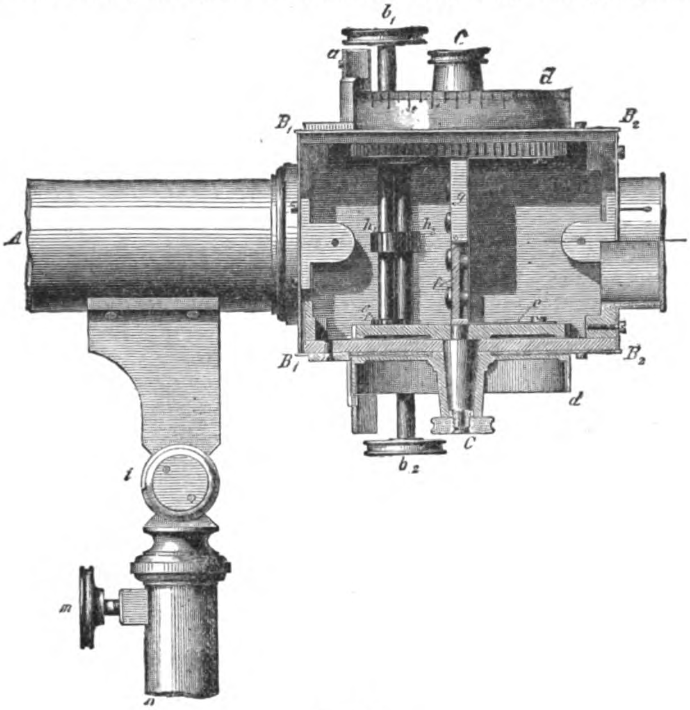

Fig. 4.

plate. A complete metal frame was made for this plate and fastened between the surfaces of the two toothed wheels. Then the axles were rounded off on a lathe and finally the frame was cut through in the middle. The glass plate was cut through in the same manner, and each half fastened in its corresponding half of the frame. Thus an exact adjustment of the positions of the plates on the two axles was accomplished. The toothed wheels are moved by the sprockets c_1 and c_2 which are rigidly connected with the axles b_1c_1 and b_2c_2. Moreover, each of these axles has a sprocket h in the middle. If the knob at b_1 is turned, the lower toothed wheel controlling the lower glass plate will be moved by means of the sprocket c_1. Furthermore, the sprocket h_1 engages with the sprocket h_2 and rotates the second axle b_2c_2 by an equal amount in the opposite direction. As a result of this the sprocket c_2 acts upon the upper toothed wheel connected with the upper glass plate and turns it through an angle almost as large as that of the lower plate. The rotation of each plate is measured by the graduated discs attached to the axles on the outside of the box.

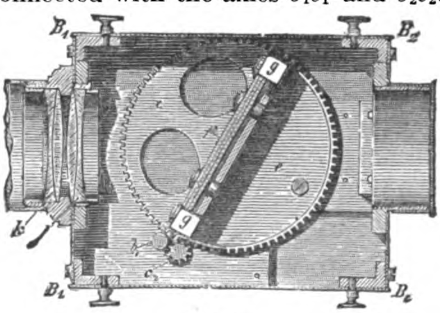

Fig. 5.

It is necessary to employ two plates which are rotated at approximately the same angle because the images of the object viewed through the plates will not only be displaced laterally, but will also be brought a little nearer together. If the amount of this approximation is unequal for the two images of the same object, the telescope cannot be focused exactly on both simultaneously.

The objective of the telescope is composed of two lens-systems k and l. The achromatic doublet k is used by itself when the telescope is to be focused on a distant object. Its double convex crown glass lens, as usual, is turned towards the object. On the other hand, when the object is very close, a single lens-system does not give a good image, because it is designed to concentrate parallel rays at a point. Hence, the writer has inserted a second achromatic doublet l, with its crown glass lens turned towards the other doublet. If now the object is in the first focal plane of this second system, the rays that issue from it will be bundles of parallel rays, which in turn will be concentrated by the first system at its second focal point. Thus sharper images are obtained. In the writer's own instrument the focal lengths of k and l are 6 inches and 16 inches, respectively. The telescope is supported on an upright n in which a cylindrical bar fits so that the instrument can be turned and at the same time raised or lowered. The telescope itself is fastened to this bar by a hinge joint i. Thus the axis of the telescope can be pointed in any direction. And, finally, the box with the glass plates can be turned around the end of the telescope.

Let us proceed now to show how the shifting of the images may be found from the angle of rotation of the glass plates.

In Fig. 6 $A_1A_1A_2A_2$ represents one of the glass plates; and the straight lines a_1c_1, c_1c_2, and c_2a_2 show the path of a ray which traverses it. The normals to the two faces at entrance and emergence are $b_1c_1d_2$ and $b_2c_2d_1$, respectively. The angle of incidence $b_1c_1a_1$, which is equal to the angle $b_2c_2a_2$, is marked α, and the angle of refraction $d_2c_1c_2$, which is equal to $c_1c_2d_1$, is marked β. The thickness of the plate is denoted by h. The luminous point a_1 appears to an eye below the plate to lie in the prolongation of a_2c_2 backwards. If x denotes the length of the perpendicular a_1f drawn from a_1 to the prolongation of the emergent ray, this distance x will be the apparent lateral shifting of the luminous point. Now

$$x = c_1c_2 . \sin \angle c_1c_2f$$

$$c_1c_2 = \frac{h}{\cos \beta}$$

$$\angle c_1c_2f = \angle d_1c_2f - \angle d_1c_2c_1$$

$$= \alpha - \beta$$

$$x = h . \frac{\sin(\alpha - \beta)}{\cos\beta} .$$

The angle α is measured by the instrument. The thickness of the glass plate h must be known, as well as its index of refraction n with respect to air. Then

$$\sin \alpha = n . \sin \beta ,$$

from which β may be calculated, and then all the factors for computing the value of x are known. If two rotatable glass plates are used, as in the instrument above described, the separation E of two luminous points, whose images have been superposed, will be twice as great as x; that is,

$$E = 2h \frac{\sin (\alpha - \beta)}{\cos \beta} .$$

Lacking other ways of doing it, the magnitudes denoted by n and h can be ascertained by measurements made with the instrument itself, by finding the angle through which the plates have to be turned to superpose each division of an accurate scale on the next following division or on every second or third

division, etc. In this way a series of correlated values is obtained for x and α, from which h and n may be calculated by a suitable process of elimination. If many observations are to be made, it is well to prepare a table for E for every degree from 0° to 60°.[1]

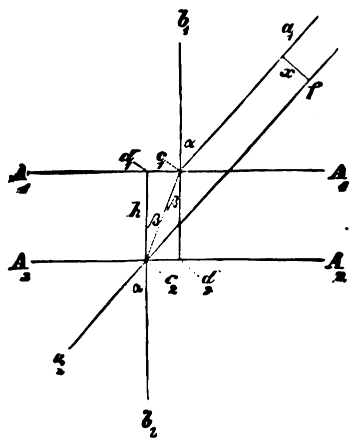

Fig. 6.

The same adjustment of the double images that is found for a rotation of α degrees occurs also for a rotation of $-\alpha$ and likewise for $(180°-\alpha)$ or $(\alpha-180°)$. In order to eliminate errors in the disc scale and in the parallelism of the faces of the glass plates, it is advisable to repeat each measurement in these four positions and to take the average of the four results thus obtained.

One of the most important advantages of the ophthalmometer is that the linear magnitude of the apparent distance of its double images is independent of the distance of the object, so that it is not necessary to know the latter in order to make the measurements.

If the instrument described is used for the measurement of a corneal image, small movements of the head of the subject do not introduce a disturbing factor, inasmuch as both images will always move in the same way, and their relative positions will not be altered. If at the same time the object which is responsible for the corneal image is far enough away, so that small movements of the head are negligible in comparison with its distance, the size of the image also will not be noticeably affected by these movements, and the head can be kept steady enough by providing an easy support for the chin.

A bright window may be chosen as object for the image in the cornea. If the parallel boundaries of two images of such a bright surface are adjusted in contact in the ophthalmometer, the eye of the observer will be quite sensitive to any overlapping or separation of the two images, which will be noticeable at once in virtue of a white or black line between the two uniformly illuminated fields. On the other hand, a scale placed far enough away from the eye may be used as object if one of its divisions is marked by a small lamp and another preferably by two such lamps placed close together. The measurement in this case consists in adjusting one of the images of the single lamp midway between that of the pair of lamps. This method of adjustment is capable of great accuracy, as has been observed by Bessel in the measurement of stellar parallaxes with the heliometer.

The calculation of the radius of curvature of the cornea is very simple provided the reflex image as measured is relatively small in comparison. In this case the size of the object is to its distance from the eye in the same ratio as the size of the image is to half the radius of curvature; so that the latter may then be calculated from this proportion. The ellipticity of the cornea may also be determined in this way, by measuring the size of the reflex image in each position of the eye when, by appropriate variations of the point of fixation, the eye is made to turn through successive known angles, both laterally and vertically. Thus the various values of the radius of curvature over the different parts of the cornea are calculated first of all, whence are computed the elements of the ellipsoid which the cornea approximately resembles in form.

[1] In the first edition h occurs here instead of E, evidently an error in the manuscript or a typographical error. N.

Appended below are the elements of the horizontal sections of the cornea of three women between 25 and 30 years of age, with whose eyes a series of measurements were made by the author.

Designation of the Eye	O.H.	B. P.	J. H.
Radius of curvature at the vertex	7.338	7.646	8.154
Square of the eccentricity	0.4367	0.2430	0.3037
Semi-axis major	13.027	10.100	11.711
Semi-axis minor	9.777	8.788	9.772
Angle between axis major and visual axis	4. 19′	6° 43′	7° 35′
Horizontal diameter of the area	11.64	11.64	12.092
Distance of the vertex from the base	2.560	2.531	2.511

The centre of the anterior surface of the cornea in all three of these eyes corresponds almost exactly with the vertex of the ellipse. The visual axis lies on the nasal side of the forward extremity of the axis major of the corneal ellipsoid.

Measurements of the eyeball will be found in the following:

1723–30. PETIT in *Mem. de l'Acad. des sciences de Paris.* 1723. p. 54.—1725. p. 18.—1726. p. 375.—1728. p. 408.—1730. p. 4.

1738. JURIN, *Essay upon distinct and indistinct vision.* p. 141 in SMITH's *Compleat System of Optics.*

1739. HELSHAMA, *Course of Lectures on Natural Philosophy.* London 1739.

1740. WINTRINGHAM, *Experimental Inquiry on some parts of the animal structure.* London, 1740.

1801. TH. YOUNG, *Philos. Transact.* 1801. p. 23.

1818. D. W. SOEMMERING, *De oculorum hominis animaliumque sectione horizontali.* Göttingen 1818. p. 79*.

1819. BREWSTER in *Edinburgh Philos. Journal.* 1819. No. I. p. 47.

1828. G. R. TREVIRANUS, *Beiträge zur Anat. und Physiol. der Sinneswerkzeuge.* Bremen 1828. Heft I. S. 20*.—This work contains a summary of the results of earlier investigators.

1832. C. KRAUSE, Bemerkungen über den Bau und die Dimensionen des menschlichen Auges, in MECKELS *Archiv. für Anatomie und Physiol.* Bd. VI. S. 86* (Description of the methods and measurements in two eyes.) Extract from this in POGGENDORFFS *Ann.* XXXI. S. 93*.

1836. C. KRAUSE in POGGENDORFFS *Ann.* XXXIX. S. 529* (Measurements of 8 human eyes).

1839. KOHLRAUSCH über die Messung des Radius der Vorderfläche der Hornhaut am lebenden menschlichen Auge, in OKENS *Isis.* Jahrg. 1840. S. 886*.

1846. SENFF in R. WAGNERS *Handwörterbuch der Physiol.* Bd. III. Abt. 1. Art.: Sehen. S. 271*.

1847. E. BRÜCKE, *Beschreibung des menschl. Augapfels.* S. 4 and 45*.

1854. H. HELMHOLTZ, in GRAEFES *Archiv. für Ophthalmologie.* II. S. 3.

1855. SAPPEY, *Gazette medicale.* No. 26, 27.

1857. ARLT, *Archiv. f. Ophthalmologie.* III, 2. S. 87.

1858. NUNNELEY, *On the organs of vision.* London. p. 129.

1859. J. H. KNAPP, *Die Krümmung der Hornhaut des menschlichen Auges. Habilitationsschrift.* Heidelberg 1859. Also: *Arch. f. Ophthalm.* VI, 2. S. 1-52.

1860. MEYERSTEIN, Beschreibung eines Ophthalmometers nach HELMHOLTZ. POGGENDORFFS *Ann.* CXI. S. 415-425, and HENLE u. PFEUFERS *Zft.* XI. S. 185-192.

1861. v. JÄGER, *Über die Einstellung des dioptrischen Apparates im menschlichen Auge.* Wien.

1864. R. SCHELSKE, Über das Verhältnis des intraocularen Druckes zur Hornhautkrümmung. *Arch. f. Ophthalm.* X, 2. S. 1-46.

§3. The Uvea

The *tunica uvea*[1] derives its name from its resemblance to a dark grape from which the stem has been removed. The opening for the stem corresponds to the pupil. The dark colour of this tunic is produced by the layer of pigment cells on its inner surface and in its stroma. The uvea is attached to the sclerotica in two places, namely, in the rear at the entrance of the optic nerve (Fig. 2, *d*) and in front at the inner wall of SCHLEMM's canal *a*. The portion marked *abba*, which is the iris, lies in front of and within this latter line of attachment, right behind the cornea. The posterior portion, which lies in contact with the inner surface of the sclerotica, is called the *choroid* (*chorioidea*).

The choroid, composed chiefly of blood vessels bound together by a characteristic connective tissue, forms a thin, dark membrane in the posterior portion of the eyeball. KÖLLIKER considers this connective tissue to be undeveloped elastic tissue. It consists of branched cells, some of which are pigmented, having very finely divided processes which are felted together. This peculiar stroma binds the arteries and veins of the choroid together beneath the sclerotica.[2] Inside this layer lies the looser layer of capillary vessels (*membrana chorio-capillaris*), which in turn is covered by the pigment cells on the surface towards the retina.[3] These pigment cells form a single layer towards the posterior part of the choroid, but become stratified[4] towards the ciliary body. Their nuclei are to be seen usually as a lighter area between the black pigment granules. In Fig. 7 (copied from KÖLLIKER) a surface view of these cells is shown at *a* and a side view at *b*. The pigment granules, which are small, flattened and rod-like, 0.0016 mm in length, are shown at *c*. These granules may be destroyed by chlorine and potassium hydroxide.

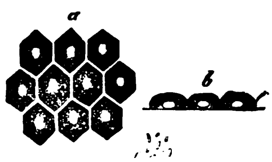

Fig. 7.

In front, the *ciliary muscle* (*tensor chorioideae, musculus* BRÜCKIANUS) is attached to the outer surface of the choroid, while the *ciliary proc-*

[1] ¶*Tunica uvea* (*uva*, L., a grape), uveal tract, more frequently termed *tunica vasculosa*, includes the *tunica chorioidea* (Choroid from Greek *chorion*, meaning a membrane, formerly spelled chorioid), the ciliary body and the iris. (D. H.)

[2] ¶The tunica choroidea is usually divided into three layers, from without inwards: *lamina suprachorioidea, lamina vasculosa, lamina choriocapillaris.* (D. H.)

[3] ¶The pigment cells, which are the sole derivatives of the outer layer of the optic cup, are, in consequence of this fact, now considered the outermost layer of the retina. (D. H.)

[4] ¶This is an error. The retinal pigment cells are not stratified at any point, though they may appear so in oblique sections. (D. H.)

esses (*processus ciliares*) arise from its inner surface. They are plaited and contain a network of blood vessels. The section in Fig. 2 is represented as passing through a ciliary process *c* on the left side, while on the right side it passes between two such processes, so that there only the ciliary muscle *h* is visible. The fibres of the ciliary muscle originate at the inner wall of SCHLEMM's canal where the elastic and collagenous tissues unite (at *a* in Figs. 2 and 3); whence they extend backwards along the external surface of the choroid and are inserted in this membrane. The fibres of this muscle are of the so-called visceral type, such as we find in most of the involuntary muscles. They are provided with longitudinally oval nuclei and are not cross-striated. BRÜCKE, who discovered this muscle, maintained that it stretched the choroid (together with the retina and hyaloid membrane closely attached to it at *g*) over the vitreous humor. DONDERS, on the other hand, supposes that the choroid constitutes its fixed point of origin, and that it stretches the elastic part of the inner wall of SCHLEMM's canal and consequently pulls the base of the iris backwards. Both of these actions probably occur together (see §12).

The ciliary processes are membranous folds of the choroid, which lie in the meridional planes of the eye. They are from 70 to 72 in number. They arise in the neighbourhood of the anterior border of the retina[1] (Fig. 2, *g*), increasing in height as they pass forwards and attain their greatest height near the external edge of the lens. From this point their height rapidly falls off, and their anterior extremities blend with the posterior surface of the iris. Their projecting sharp boundaries are often free from pigment and stand out as white lines when the ciliary region is viewed from behind through the vitreous humor. The ciliary processes contain a large number of blood vessels bound together by a stroma similar to that found in the choroid.

The *iris*, the most anterior part of the tunica uvea, forms a movable diaphragm for the eye. It arises in common with the ciliary muscle at the inner wall of SCHLEMM's canal at the posterior edge of the fibrous portion and is bound to the elastic portion of this inner wall (Fig. 3, *b*) by a network of elastic fibres which run freely through the aqueous humor. These elastic fibres form the *ligamentum iridis pectinatum*. Thence the iris extends medially to its pupillary margin, lying on the anterior surface of the lens, and consequently is slightly arched in front. It contains smooth muscle fibres, forming two muscles:

1. The *pupillary sphincter* (*musculus contractor sive sphincter pupillae*), surrounds the pupillary margin in the form of a ring about 1 mm wide. It lies in front of the pigment layer, but behind the chief nerves and vessels passing to the pupillary margin. Its fibres run in

[1] ¶At the *ora serrata*. (D. H.)

concentric circles and so decrease the size of the pupil by their contraction.

2. The *dilatator of the pupil* (*musculus dilatator pupillae*). The fibres of this muscle originate in the inner wall of SCHLEMM's canal and perhaps also in the fibres of the *ligamentum pectinatum*. They proceed to the posterior surface of the iris, bound together in a network passing medially to lose themselves in the substance of the sphincter.

The stroma of the iris is connective tissue. It is covered behind by the layer of pigment cells and in front by epithelium. This stroma frequently contains pigment cells, in which case the colour of the iris is brown. In their absence from the stroma, this semi-opaque medium in front of the dark posterior pigment cells causes the iris to appear blue.

The arrangement of the blood vascular system of the tunica uvea is very characteristic. As already indicated, the vessels make up the greatest portion of the mass of this coat. Its arteries (*arteriae ciliares posticae breves*, for the choroid and ciliary processes; *arteriae posticae longae* and *anticae*, for the iris) enter through the sclerotica, and communicate with the veins not only through a fine capillary network, as elsewhere in the body, but also through rather large connecting vessels which arise in the arteries of the choroid in delicate, fan-like arches and reunite to enter the veins (*venae vorticosae*). The *arteriae ciliares posticae breves*, about 20 small twigs, penetrate the sclerotica on its posterior surface, and branch dichotomously as they pass forwards. One portion pours blood through a capillary network which lies on the inner side of the choroid to supply the retina and the other supplies the veins through the wide communicating vessels of the *venae vorticosae*. Some of these veins (*vasa vorticosa*) leave the eyeball at its equator, and others (*venae ciliares posticae*) at the posterior portion of the sclerotica. Most of the branches of these arteries, however, pass forwards into the ciliary processes and form there a vascular network, from which recurrent branches pass into the anterior arches of the vortices. The vascular net of the iris receives some tributaries from that of the ciliary processes, but gets the greater volume of its blood supply through special vessels. Some of these penetrate the sclerotica at the back (*arteriae ciliares posticae longae*) and run forwards between the choroid and the sclerotica as far as the ciliary muscle, while others penetrate in front (*arteriae ciliares anticae*). These vessels form two anastomosing or interlacing vascular circles, one on the peripheral border of the iris (*circulus arteriosus iridis major*), the other near the pupillary margin (*circulus arteriosus iridis minor*). The iris is thickest over the latter and has a bulge on its anterior surface at this place.

In an uninjured eye the iris is visible through the cornea. Refraction causes it to appear nearer the cornea and therefore more convex than it actually is. On the other hand, if the eye of a cadaver is placed under water (which has approximately the same index of refraction as the aqueous humor), refraction is almost abolished, and then the iris appears but slightly convex in its natural form and position. In order to obtain a correct image of the iris of the living eye, J. CZERMAK[1] invented an instrument called the *orthoscope*. For all practical purposes it is a small box with glass walls which is used in such a manner that the eye to be examined forms its back side. Having been applied to the eye, it is then filled with water. The instrument illustrated in Fig. 8 has a bottom *fcb*, and a side wall *gab*, placed next the nose, both made of sheet metal. Their free edges are shaped to fit the face closely. The front *abcd* and the side *cdef* are made of flat glass plates. In order to make the edges against the face water-tight, CZERMAK recommended placing kneaded bread-crumbs on the face and pressing the rim of the instrument into them. The eye being shut at first, water from 29 to 33° C is poured into the box and the eye opened. Seen from the side, the cornea appears as a transparent, convex bladder, while the iris is seen to be an almost flat curtain across its base.

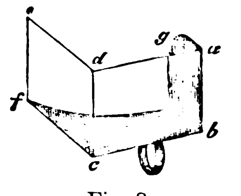

Fig. 8.

In this method there might be some doubt as to whether the form of the iris were not changed a little either by the refraction between the cornea and the water or by that between the cornea and the aqueous humor. A correct knowledge of the form and position of the iris is very important for the theory of the accommodation of the eye. Therefore, other methods of examination will be described here. An easy way of observing the iris relief is as follows: A light is placed a little to one side in front of the eye to be observed, and with the aid of a convex lens of 2-inch focus and relatively large aperture its rays are concentrated on a point of the cornea, so as to form there an image of the source. The cornea appears opaque at the illuminated spot. The focal point on the cornea constitutes a new source of illumination, from which the rays proceed directly to the iris without further refraction. If they fall on it obliquely, shadows of various lengths will be seen on the iris, due to the thickening which contains the *circulus arteriosus minor*. By means of these shadows, the amount of the forward or backward displacement of individual parts of the iris may easily be determined. When this method is used, the iris of a myopic eye is often so flat that there are no deep shadows on it. In normal eyes, on the contrary, prominent shadows are found surrounding the pupil. If the illuminating focal point is about 1 mm from the edge of the cornea, the shadows will frequently extend to the peripheral edge of the iris.

In order to realize the important fact that the iris lies close against the lens in the living eye, the same process may be employed, except that the focal point of the convex lens should be made to fall a little to one side on the anterior surface of the crystalline lens. Strongly illuminated in this way, the substance of the crystalline lens appears milky, and no shadows are cast by the iris. This is even better shown by the reflex from the anterior surface of the lens. In Fig. 9, C_1C_2 represents a convex spherical mirror, with a dark screen DE having an opening FG placed in front of it. The observer's eye is supposed to be at A and a source of light at B. If the ray of light BF passing the edge of the aperture at F is reflected at H along the line HA, the eye can get no light reflected from the part of the mirror between H and C_1 except such light as may come from the dark rear surface of the diaphragm

[1] J. CZERMAK, *Prager Vierteljahrsschrift für prakt. Heilkunde*. Bd. XXXII. S. 154. 1851.

For example, light emanating from the point K of the diaphragm will be reflected into the eye along JA. Therefore, the eye must see a dark area of the surface of the mirror between F and H whenever the edge of the diaphragm is not in intimate contact with the reflecting surface. The validity of this statement may be tested on any convex reflecting surface, as for example a convex metal knob, for which there has been made a suitable black diaphragm with round opening. Only in case the edge of the opening lies close against the surface can the reflex images of surrounding objects in the mirror reach the edge of the diaphragm. If, however, there is a little space between the diaphragm and the reflecting surface, a dark line will be seen between the edge of the opening next the eye and the image in the mirror.

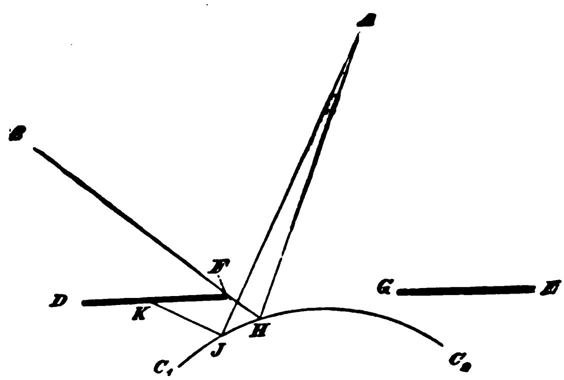

Fig. 9.

The surfaces of the crystalline lens also reflect some light, but not very much. These reflexes[1] may be seen when the eye is in a dark room where there is only one source of illumination. The light is adjusted in front of the eye a little to one side of the prolongation of the optical axis. The observer looks into the eye from the other side, in such fashion that his visual axis makes about the same angle with the optical axis as the incident rays. Near the prominent bright corneal reflex two other much fainter reflex images may be seen. The larger of the two is an erect, rather indistinct image of the flame reflected in the anterior surface of the lens. The smaller one is a sharp, inverted image reflected in the posterior surface of the lens. These reflex images are known to ophthalmologists as SANSON's images. If either the light or the observer's eye is moved during the examination, the position of the image also changes, and hence the first of these reflex images may be shifted voluntarily along the anterior surface of the lens to any desired spot in the edge of the pupil. In this case the image will always be seen without any black line between it and the pupillary margin on the side next the observer. At least, the writer has always found this to be the case under normal conditions without artificial dilatation of the pupil, and this fact unequivocally demonstrates that the pupillary margin of the iris rests against the lens.

The distance from the pupillary plane to the vertex of the cornea has been measured by C. KRAUSE on sectioned eyes. However, it should be borne in mind that the attachment of the lens to the sclerotica by means of the ciliary processes is not sufficiently strong to prevent considerable change of position in the process of sectioning.

If the living eye is observed from the side so that the pupil is just visible in front of the edge of the sclerotica, convincing evidence will be obtained that the pupillary plane lies behind a plane passed through the external cornea-scleral junction. On the edge of the cornea there will be seen in perspective

[1] Discovered by PURKINJE. See his treatise: *De examine physiologico organi visus et syst. cutanei.* Vratisl. 1823. In the diagnosis of diseases as used by SANSON (*Leçons sur les maladies des yeux.* Paris. 1837). Its origin was more exactly ascertained by H. MEYER (HENLES und PFEUFERS *Zeitschrift f. rationelle Medizin.* 1846. Bd. V.).

as shown in Fig. 10, two streaks in front of the pupil. The one nearer the pupil is bright and is a distorted image of the iris. The other, a dark streak,

Fig. 10.

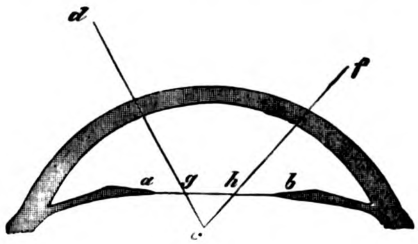

Fig. 11.

is the edge of the sclerotica as it projects over the cornea on the opposite side. As the observer moves his eye farther back still, the pupil and iris disappear entirely and then only the scleral margin can be seen behind the still visible edge of the cornea. As the rays of light, after penetrating the cornea, proceed in straight lines through the aqueous humor, it naturally follows that the iris lies farther back than a line joining a pair of opposite points on the external margin of the cornea.

If the radius of curvature at the vertex of the cornea is known, the distance of the pupillary plane from the vertex of the cornea can be calculated quite accurately by finding the apparent position of the iris as compared with the apparent position of the image of a luminous point in the corneal mirror. The reflex image of a distant luminous point lies a little behind the plane of the pupil, as may easily be verified by looking at the eye from different sides and noting the position of the image with respect to the border of the pupil.

In Fig. 11, let ab represent the pupil, c the apparent position of the reflex image, and dc and fc two different directions in which the observer looks at the point c. From d the point c will be seen to be beyond the point g in the pupillary plane, and therefore apparently nearer a. From f it will appear behind the point h, and therefore apparently nearer b. The simplest way to ascertain the position of the point c would be to measure its apparent distance in perspective from both edges of the pupil, which might be done with the ophthalmometer. However, the almost continual variations in the diameter of the pupil render this difficult.

Accordingly, the writer found it better to proceed a little differently. Suppose that the elliptical axes of the eye to be examined have been measured and the position of the visual axis determined with respect to them. A lamp may then be placed in front of the eye in a given position with respect to the visual axis; and the position of its reflex image in the cornea can be calculated according to the theory of the imagery of a spherical mirror. Consequently, it will be assumed in what follows that the position of this reflex image is known. If the lamp, the point of fixation and the ophthalmometer are all so adjusted that the two corneal images of the source as seen in the ophthalmometer can be so arranged that one is on one edge of the pupil and the other on the other edge, it may be inferred that, as seen from the ophthalmometer, the reflex image lies in perspective beyond the centre of the pupil. In Fig. 12 ed and $\epsilon\delta$ are two straight lines supposed to be parallel to the axis of the telescope of the ophthalmometer;

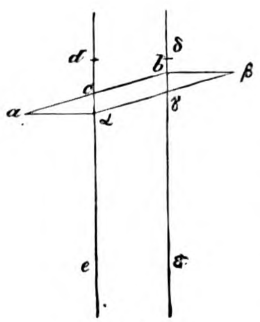

Fig. 12.

and ab and $\alpha\beta$ represent the double images of the horizontal diameter of the pupil. It is assumed that the centre of the pupil, the lamp, the axis of the telescope and the visual axis of the eye that is being measured are all in the same horizontal plane. According to the theory of this instrument as given in §2 above, all lines connecting corresponding points of the double images must be of equal length and perpendicular to the axis of the telescope, and the double images themselves must be geometrically congruent. Hence it follows that $a\alpha$ and $b\beta$, and also ab and $\alpha\beta$, are equal and parallel to each other. Now suppose that d and δ are the corresponding double images of the luminous point, and that such a position of the eye has been found that d is covered by α and δ by b, in other words, so that the lines de and $\delta\epsilon$, which are parallel to the axis of the telescope, pass through α and b, respectively. According to the theory of parallel lines,

$$d\delta : b\beta = \alpha\gamma : \gamma\beta$$
$$d\delta : a\alpha = cb : ac.$$

But since the distances between corresponding points of the double images are equal, therefore:

$$d\delta = a\alpha = b\beta.$$

Consequently,

$$\alpha\gamma = \gamma\beta$$

and

$$cb = ac.$$

The points c and γ, behind which the points d and δ appear in perspective, are therefore the centres of the pupillary images.

By suitable measurements it is now easy to determine what angle the line *ed*, or the axis of the telescope, makes with the visual axis of the observed eye. Thus the position of the line *ed* in the horizontal section of the eye is given by a point and the angle which it makes with another line of known direction, namely, with the visual axis. The centre of the pupil also lies in the line *ed*.

All that remains to be done now is to make a second observation of the same sort in another direction. Thus we find a second straight line of known direction, along which the centre of the pupil lies. Accordingly, the centre of the pupil must be at the point of intersection of this pair of lines, and its distance from the cornea can be found by geometrical construction or by calculation.

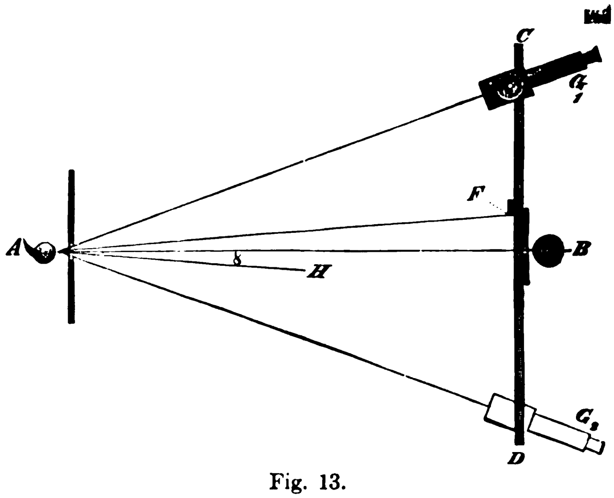

Fig. 13.

The method of observation is as follows: In Fig. 13 A represents the eye to be measured, which gazes through a hole in a screen so as to keep its position fairly fixed. At some distance from it there is a horizontal scale CD. Suppose a perpendicular is drawn from the eye to the scale meeting it at B. Here a diaphragm is placed with a small hole in it, and a lamp behind the hole. The light from the lamp

goes through the hole and reaching the eye is reflected from the cornea. At F there is an adjustable mark which is used as the point of fixation of the eye. The ophthalmometer is placed first at G_1 and then at G_2, equidistant from B in both positions. Marks may be made on the table for the tripod of the ophthalmometer, as the adjustment of the telescope has to be changed during the experiment. The patient is told to look steadily at the fixation mark F, and to follow all its movements. The observer, making his first observation at G_1, adjusts the glass plates of the ophthalmometer until one of the double images of the luminous point in the cornea coincides with the edge of the pupil. If now the other double image does not also coincide with the other edge of the pupil, the marker F is moved along the scale until this occurs, and the scale division noted. This same procedure is repeated from the second position of the ophthalmometer at G_2.

The distance AB must be measured in divisions of the scale CD; and then the angle FAB can be found by the formula:

$$\frac{FB}{AB} = \tan \angle FAB.$$

Knowing the axis major AH of the corneal ellispsoid and the angle FAH, the angle BAH may be calculated. This angle is needed to find the position of the reflex image in the cornea. In the same way the angle G_1AH is found, which establishes the direction along which the observer viewed the eye at A. The centre of the apparent pupil (as it looks through the cornea) will lie therefore in a line parallel to G_1A, which passes through the apparent position of the corneal image.

The method by which the actual position of the centre of the pupil may be calculated from its apparent position will be given in §9 and §10.[1]

The results of the measurements made with the ophthalmometer on the corneae of three eyes are as follows:

		O. H.	B. P.	J. H.
Distance of the pupillary plane from the vertex of the cornea	apparent	3.485	3.042	3.151
	actual	4.024	3.597	3.739
Distance of the centre of the pupil from the corneal axis on nasal side	apparent	0.037	0.389	0.355
	actual	0.032	0.333	0.304

That the iris lies in contact with the lens and is convex in front, has been a subject of much controversy among anatomists.[2] The older anatomists assumed this to be true, until PETIT denied it and asserted that the so-called *posterior chamber* of the eye lies between the iris and the lens, as a result of his investigations on frozen eyes. In frozen eyes, thin sheets of ice are sometimes found between iris and lens. Almost all later anatomists accepted PETIT's view, until very lately STELLWAG VON CARION and CRAMER again asserted the close apposition of iris to lens. The author has been able to make direct observations by the method mentioned above which appear to confirm this. Nevertheless, BUDGE (1855) has defended the work of PETIT.

1728. PETIT in *Mém. de l'Acad. Roy. des sciences.* 1728. p. 206 and p. 289.

1850. STELLWAG VON CARION in *Zeitschrift d. Wiener Ärtze.* 1850. Heft 3, S. 125.

1852. CRAMER in *Tijdschrift der Nederl. Maatschappij tot bevord. der Geneeskunst.* 1852. Jan.

[1] HELMHOLTZ in GRAEFES *Archiv. für Ophthalmologie.* Bd. 1. Abt. 2, S. 31.

[2] ¶See the supplement to this section which quite adequately presents the present viewpoint. (D. H.)

1853. CRAMER: *Het Accommodatievermogen der Oogen.* Haarlem. bl. 61*.
1855. J. BUDGE *über die Bewegung der Iris.* Braunschweig. S. 5-10 (also gives earlier literature of the subject.)
HELMHOLTZ in GRAEFES *Archiv. für Ophthalmologie.* Bd. I. Abt. 2, S. 30.

Supplement (from the first edition, pp. 820 et seq., 1867)

There now appears to exist a general agreement that the central part of the iris lies in apposition to the lens in the normal eye. The only differences of opinion are in regard to the extent of the free space between the peripheral portion of the iris and the anterior borders of the ciliary processes and the folds of the zonule. The difference is on the question as to whether this space is only a cleft, as CRAMER, VAN REEKEN, ROUGET and HENKE maintain, or whether there is an open annular space, corresponding to a posterior chamber, as ARLT supposes. The ciliary processes are empty of blood after death and have collapsed. As it is impossible to know exactly how much they would be enlarged were the vessels filled with blood, the matter is difficult of solution.

In Figs. 2 and 3 above the ciliary processes are perhaps represented as being too close to the iris. The relations of these parts were taken from dried specimens (as Fig. 3) in which the angle of the pigment layer between the ciliary processes and iris appears to have been drawn out and flattened by the drying. In fresh preparations, the ciliary processes are separated from the iris at their anterior ends by a much deeper cleft than is shown in the diagrams.

1855. VAN REEKEN. Onteedkundig onderzoek van den toestel voor accommodati van het Oog. *Onderzoekingen gedaan in het Physiol. Laborat. der Utrechtsche Hoogeschool.* Jaar VII 248-586
—— ROUGET in *Gaz. med. 1855.* No. 50
1860. W. HENKE. Der Mechanismus der Akkommodation für Nähe und Ferne. *Archiv. für Ophthalm.* VI, 2. S. 53-72.
1863. O. BECKER. Lage und Funktion der Ciliarfortsätze in lebenden Menschenauge. *Wien. Mediz. Jahrbücher.* S. 159.

The discovery of H. MÜLLER and ROUGET concerning the ciliary muscle should be mentioned. They have shown that there is a large mass of circularly arranged muscle fibres, running parallel to the equator of the lens, on the medial side of the muscle toward the ciliary processes and between its previously described meridional fibres. The action of these fibres will be further discussed in the supplement to §13.

1856. C. ROUGET. Recherches anatomiques et physiologiques sur les appareils érectiles. Appareil de l'adaptation de l'oeil. *C. R.* XLII, 937-941. *Institut.* 1856. p. 193 to 194. *Cosmos.* VIII, 559-560.
—— H. MÜLLER. Réclamation de priorité. *C. R.* XLII, 1218-1219.
—— C. ROUGET. Résponse à une réclamation de priorité addresée par M. MÜLLER. *C. R.* XLII, 1255-1256. *Instilt.* 1856. p. 245. *Cosmos.* IX, 9.
1857. H. MÜLLER. Über einen ringförmigen Muskel am Ciliarkörper. *Archiv. für Ophthalmol.* III, 1.
—— ARLT. *Zur Anatomie des Auges.* Ibid., III, 2.
1858. H. MÜLLER. Einige Bemerkungen über die Binnenmuskeln des Auges. Ibid. IV, 2. p. 277-285.

The existence and position of the *dilatator pupillae* is still subject to much discussion.[1] The blood vessels of the iris are plentifully supplied with muscle

[1] ¶The existence of the *musculus dilatator pupillae* is amply proved by the work of GRYNFELTT (*Le muscle dilatateur de la pupille chez les Mammiferes*, 1899) and others. (D. H.)

fibres. Some anatomists have described various layers of fibres in addition to those of the vessels, which they considered as forming a *dilatator pupillae*. Others have denied the existence of such fibres.

J. Henle. *Handbuch der systematischen Anatomie des Menschen.* II, 635. Braunschweig. 1866.

§4. The Retina

The retina is a superficial expansion of nerve substance spread over the fundus of the eye between the choroid and the vitreous humor. It is quite transparent when fresh, but a cloudy white *post mortem*. It is thickest (0.22 mm) at the back of the eye. The place where the optic nerve enters, situated somewhat towards the nasal side (Fig. 2, *d*), is white, whereas a little towards the temporal side (at *p*) there is a yellow spot (*macula lutea retinae*), the seat of clearest vision. Towards the front the retina gets thinner (0.09 mm at the anterior edge) and ends in a serrated edge (*ora serrata retinae*), where the ciliary processes begin, or at any rate its nervous elements are not found beyond here.[1] It is closely attached at this place to the choroid and the hyaloid membrane which is a transparent structureless sac enclosing the vitreous humor. The membranous parts which form its anatomical continuation from here on (*pars ciliaris retinae* and *Zonula Zinnii*) have an entirely different structure and physiological significance.

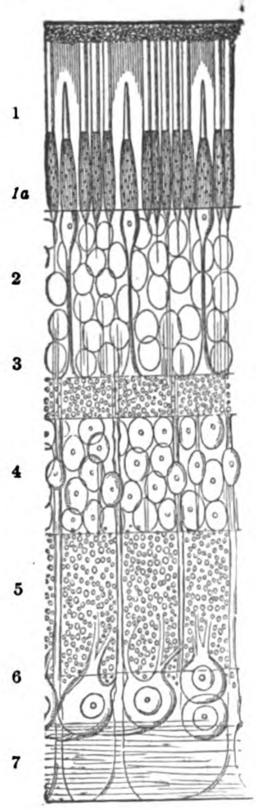

Fig. 14.

The retina is composed partly of the usual microscopical components of the nervous system (nerve fibres, ganglion cells and nuclei), and partly of certain characteristic elements, the so-called rods (*bacilli*) and cones (*coni*). Fig. 14 represents a cross section through the layers of the retina at the equator of the eye. It is taken from Max Schultze, but with its dimensions altered by Schwalbe.[2] The various layers in the

[1] ¶The pigment layer of the retina (*pars ciliaris retinae*) is contained over the ciliary body and the posterior surface of the iris as far as the *rima pupillae*. (D. H.)

[2] In the first edition reference was made to a figure after Kölliker on a special plate. This was changed in the second edition by Helmholtz himself to M. Schultze's semi-schematic figure as shown above. N.

order that they occur beginning with the one next the choroid are as follows:[1]

1. The *layer of rods and cones* (Fig. 14, 1). The rods are cylindrical in shape, from 0.063 to 0.081 mm in length and 0.0018 mm in diameter, made of a substance of high index of refraction. They are arrayed close together like the stakes in a palisade. Their outer knobs end abruptly; inwardly, they are continued as fine fibres which pass into the next layer. The cones are found between the rods. They are thicker (from 0.0045 to 0.0065 mm) and shorter than the rods and are composed of similar substance. The external extremity of each cone runs out into an ordinary rod (*cone rod*), while at the inner end it is continued as a pear shaped nucleated cell-body which is separated from the cone by a slight constriction lying in the next succeeding layer ("Zapfenkorn," as KÖLLIKER called it or cone nucleus according to VINTSCHGAU).

The cones, which are distributed between the rods, occur less frequently out towards the periphery of the retina and are far more concentrated towards the yellow spot, where there are no rods at all. In Fig. 15 *A* represents a surface view of this layer at the equator of the eye, *B* the edge of the yellow spot, and *C* the yellow spot itself. The little circles indicate rods and the larger ones cones, in which the cross section of the cone rod is shown. This layer is probably the one where the impression of light is obtained.

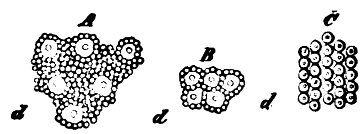

Fig. 15.

The layers of the retina which come next are:

2. The *outer nuclear layer* (Fig. 14, 2). (Separated from the layer of rods and cones by the *membrana limitans externa*, 1 *a*;—an observation inserted in the text by NAGEL.)

3. The *outer reticular* (*molecular*) *layer* (Fig. 14, 3).

4. The *inner nuclear layer* (Fig. 14, 4).

5. The *inner reticular* (*molecular*) *layer* (Fig. 14, 5).

These layers consist of fine fibres proceeding from the rods and cones (*radial fibres*, fibres of MÜLLER[2]). They are imbedded in a fine granular

[1] ¶The retinal pigment cells, indistinctly shown at the top of Fig. 14, form the outermost layer. Modern names of the layers have been substituted in the English translation. (D. H.)

[2] ¶The fibres of MÜLLER (radial fibres, sustentacular cells) extend throughout the entire retina from the *membrana limitans externa* to the *membrana limitans interna*, which appear to be formed by the ramifications of the terminal fibres of these cells. Their nuclei lie in the inner nuclear layer. These cells have no nervous function. Consequently, the statement in the next paragraph that the processes of the ganglion cells "in part seem to be united with the fibres of MÜLLER," is evidently a mistake. (D. H.)

substance and are much branched. Between them lie the *molecular bodies*, from 0.004 to 0.009 mm in diameter, attached to the fibres of MÜLLER.

6. The *layer of ganglion cells* (Fig. 14, 6) consists of large nerve or ganglion cells with many processes. One such cell, from the eye of an elephant, is shown in Fig. 16, which is copied from CORTI. Each cell contains a nucleus (Fig. 16, *a*). The processes of these cells in part pass out as fibres of the optic nerve and in part seem to be united with the fibres of MÜLLER. This layer is thickest in the yellow spot where it may be from eight to ten cells deep. It becomes thinner towards the periphery of the retina, where its cells cease to form a continuous layer.

7. The *layer of nerve fibres*. From the place where the optic nerve enters the eye its fibres spread out radially all over the retina, except in the yellow spot, which they go around. This layer is, naturally, thickest (0.2 mm) near the entrance of the optic nerve and becomes thinner towards the outer limit of the retina (0.004 mm at its margin). These fibres are of extremely fine calibre. They usually become varicose after death. Their thickness varies greatly (from 0.0005 to 0.0045 mm). As to their final terminations, nothing definite is known so far. Some of them are known to unite with the processes of the ganglion cells, and probably this is true of all of them.[1]

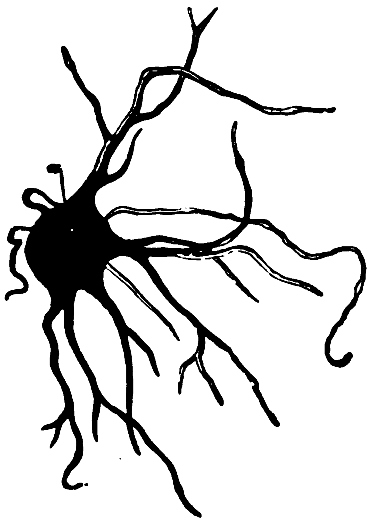

Fig. 16.

The inner ends of the fibres of MÜLLER, which have dense arborizations here, pass through the layer of nerve fibres. Their ultimate ends are attached to a glassy membrane which clothes the retina on its inner surface (*membrana limitans interna*).

The *yellow spot*, the most important part of the entire retina for vision, may be differentiated from the surrounding portions by its yellow colour, which is due to a pigment found in all layers except the layer of rods and cones. The layer of nerve fibres is lacking here, and in the layer of rods and cones the cones

[1] Except for a few centrifugal fibres, probably vaso-motor in function, all the fibres of this layer are axones of the ganglion cells. (D. H.)

alone are present. In its centre there is an extremely transparent indented place or hollow, the *fovea centralis*, which is very easily torn and consequently was considered for some time to be a foramen. At the margin of the yellow spot the layer of ganglion cells is thicker than in all the other parts of the retina, but it gets thinner again in the *fovea centralis* and consists of only a few layers of cells at this point. The inner granular layer is probably absent entirely at the centre of the fovea. The inner nuclear layer and the outer granular layer become considerably thicker, whereas the outer nuclear layer gets thinner, towards the yellow spot. According to H. Müller, the inner nuclear layer is also thinner in the fovea. According to Remak and Kölliker, all layers except the ganglion cells and cones are absent in the *fovea centralis*.[1] Remak also states that here there is a very yellow glassy substance interposed between the cones and the choroid.

In spite of their importance, the relations of the structures in the yellow spot are in many respects but little understood. The delicate structures of this region disintegrate very shortly after death. Besides, owing to the fact that the macula has hitherto been found only in the human eye,[2] all the more delicate investigations of this area have been necessarily made on the eyes of executed criminals, and such opportunities are rare.

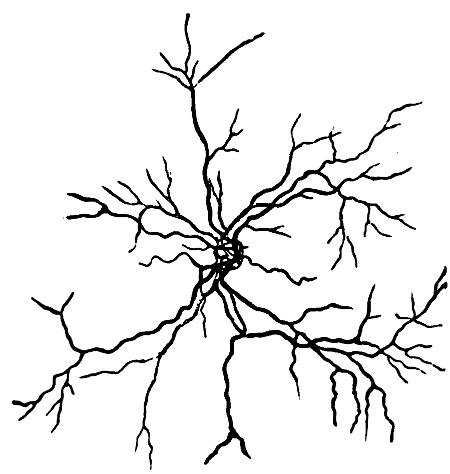

Fig. 17.

The *fovea centralis* is readily seen with an ophthalmoscope by virtue of its peculiar light reflex (see §16). Here is situated the point of the retina for direct vision, where the image of the point of fixation of the field of view is focused.

The *vessels of the retina* (*arteria* and *vena centralis retinae*) enter the eye through the centre of the optic nerve and branch out in all directions from this point. At first, they lie directly under the *membrana limitans interna* in the layer of nerve fibres. Then they penetrate into

[1] ¶There is some question as to the constitution of the tunica retina at the fovea. It would appear from sections that the cones, a few elements of the outer nuclear and the outer reticular layers constitute with the external and internal limiting membranes the entire retina at this point. (D. H.)

[2] ¶The macula and the fovea have been found in all classes of vertebrates by a number of investigators, though it is lacking in most fishes, in many amphibia and reptiles and in a few mammals. (D. H.)

the layer of ganglion cells and even into the external molecular layer, in both of which they form a large-meshed capillary net. The position and form of this vascular arborization is important for certain optical phenomena (see §15), and therefore a drawing is inserted here (Fig. 17), which was made by DONDERS from an injection preparation. The arteries are shown in outline, the veins in solid black. In the yellow spot there are no larger vessels and in the fovea even capillaries are absent. The fovea is surrounded by a wreath of terminal capillary branches.

At the anterior edge (*ora serrata*) the retina passes over into a layer of cells (*pars ciliaris retinae*) which together with the *membrana limitans* (*interna*) covers over the ciliary processes and the posterior surface of the iris. Here it appears to become changed into pigment cells, and is closely attached to the underlying structures.

Inasmuch as the dimensions of the retina and its elements are of much importance in connection with many optical phenomena, attached herewith is a résumé of certain measurements, as made by various observers, the results being all expressed in mm. These data derived from the works of C. KRAUSE, E. H. WEBER, BRÜCKE, KÖLLIKER and VINTSCHGAU are indicated below by the abbreviations *Kr.*, *W.*, *B.*, *Ko.* and *V.*, respectively.

Diameter of the optic nerve at its point of entrance: *Kr.*, 2.7 and 2.14; *W.*, 2.09 and 1.71.

Diameter of the vessels contained in it: *W.*, 0.704 and 0.63.

Distance of the centre of the optic nerve from the centre of the yellow spot: *W.*, 3.8. *Kr.*, 3.28 and 3.6. Distance between the centre of the optic nerve and the inner edge of the yellow spot: *Ko.*, 2.25 to 2.7.

Horizontal diameter of the yellow spot: *Kr.*, 2.25; *W.*, 0.76; *Ko.*, 3.24.

Vertical diameter of the yellow spot: *Ko.*, 0.81.

Diameter of the *fovea centralis: Ko.*, 0.18 to 0.225.

Distance between the *ora serrata* and the edge of the iris on nasal side: *B.*, 6; on temporal side, 7.

Thickness of the retina near the optic nerve: *Ko.*, 0.22.

Thickness of the retina at the back of the eye: *Kr.*, 0.164, *Ko.*, 0.135.

Thickness of the retina at the equator: *Kr.*, 0.084.

Thickness of the retina at its anterior margin: *Ko.*, 0.09.

Thickness of the layers in the yellow spot: *Ko.*, ganglionic layer, 0.101 to 0.117; inner molecular layer, 0.045; inner nuclear layer, 0.058; outer molecular layer, 0.086; outer nuclear layer, 0.058; layer of rods and cones, 0.067.

Diameter of the ganglion cells: *B.*, 0.01 to 0.02; *Ko.*, 0.009 to 0.036, usually between 0.013 and 0.022.

Diameter of the nuclei: *B.*, 0.006 to 0.008; *Ko.*, 0.004 to 0.009. Of the cone-cells: *V.*, 0.0068.

Diameter of the rods: *B.* and *Ko.*, 0.0018; *V.*, 0.0010.

Length of the rods: *B.*, 0.027 to 0.030; *Ko.*, 0.063 to 0.081.

Diameter of the cones: *Ko.*, 0.0045 to 0.0067; *V.*, 0.0034 to 0.0068. In the yellow spot: *Ko.*, 0.0045 to 0.0054.

Length of the cones: *V.*, 0.015 to 0.020.

The newer (1845–1854) works on the structure of the retina are:

1845. F. PACINI in *Nuovi Annali delle scienze nat. di Bologna.* 1845.
1851. H. MÜLLER in SIEBOLD und KÖLLIKERS *Zeitschrift für wiss. Zoologie.* 1851. S. 234. —*Verhandl. der Würzburger med. Ges.* 1852. S. 216. Ibid. III. 336 and IV. 96.

1850. CORTI in J. MÜLLERS *Archiv.* 274.—*Zeitschr. für wissensch. Zoologie. V.*— J. HENLE in *Zeitschr. für ration. Medizin.* N. F. II. 304 and 309.
1852. A. KÖLLIKER *Verhandl. der Würzburger med. Ges.* III. S. 316*.
1853. A. KÖLLIKER u. H. MÜLLER *C. R. de l'Acad. d. Sc.* 1853. Sept. 23.—Plate drawing of the retina by these same authors in ECKER *Icones physiologicae**.
R. REMAK in *C. R. de l'Acad. d. Sc.* 1853. Oct. 31. and *Allgem. med. Zentralz.* 1854. No. 1*. *Prager Vierteljahrsschr.* XLIII. S. 103.
*M. DI VINTSCHGAU in *Sitzbr. d. Wiener Akad.* XI. 943*.
1854. *A. KÖLLIKER *Mikroskopische Anatomie.* Leipzig 1854. II. 648-703*.
Some measurements have been copied from:
C. KRAUSE, *Handbuch der menschlichen Anatomie.* Hannover 1842. I, 2. S. 535*.
E. BRÜCKE, *Anat. Beschr. d. menschl. Augapfels.* Berlin 1847. S. 23.
E. H. WEBER in *Sitzber. d. Sächs. Ges. d. Wiss.* 1852. S. 149-152.

Supplement (from the first edition, pp. 822 et seq. 1867)

The more delicate anatomy of the retina has been much studied by anatomists and more complete information is at hand. As a result of his own work and that of other observers, J. HENLE distinguishes the following layers in his latest compilation:

Mosaic layer		1. Layer of rods and cones
		2. External limiting membrane
		3. Nuclear layer (granular layer)
External fibre layer		4. External fibre layer
Nervous layer	Grey substance	5. External granular layer
		6. External ganglionic layer
		7. Internal granular layer
		8. Internal ganglionic layer
	White substance	9. Layer of nerve fibres
Boundary membrane		10. *Limitans hyaloidea.*

Of these, 1, the layer of rods and cones; 3, the outer nuclear layer; 4 and 5, the outer reticular (molecular) layer; 6, the internal nuclear layer; 7, the inner reticular (molecular) layer; 8, the layer of ganglion cells; and 9, the expansion of the optic nerve are all enumerated in the list given above (p. 25).

The rods of the hindmost layer of the retina are themselves each composed of two rod-like members joined together, the inner of which is thicker (0.0018 to 0.0022 mm in diameter) and consists of a less highly refracting substance than that of the outer (0.0013 to 0.0018 mm in diameter). The basal portion of the *rods* reaches the same height as the thicker bottle shaped basal segment of the *cones*. The external portion of the cones, the *cone rods* mentioned above, lie in a row with the external portions of the rods, but they are shorter and therefore do not extend as far towards the choroid. The diameter of the thicker inner part of the cones gets to be as much as from 0.004 to 0.006 mm. Only in the fovea where there are no rods between the cones the latter are thinner. (Their basal ends measure from 0.002 to 0.0025 mm ac-

cording to M. Schultze, and in a small region from 0.0015 to 0.002 mm according to H. Müller, and between 0.0031 and 0.0036 mm according to Welcker). According to M. Schultze, the cones of the yellow spot are further differentiated by being nearly twice as long as those in other parts of the retina.

Henle states that the *outer nuclear layer* contains many superposed layers of ellipsoidal granules which, in fresh condition, exhibit a characteristic and very delicate cross striation. Each granule, as a rule, shows three bright bands separated by darker ones giving the optical effect of layers of two alternating substances passing through the granule parallel to the surface of the retina. In well fixed preparations these granules may be been in fairly regular rows perpendicular to the surface of the retina. Their reaction to reagents is so essentially different from that of nerve cells that they may be readily distinguished. Their long axis, perpendicular to the surface of the retina, measures from 0.006 to 0.007 mm. The shorter axis is not much more than half as long.

The *cone nuclei* ("Zapfenkörner," above mentioned) extend also into the nuclear layer. Each of them contains a nucleus and is continued towards the inner layers as a smooth shining cylindrical fibre, 0.0015 mm in diameter, which penetrates the thickness of the nuclear layer and enters the external granular layer with or without a cell-like enlargement.

According to M. Schultze, this fibre appears to break up here into a large number of very fine fibres which enter the external granular layer and become lost in it. In the rods likewise originate delicate nerve fibres which are connected with the granules of the external granular layer. They correspond to the cone fibres but are much finer, and have an enlargement as they approach the external granular layer in which they also become lost.

A special fibre layer (known as Henle's *external fibre layer*) is to be found in general only in and around the yellow spot and around the *ora serrata,* along the outer edge of the retina. The fibres in the yellow spot run radially out from the centre of the fovea in all directions, proceeding principally parallel to the surface of the retina. However, they sometimes pass out of the granular layer in small bundles and join the horizontal layer of fibres, or they may leave this layer to penetrate into the layer of nerve fibres and the outer granular layer. These fibres apparently represent the connections between the cones of the fovea and the nerve cells which are found in large numbers in its vicinity. On account of their great number, however, Henle doubts whether they all serve this purpose. The role that these fibres appar-

ently play in the production of HAIDINGER's tufts in polarized light is explained in §25.

Nothing of importance has been recently discovered with respect to the other front layers of the retina. A large number of the radially arranged fibres of MÜLLER, especially those which unite to form the *membrana limitans hyaloidea*, are connective tissue fibres. According to MAX SCHULTZE, the nerve fibres proper can be recognized by their varicose appearance, but, beyond their procedure in the foremost layer of the retina which forms the expansion of the optic nerve, nothing very definite is known.

In the fundus of the fovea the two layers of nerve cells have united with each other and with the nuclear layer. Behind these lie the cones. All the other layers are absent.

1856. H. MÜLLER. Anatomische Beiträge zur Ophthalmologie. *Archiv. für Ophthalmologie.* II, 2. S. 1. III, 1. S. 1. IV, 1. S. 269.
Idem, Anatomisch-physiologische Untersuchungen über die Retina bei Menschen und Wirbeltieren. SIEBOLD und KÖLLIKER *Zft. für wissensch. Zoologie.* VIII, 1. *C. R.* XLIII. Oct. 20.

1857. C. BERGMANN. Anatomisches und Physiologisches über die Netzhaut des Auges. *Zft. für rationelle Medizin.* (3) II. 83.

1858. NUNNELEY. On the structure of the retina. *Quarterly Journal of microscop. science.* 1858. July. 217.

1859. RITTER. Über den Bau der Stäbchen und äusseren Endigungen der Radialfasern an der Netzhaut des Frosches. *Archiv. für Ophthalm.* V, 2, S. 101.
M. SCHULTZE. *De retinae structura penitiori.* Bonn.

1859. E. v. WAHL. *De retinae textura in monstro anencephalo. Dissert.* Dorpat.

1860. W. MANZ. Über den Bau der Retina des Frosches. *Zft. für ration. Medizin.* (3) X, 301.
G. BRAUN. Eine Notiz zur Anatomie und Bedeutung der Stäbschenschichte der Netzhaut. *Wiener Sitzungsber.* XLII, 15-18.
W. KRAUSE. Über den Bau der Retinastäbchen beim Menschen. *Göttinger Nachrichten.* 1861. No. 2. *Zft. für ration. Medizin.* (3) XI, 175.

1861. M. SCHULTZE. *Sitzungsber. der niederrheinischen Ges.* 1861. S. 97. *Archiv. für Anatomie und Physiol.* 1861. S. 785. *Archiv. für mikrosk. Anatomie.* II, 175-286.
RITTER in *Archiv für Ophthalm.* VIII, 1.

1862. H. MÜLLER. Bemerkungen über die Zapfen am gelben Fleck des Menschen. *Würzburger naturwiss.* Zft. II, 218.
Idem, Über das Auge des Chamäleon. Ibid. III, 10.

1863. SCHIESS. Beitrag zur Anatomie der Retinastäbschen. *Zft. für ration. Medizin.* (3) XVIII, 129.
H. WELCKER. Untersuchung der Retinazapfen bei einem Hingerichteten. Ibid. XX, 173.
W. KRAUSE. Ibid. XX, 7.

1865. BLESSIG. *De Retinae textura. Dissert.* Dorpat.

1866. J. HENLE. *Handbuch der systematischen Anatomie des Menschen.* II, 636-670.

§5. The Crystalline Lens

The crystalline lens is a transparent, colourless double convex body less curved in front than behind. It is enclosed in a structureless glassy membrane (the *capsule of the lens*), which resembles in all respects the membrane of DESCEMET. Like the latter, it also, according to BRÜCKE, has a layer of epithelium on its anterior surface where it is in contact with the aqueous humor; but HENLE and KÖLLIKER deny this. Its posterior half is fused with the hyaloid membrane. The substance of the lens is of a gelatinous consistency in the outer layers, but becomes stiffer in the centre or *core* of the lens. The entire organ is highly elastic in the fresh condition, yielding easily to any external force, but quickly and completely recovering its former shape.

The crystalline humor is a double refracting medium. If examined between two crossed NICOL prisms, the black cross with coloured rings is seen which is a characteristic appearance of the section of a uni-axial crystal perpendicular to the optical axis.

The mass of the lens consists of a peculiar protein, the so-called *globulin* or *crystallin*. Its microscopical elements are fibres of hexagonal

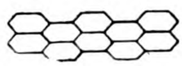

Fig. 18.

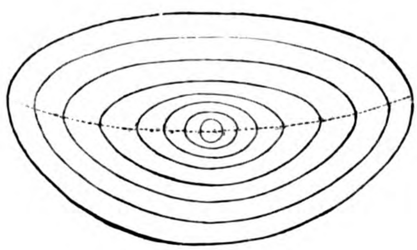

Fig. 19.

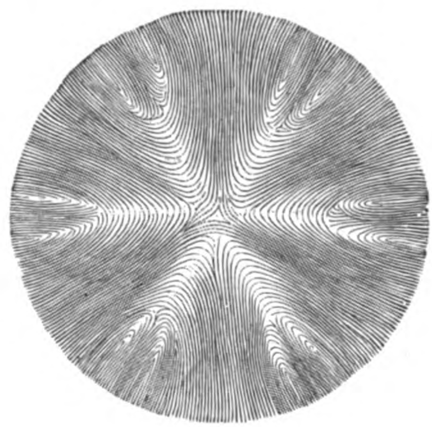

Fig. 20.

cross section, from 0.0056 to 0.0112 mm broad, and from 0.002 to 0.0038 mm thick. They are smaller and of a firmer consistency in the core of the lens than in the outer layers. Their broader surface lies parallel to the surface of the lens, and thus the lens may easily be split in this direction in concentric layers like an onion. Fig. 18 shows a cross section of the fibres in normal apposition; and Fig. 19 shows the direction of the layers in a cross section of the lens. In each individual layer the fibres extend generally from the axis of the lens to its peri-

phery. The characteristic star-shaped figures, such as that shown in Fig. 20 from the outer layers of the lens, are found only near the axis where the fibres bend back again. In the core layers the star has only three rays, which make angles of 120° with each other. The stars of the posterior and anterior surfaces are turned through 60° with respect to each other. In the outer layers, however, the three principal rays of the stars are much broken up into secondary rays so that much more intricate and irregular figures occur.

Close under the capsule there is a layer of cells instead of fibres which disintegrate after death and form the *liquor Morgagnii*. According to Brücke, similar cells also unite the ends of the fibres in the rays of the star, at any rate in the outer layers. Bowman and Kölliker, however, maintain that a structureless substance exists at these places. The latter explains also the cell-like structures on the posterior surface of the lens as swollen and flattened ends of the lens fibres that are attached here to the capsule. Thus in each half of the lens there are three planes passing through the axis, and corresponding to the principal rays of the star (*central planes*, Bowman), in which the structure of the lens is different from that found elsewhere. In the superficial layers these planes are further subdivided. Doubtless, certain irregularities in the refraction of light rays are dependent on this latter fact.

We have by no means a clear idea of the distribution of the fibres in the lens. Thomas[1] has described peculiar figures made by the ends of the fibres on the surfaces of sections of dried lenses, consisting mainly of two systems of concentric circles. These do not admit of any explanation on the basis of present knowledge as to the distribution of the fibres of the lens.

As a result of his measurements, Krause considers the anterior surface of the lens as a portion of a flattened ellipsoid of revolution and the posterior surface as a paraboloid of revolution. He gives the following values of the different constants, in Paris lines, for the eight eyes specified in §2:

No.	Axis			Anterior Surface			Posterior Surface		Diameter
	of the whole lens	of the front half	of the rear half	Semi-axis of ellipse major	Semi-axis of ellipse minor	Distance from the cornea	parameter	distance from the retina	
I.	2	0.85	1.15	2.05	0.95	1.2	4.49	6.65	4.1
II.	1.9	0.78	1.1	2	0.91	1.35	4.99	6.8	4
III.	2.4	0.98	1.42	2	1.14	1.25	4.99	6.1	4.1
IV.	2.2	0.95	1.25	2.05	1.10	1.35	4.51	5.9	4.1
V.	1.85	0.65	1.2	2.03	0.83	1.25	4.83	6.4	4
VI.	2.35	0.8	1.55	1.95	0.98	1.2	4.53	6.0	4.1
VII.	1.8	0.78	1.02	2.03	0.95	1	4.09	6.65	4
VIII.	1.85	0.85	1	2	0.94	1	3.79	6.55	4

[1] Thomas, *Prager mediz. Vierteljahrsschr.* 1854. Bd. I. Ausserord. Beilage S. 1.

KRAUSE's data on the distances of the two surfaces of the lens from the cornea and retina are included above, although, as formerly observed, the correctness of these results appears very dubious to the writer. Moreover, the latter's measurements of the thickness of the lens in the eyes of living persons do not agree with those made on lenses of cadavers. Inasmuch as the thickness of the lens is changed in near and far vision, the discussion of these investigations will be postponed until we come to the theory of accommodation in §12.

Literature concerning the structure of the lens:

1845. A. HANNOVER in J. MÜLLERS *Archiv.* 1845. S. 478*.
1846. HARTING in VAN DE HOEVEN EN DE VRIESE *Tijdschrift* XII. S. 1.
1847. *E. BRÜCKE, *Beschr. d. menschl. Augapfels.* Berlin. S. 27-30*.
1849. W. BOWMAN, *Lectures on the parts concerned in the oper. on the eye.* London.
1851. H. MEYER in J. MÜLLERS *Archiv.* 1851. 202*.
1852. GROS in *C. R. de l'Acad. d. Sciences.* 1852. April.
1852. D. BREWSTER. On the development and extinction of regular doubly refracting structures in the crystalline lenses of animals after death. *Phil. Mag.* (4) III, 192-193.
1854. *A. KÖLLIKER, *Mikroskopische Anatomie.* Leipzig. II. 702-713*.
THOMAS in *Prager med. Vierteljahrsschrift.* 1854. Bd. 1. S. 1.*
1859. G. VALENTIN. Neue Untersuchungen über die Polarisationserscheinungen der Kristallinsen des Menschen und der Tiere. *Archiv für Ophthalm.* IV, I, 227-268.
——. D. BREWSTER, On certain abnormal structures in the crystalline lenses of animals and in the human crystalline. *Rep. of Brit. Assoc.* 1858. 2, p. 7.
1863. F. J. v. BECKER. Über den Bau der Linse bei dem Menschen und den Wirbeltieren. *Archiv für Ophthalm.* IX (2), 1-42.

§6. The Aqueous and Vitreous Humors

The aqueous humor (*humor aqueus*) fills the space between the cornea, iris and lens. The space comprised between the posterior surface of the cornea, the anterior surface of the iris and the pupillary plane is called the *anterior chamber of the eye.* On the other hand, the space which was supposed to lie between the pupillary plane, the posterior surface of the iris and the anterior surface of the lens was called the *posterior chamber of the eye.* However, as a matter of fact, this is only a potential space or capillary gap under normal circumstances,[1] as the posterior surface of the iris is close against the anterior surface of the lens. The iris seems to become separated from the lens only after strong artificial dilatation of the pupil with belladonna.

The aqueous humor therefore fills the anterior chamber of the eye.[2] It is clear, colourless and consists of water containing about 2% of solids the chief of which are sodium chloride and extracts. Its index of refraction differs very little from that of water.

The cavity of the eyeball between the lens and the retina is filled with the *vitreous humor* (*corpus vitreum, humor vitreus*) which in turn

[1] ¶Except at its base. (D. H.)
[2] ¶The aqueous humor also fills the posterior chamber of the eye. (D. H.)

is contained in the *hyaloid membrane* (*membrana hyaloidea*). The vitreous humor is a gelatinous mass of slight coherence. When it is cut, a thin non-viscous fluid drips from it, which has an alkaline reaction and contains from 1.69 to 1.98% of solids, about half of which is inorganic material (sodium chloride, a little sodium carbonate, traces of lime, sulphuric acid and phosphoric acid). The organic part seems to consist chiefly of mucin containing traces of some protein compound. The index of refraction of the vitreous humor likewise differs little from that of water, but is somewhat higher than that of the aqueous humor.

In the embryo the vitreous humor has a cellular structure, but subsequently only a few remnants of these cells are found, such as membranes, granules, and granular lumps[1] that move about in it but not with perfect freedom. The vitreous humor apparently owes its consistency to a rather small amount of very much swollen organic substance (mucin or fibrous material). Small amounts of fibrous tissue which have differentiated from a watery fluid often give rise to similar soft gelatinous masses from which fluid runs out if the continuity of the coagulum is mechanically disturbed. If the vitreous humor is hardened in reagents that precipitate the mucin, as, for example, in a solution of acetic oxide of lead or of chromic acid, the sections of it are found to contain sometimes uniform condensations. However, it is still extremely doubtful whether these can be considered as membranes that pass through the vitreous humor.

HANNOVER infers that the presence of these bands indicates that flat membranes are present in the vitreous humor of the human eye which all intersect in a line proceeding from the place where the optic nerve enters the eye to the posterior surface of the lens; and that these membranes extend from this line to the exterior of the vitreous humor where they are inserted in the hyaloid membrane, so that the structure of the vitreous humor would be like that of an orange.

The deductions as to the structure of the vitreous body which may be drawn from entoptical phenomena will be considered later.

The *hyaloid membrane* is a very delicate, glassy membrane without structure which at the back part of the eye rests against the *membrana limitans interna* of the retina, being attached thereto over the entire surface during life,[2] but after death only at the place of entrance of the optic nerve and at the *ora serrata*. From the *ora serrata* it is continued as a thinner membrane over the posterior surface of the capsule of the

[1] ¶These remnants are frequently noticeable as the shadowy "*muscae volitantes*," noted at times by the observer in his own eye. (D. H.)

[2] VINTSCHGAU in *Sitzbr. d. Wiener Akad.* XI. 943 and BUROW in J. MÜLLERS *Archiv.* 1840.

lens with which it is united (Fig. 2, *k*). Another membrane, the so-called *zonula Zinnii* (*ligamentum suspensorium lentis*) is inserted between it and the ciliary portion of the retina. This is regarded by many anatomists as an anterior layer of the hyaloid membrane.

The zonule is folded like a ruff so as to follow the surface of the ciliary processes. The anterior or outer border of its folds is closely attached to the *membrana limitans* in the hollows between the folds of the ciliary processes. The posterior or inner border of its folds, corresponding to the summits of the ciliary processes, lies in apposition to the hyaloid membrane. In Fig. 2 the zonule is indicated by the line *e*. On the right it passes between two ciliary processes, while on the left it goes over the summit of one of them. In this manner it reaches the border of the lens and is attached to its capsule in a wavy line. A quadrant of the lens is shown in Fig. 21, projected on a plane passed through the axis of the lens *ab*. The line of attachment of the hyaloid membrane is indicated by *cd*. Above this is seen the serrated line of attachment of the zonule.

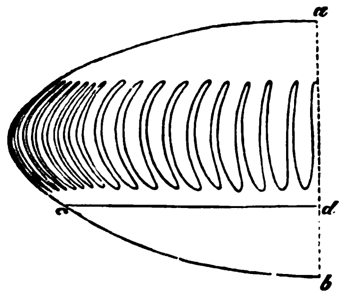

Fig. 21.

The cleft-like space between the zonule and the hyaloid membrane is called the *canal of* Petit (*canalis Petiti*). If it is inflated after the zonule has been laid free from the anterior surface, the invaginated folds of the zonule bulge outwards so that the whole has the appearance of an Ionic egg-molding, and hence Petit, who discovered it, named it "*canal godronne.*" With stronger inflation, the evaginated portions of the membrane tear, leaving only the tougher anterior edges of the folds as bands which bind the lens to the vitreous humor. The anterior edges of the folds are attached also to the ciliary portion of the retina which dips down between the ciliary processes; and this in turn is attached to the pigment layer. Fibres of attachment are also present here. According to Brücke, they arise from the fibres between which the nerve cells of the retina are embedded. The fibres are attached to the *ora serrata* at those places which correspond to the interval between each pair of ciliary processes and pass forward in the bottom of these grooves. Brücke thinks that the zonule itself is a structureless membrane, but Henle and Kölliker maintain that it is fibrous. The zonule and its fibres are as resistant to chemicals as is elastic tissue.

The zonule secures the position of the lens inasmuch as it attaches the latter to the ciliary body. Under tension the zonule may exert

a pull on the equatorial edge of the lens thereby tending to elongate the equatorial diameter, reduce the axial thickness, and flatten the surfaces of the lens.

Literature on the structure of the vitreous humor:

PAPPENHEIM, *Spezielle Gewebelehre des Auges.* 842. S. 181.

E. BRÜCKE in J. MÜLLERS *Archiv.* 1843. S. 345 and 1845. S. 130.

HANNOVER, ibid. 1845. S. 467 and in: *Das Auge.* Leipzig. 1852.

BOWMAN in *Dublin Quarterly Journal of Med. Science.* 1848. Aug.; also in *Lectures on the Parts conc. in the oper. on the eye.* London 1849. p. 94.

*E. BRÜCKE, *Beschr. d. menschl. Augapfels.* Berlin 1847.

VIRCHOW in *Verhandl. d. Würzburger phys. med. Ges.* II. 1851. 317 and in *Archiv für pathol. Anat.* IV. 468 and V. 278.

*KÖLLIKER, *Mikrosk. Anatomie* II. 713.

DONDERS en JANSEN in *Nederlandsch Lancet* 1846. II. 454.

*A. DONCAN, *De corporis vitrei structura. Dissert.* Utrecht 1854. Abridged in *Onderzoekingen ged. in het physiol. Laborat. der Utrechtsche Hoogeschool.* Jaar VI. S. 172.

§7. Surroundings of the Eye

The eyeball lies imbedded in loose adipose tissue in the bony orbit or *socket of the eye* (*orbita*). This is very nearly conical in form. The base of the cone is the external opening of the orbit in the face; its apex lies posteriorly and somewhat medially. Fig. 22 shows the positions of

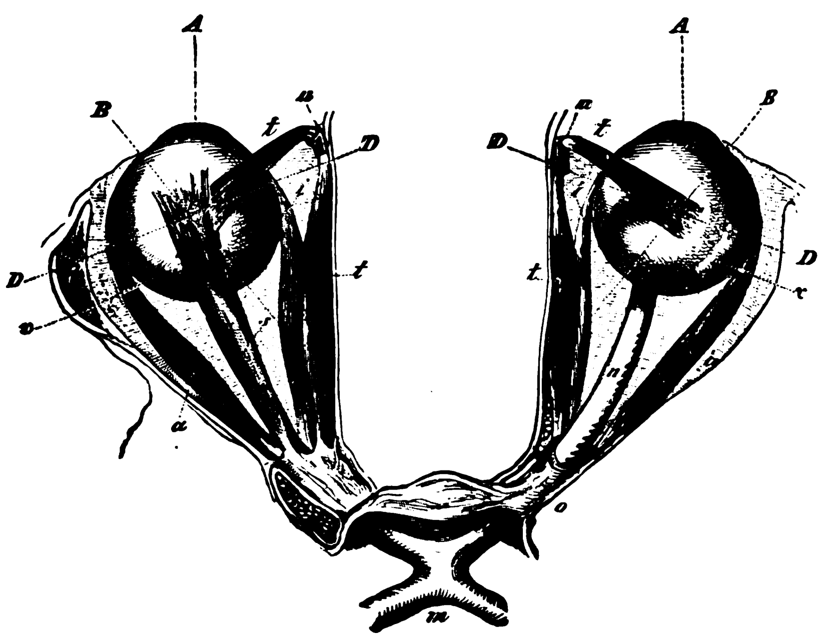

Fig. 22.

the eyes in both orbits. From the posterior side of the right eyeball in the figure, the optic nerve n may be seen to arise, entering the cranial cavity through a hole o (*foramen opticum*) situated at the apex

of the orbit, and uniting and crossing with the nerve from the opposite side in the *chiasma nervorum opticorum* at *m*. The continuations of the optic nerve from the chiasma to the brain are known as the optic tract (*tractus opticus*). The fibres of each *tractus opticus* pass partly into the optic nerve of its own side, partly into that of the opposite side, and a small part through the *tractus opticus* of the opposite side back into the brain. Some observers have also found fibres which pass from one optic nerve through the chiasma into the other.

There are six muscles whose function is to turn the eyeball in its socket, namely:

1. The *internal rectus i*, and

2. The *external rectus a*. Both of these originate in the vicinity of the *foramen opticum* at the apex of the orbit and are inserted on the inner (medial) and outer (lateral) sides of the eyeball, respectively. They turn the eye around its vertical axis.

3. The *superior rectus*, removed from the right side of Fig. 22 in order to show the optic nerve, marked *s* on the left side; and

4. The *inferior rectus*, which lies on the under side of the orbit, just as the superior rectus shown in the figure lies on the upper side. These two muscles also originate in the vicinity of the *foramen opticum* and are inserted in the upper and lower sides of the eyeball. They turn it around a horizontal axis, indicated by the line *DD* in Fig. 22, which passes from the nasal side of the eye, a little to the front, to the temporal side, a little to the back, making an angle of about 70° with the optical axis (*A*) of the eye.

5. The *superior oblique muscle t* arises from the edge of the *foramen opticum* and proceeds to the upper nasal side in the front part of the orbit, where its tendon passes through a small pulley *n* (*trochlea*), which is attached to the upper anterior edge of the orbit. Here it turns at an angle and is inserted in the upper side of the eyeball at *C*. The muscle exerts a pull in the direction of its tendon.

6. The *inferior oblique muscle*, not shown in the figure, arises from the front nasal border of the orbit, passes under the eyeball towards the temporal side, and is inserted in the posterior outer side of the eyeball at *v* in Fig. 22. The axis of rotation (*BB*) of the oblique muscles of the eye is likewise horizontal and passes from the outside in front to the inside behind, making an angle of about 75° with the axis of rotation of the superior and inferior recti and an angle of 35° with the optical axis of the eye.

The optical axis of the eye may be turned in any desired direction by the action of these six muscles combined in different ways. The eyeball is capable of rotation also about the optical axis itself. As an intro-

duction to the subject, it would appear to be justifiable to assume provisionally a common axis of rotation for the two muscles of a pair. Anyhow it simplifies very much the comprehension of the movements which the muscles of the eye have to produce.

The eyeball is protected in front by two *eyelids* (*palpebrae*). Each of them contains a cartilaginous plate which is covered on the outer surface by the external skin and on the inner by a mucous membrane, the *conjunctiva*, which is continued over the eyeball. It is loosely attached to the white of the eye but fuses with the cornea at its border. The surface of the *conjunctiva* and the anterior surface of the cornea are kept continually moist by three different secretions. These are: (1) the secretions of the Meibomian glands which lie on the inner surface of the eyelids under the *conjunctiva*. This fatty secretion perhaps affects mainly the edges of the lids and prevents the overflow of tears, but it may also spread over the cornea in oily drops, especially when the lids are moved strongly. (2) The mucous of the mucous glands of the conjunctiva which are most numerous in the folds between the lids and the eyeball. (3) The tears, secreted by the lachrymal glands. There are two of these glands situated in the upper outer side of each orbit. They pour out their watery secretion, which contains only about 1% of solids, through from seven to ten fine ducts above the outer angle of the eye between the upper lid and the eyeball. From this point the tears spread over the entire surface of the *conjunctiva* and are taken up at the inner angle of the eye by two small openings, the *puncta lacrymalia*. These are the orifices of the two *lachrymal canals* which empty into a wider canal, *ductus nasolacrymalis*, from which they finally reach the nose.

The *conjunctiva* of the eye is extraordinarily sensitive. The slightest contact with a foreign body causes pain and involuntary movement of the eyelids or *winking*. By this means and by the continuous flow of tears across the *conjunctiva*, the anterior surface of the cornea is kept clean and bright at all times, which is a necessary condition for clear vision. The larger particles of dust floating in the air, insects, etc., are kept out by the *eyelashes*.

MORE RECENT BIBLIOGRAPHY ON ANATOMY OF THE EYE[1]

ADDARIO, C., 1902: Ueber die Matrix des Glaskörpers im menschlichen und tierischen Auge, *Anat. Anz.*, Bd. 21, S. 9-12.

1904–05: La matrice ciliare delle fibrille del vitreo, loro forma e disposizione, nonche lore rapporti colla neuroglia della retina visiva periferica nell'occhio umano adulto, *Arch. di Ottalmol.*, vol. 12, p. 206-262.

[1] ¶This alphabetical list, compiled by Dr. HOOKER, for the English translation, is inserted here for convenience of reference. Although it does not pretend to be a complete list by any means, it will undoubtedly be useful for the purpose in view. (J. P. C. S.)

AGABABOW, A., 1904: Ueber die nerven des Sclera, *Arch. f. mikr. Anat.*, Bd. 63, p. 701-709.

AREY, L. B., 1915: The occurrence and the significance of photo-mechanical changes in the vertebrate retina—an Historical Survey, *Jour. Comp. Neur.*, vol. 25, p. 535-554.

1916: The function of the efferent fibers of the optic nerve of fishes, *Jour. Comp. Neur.* vol. 26, p. 213.

1919: A retinal mechanism of efficient vision, *Jour. Comp. Neur.*, vol. 30, p. 343-353.

BACH, L. and SEEFELDER, R., 1911: Atlas zur Entwicklungsgeschichte des menschlichen Auges, Part I, Leipzig.

BALDWIN, W. M., 1912: Die Entwicklung der Fasern der Zonula Zinnii im Auge der weissen Maus nach der Geburt, *Arch. f. mikr. Anat.*, Bd. 80, S. 274-305.

BARATZ, W., 1902: Das Wachstum des Auges und seine Besonderheiten beim Neugeborenen, *Inaug. Diss.*, St. Petersburg, (Abstr. in *Schwalbe's Jahresberichte*, *N. S.*, Bd. 9).

BARTELS, M., 1920: Aufgaben der vergleichenden Physiologie der Augenbewegungen, *Arch. f. Ophthal.*, Bd. 101, S. 299-332.

BELL, E. T., 1906: Experimentelle Untersuchungen ueber die Entwicklung des Auges bei Froschen, *Arch. f. mikr. Anat.*, Bd. 68, S. 279-296.

1906: Experimental studies on the development of the eye and the nasal cavities in frog embryos, *Anat. Anz.*, Bd. 29, 185-194.

BELL, E. T., 1907: Some experiments on the development and regeneration of the eye and nasal organ in frog embryos, *Arch. f. Entw.-Mech.*, Bd. 23, p. 457-478.

BERNARD, H. M., 1904: Studies in the retina. 4. The continuity of the nerves through the vertebrate retina, *Quart. Jour. Micr. Sci.*, *N. S.*, vol. 47, p. 303.

BIELSCHOWSKY, M., und POLLACK, B., 1904: Zur Kenntnis der Innervation des Säugetierauges, *Neurol. Centralbl.*, Jg. 23, S. 387.

BODDAERT, G., 1871: Zur Histologie der Cornea, *Centralbl. f. d. med. Wissen.*, 9 Jahrg., S. 337-339.

BODEN AND SPRAWSON, 1892: The pigment cells of the retina, *Quart. Jour. Micr. Sci.*, *N. S.*, vol. 33.

BONNEFON, 1920: Le régime circulatoire dans la choroïde et le corps ciliare, ses rapports avec l'ophthalmotonus et la sécrétion de l'humeur aqueuse, *Ann. d'ocul.*, T. 157, p. 504-507.

BRUCH, C., 1854: Ueber Bindegewebe, *Zft. f. wiss. Zool.*, Bd. 6, S. 145-207.

BRUECKNER, A., 1907: Ueber Persistenz von Testen der tunica vasculosa lentis, *Arch. f. Augenheilk.*, Bd. 56, Suppl., S. 5-149.

BRYCE, T. H., 1908: Embryology, in QUAIN'S *Anatomy*, Eleventh Edition, vol. 1.

CAJAL, S. RAMON Y., 1893: La rétine des vertebrés, *La Cellule*, T. 9, p. 119-257.

1896: Nouvelles contributions a l'étude histologique de la rétine, etc., *Jour. de l'Anat. et de la Physiol.*, T. 32.

CAMERON, J., 1905: The development of the retina in Amphibia: An embryological and cytological study, *Jour. of Anat. and Physiol.*, vol. 39, p. 332.

CARINI, A., 1899: Osservazioni sull'origine del vitreo, *Monitore Zool. Ital.*, T. 10, p. 33-39.

CHIEWITZ, J. H.: 1887: Die Area and Fovea centrales retinae beim menschlichen Fetus, *Internat. Zeitschr. f. Anat. u. Physiol.*, Bd. 4, S. 201-226.

CHILD, C. M., 1921: *The Origin and Development of the Nervous System*, Chicago.

CIRINCIONE, G., 1903: Ueber die Genese des Glaskörpers bei Wirbeltieren, *Anat. Anz.*, Bd. 23, Erg. Heft, S. 51-60, Verh. Anat. Gesell., 17 Vers.

Ueber die Genese des Glaskörpers bei Wirbeltieren, *Centralbl. prakt. Augenheilk.*, Bd. 27, S. 161-169.

1904: Ueber den gengewärtigen Stand der Frage hinsichtlich der Genesis der Glaskörpers, *Arch. f. Augenheilk.*, Bd. 50, S. 201-217.

COLE, W. H., 1922: The transplantation of skin in frog tadpoles, with special reference to the adjustment of grafts over eyes and to the local specificity of integument, *Jour. Exp. Zool.*, vol. 35, p. 353-410.

CORNING, H. K., 1900: Ueber die vergleichende Anatomie der Augenmuskulatur, *Morph. Jahrb.*, Bd. 29, S. 94-140.

COSMETTATOS, G. F., 1910: Récherches sur le développement de la membrane pupillaire chez l'homme, *Arch. d'Ophthal.*, T. 30, p. 480-498.

DEDEKIND, F., 1908: Beiträge zur Entwicklungsgeschichte der Augengefässe des Menschen, *Anat. Hefte*, Bd. 38, S. 1-29.

DEMOURS, 1741: Observations sur la cornée, Histoire d. l'Acad. roy. d. sciences avec les *Mem. d. Math. et d. Physique*, p. 68-71.

DESCEMET, 1758: An sola lens crystallina cataractae sedes, *Dissertatio*, Paris.

1768: Observations sur la choroïde, *Mem. d. math. et de phys. presentés à l'Acad. roy. des Sciences*, T. 5, p. 177-189.

DETWILER, S. R., 1916: The effect of light on the retina of the tortoise and the lizard, *Jour. Exp. Zool.*, vol. 20, p. 165-191.

1920: Studies on the retina; The structure of the retina of Phrynosoma cornutum, *Jour. Comp. Neur.*, vol. 32, p. 347-356.

1921: Studies on the retina; Histogenesis of the visual cells in Amblystoma, *Jour. Comp. Neur.*, vol. 33, p. 493-508.

DEWEY, K. W., 1920: A contribution to the study of the lymphatic system of the eye, *Anat. Rec.*, vol. 19, p. 125-143.

DEYL, J., 1896: Ueber den Eintritt der arteria centralis retinae beim Menschen, *Anat. Anz.*, Bd. 11, S. 687-692.

DOGIEL, A. S., 1890: Die Nerven der Cornea des Menschen, *Anat. Anz.*, Bd. 5, S. 483.

1895: Die Structure des Nervenzellen der Retina, *Arch. f. mikr. Anat.*, Bd. 46, S. 394.

DRAGENDORF, O., 1903: Experimentelle Untersuchungen ueber Regenerationsvorgänge am Auge und an der Linse bei Hühnerembryonen, Univ. Rostock. i. M.

DRYAULT, A., 1904: Appareil de la vision; in POIRER and CHARPY, *Traité d'anatomie humaine*, vol. 5.

DUBOIS, E. et CASTELAIN, F., 1907: Contribution a l'étude de l'innervation motrice de l'Iris, *Arch. d'Ophthal.*, T. 27, p. 310.

DUBREUIL, G., 1907: La glande lacrymale de l'homme et des mammiferes, *Rev. gen. Ophthal.*, Année 26, p. 339.

DUCLOS, J., 1895: *Étude sur les dimensions du cristallen* (thesis), Bordeaux.

VAN DUYSE, 1904: *Elements d'embryologie et teratologie de l'oeil*, Paris.

DWIGHT, T., 1897: The anatomy of the orbit and the appendages of the eye; in *System of Diseases of the Eye*, ed. by NORRIS and OLIVER, vol. 1, pp. 69-108.

EKMAN, G., 1914: Experimentelle Beiträge zum Linsenbildungsproblem bei den Anuren mit besonderer Berücksichtigung von Hyla arborea, *Arch. f. Entw.-Mech.*, Bd. 39, S. 328-351.

1914: Zur Frage nach der frühzeitigen Spezifizierung der verschiedenen Teile der Augenanlage, *Arch. f. Entw.-Mech.*, Bd. 40, S. 122-130.

EYCLESHYMER, A. C., 1893: The development of the optic vesicles in Amphibia, *Jour. Morph.*, vol. 8, p. 189-193.

FALCHI, F., 1888: Sur l'histogenese de la retine et du nerf optique, *Arch. Ital. de biol.*, T. 9, p. 382-399.

1888: Ueber die Histogenese des retina und des nervus opticus, GRAEFES *Arch. f. Ophthal.*, Bd. 34, (2), S. 67-108.

FESSLER, F., 1920: Zur Entwicklungsmechanik des Auges, *Arch. f. Entw.-Mech.*, Bd. 46, S. 169-201.

FISCHEL, A., 1900: Ueber die Regeneration der Linse, *Anat. Hefte*, Bd. 14.

FISCHEL, A., 1921: Ueber normale und abnorme Entwicklung des Auges; I. Ueber Art und Ort der ersten Augenanlage sowie die formale und kausale Genese der Cyclopie; II. Zur Entwicklungsmechanik der Linse, *Arch. f. Entw.-Mech.*, Bd. 49, S. 383-462.

FRITSCH, G., 1900: Vergleichende Untersuchungen menschlicher Augen, *Sitzungsber. d. Königl. Preuss. Akad. d. Wissen. zu Berlin*, Bd. 30, S. 1-18.

1908: Ueber den Bau und die Bedeutung der histologischen Elements in der Netzhaut des Auges besonders am Ort des deutlichsten Sehens bei verschiedenen Menschenrassen, *Anat. Anz.*, Bd. 32, Erg.-heft, S. 141-145, Verhand. d. Anat. Gesell.

Fritz, W., 1906: Ueber die membrana Descemetii und das ligamentum pectinatum iridis bei den Säugetieren und beim Menschen, *Sitzber. k.k. Akad. Wiss. Wien. Math-naturwiss.*, Kl., Bd. 115.

Froriep, A., 1891: Ueber die Entwicklung des Sehnerven, *Anat. Anz.*, Bd. 6, S. 155.

1905: Ueber die Einstülpung der Augenblase, *Arch. f. mikr. Anat.*, Bd. 66, S. 1-11.

1905: Die Entwicklung des Auges der Wirbeltiere; in Hertwig's Handbuch, Bd. II, 2.

Fuchs, H., 1905: Zur Entwicklungsgeschichte des Wirbeltierauges; I. Ueber die Entwicklung der Augengefässe des Kaninchens, *Anat. Hefte*, Bd. 28, S. 1-251; 194-216.

Gabrielides, A., 1895: Récherches sur l'embryogenie et de l'anatomie comparée de l'angle de la chambre anterieure chez le poulet et chez l'homme. Muscle dilatateur de la pupille; Thesis, *Arch. d'ophthal.*, T. 15, p. 176-193, Paris.

Grynfellt, E., 1898: Sur le développement du muscle dilatateur de la pupille chez le lapin, *Comptes Rend. Acad. Sci.*, T. 127, p. 966-968, Paris.

1899: *Le muscle dilatateur de la pupille chez les Mammiferes*, Firmin-Montane, Montpellier.

Gutmann, G., 1888: Ueber die Lymphbahnen der Cornea, *Arch. f. mikr. Anat.*, Bd. 32, S. 593-602.

Haemers, A., 1903: Régeneration du corps vitré, *Arch. d'ophthal.*, T. 23, p. 103-114.

Hamburger, C., 1921: Einiges Kritische und Experimentelle zur Ernährung des Auges, *Klin. Monatsbl. f. Augenh.*, Bd. 66, S. 403-409.

Hartinger, H., 1921: Zur Messung der Kammertiefe und des Irisdurchmessers, *Zft. f. ophthal. Optik*, Bd. 9, S. 135-143.

Heerfordt, C., 1900: Studien ueber den musculus dilator pupillae, samt Angabe von gemeinshaftlichen Kennzeichen einiger Fälle epithelialer Muskulatur, *Anat. Hefte*, Bd. 14, S. 487-558.

1900: Berichtigung zum Aufsatz ueber den musculus dilatator pupillae, etc., *Anat. Hefte*, Bd. 15, S. 721.

Henckel, Fr., 1898: Beiträge zur Entwicklungsgeschichte des menschlichen Auges, *Anat. Hefte*, Bd. 10, S. 485-510; also as *Med. Dissert. Giessen.*

Herzog, H., 1902: Ueber die Entwicklung der Binnenmuskulatur des Auges, *Arch. f. mikr. Anat.*, Bd. 60, S. 517-586.

Hesse, R., 1898: Die Sehorgane des Amphioxus, IV Abt. in Untersuchungen ueber die Organe der Lichtempfindung bei niederen Tieren, *Zft. f. wiss. Zool.*, Bd. 63.

1904: Ueber den feineren Bau des Stäbchen und Zapfen einiger Wirbeltiere, *Zool. Jahrb.*, Suppl. 7, (*Fetschr. zum. Geburtst.* A. Weismann), S. 471.

Hirsch, K., 1906: Ist die fetale Hornhaut vascularisiert? *Klin. Monatsbl. f. Augenheilk.*, Bd. 44, N. S., Bd. 2, S. 13-30.

Howard, A. D., 1908: The visual cells in vertebrates, chiefly in Necturus maculosus, *Jour. Morph.*, vol. 19, p. 561-631.

Jacobi, E., 1905: Ueber die Neuroglia des Sehnerven, *Klin. Monatsbl. f. Augenheilk.*, Bd. 43, S. 129-137.

Jeannulatos, P., 1896: Étude de la formation de la chambre anterieure. Embryogenie de la membrane pupillaire, part qu'elle prend dans l'évolution d'iris, *Arch. d'ophthal.*, T. 16, p. 529-554.

1896: *Récherches embryologiques sur la mode de formation de la chambre anterieure chez les mammiferes et chez l'homme. Embryogenie de la membrane pupillaire, part qu'elle prend dans l'évolution de l'iris*, (thesis), Paris.

Johnston, J. B., 1905: The morphology of the vertebrate head from the viewpoint of the functional divisions of the nervous system, *Jour. Comp. Neur.*, vol. 15, p. 175-275.

Keibel, F., 1886: Zur Entwicklung des Glaskörpers, *Arch. f. Anat. u. Physiol.*, Anat. Abt., S. 358-368.

1889: Ueber die Entwicklung des Sehnerven, *Deutsche med. Wochenschr.*, Bd. 15, S. 116.

1906: Die Entwicklungsgeschichte des Wirbeltierauges, *Klin. Monatsbl. f. Augenheilk.*, Bd. 44, N. S. Bd. 2, S. 112-132.

Keil, R., 1906: Beiträge zur Entwicklungsgeschichte des Auges vom Schwein, mit besonderer Berücksichtigung des Verhaltens der fetalen Augenspalte, *Anat. Hefte*, Bd. 32.

Kessler, L., 1897: *Zur Entwicklung des Auges der Wirbeltiere*, Leipzig.

Kiribuchi, K., 1898: Ueber das elastische Gewebe im menschlichen Auge, nebst Bemerkungen ueber den Musculus dilatator pupillae, *Arch. f. Augenheilk.*, Bd. 38, S. 177-184.

Knape. V., 1909: Ueber die Entwicklung der Hornhaut der Hühnchens, *Anat. Anz.*, Bd. 34, S. 417-424.

Kölliker, A., von, 1903: Ueber die Entwicklung und Bedeutung des Glaskörpers, *Anat. Anz.*, Bd. 23, S. 49-51, Erg.-heft, Verh. Anat. Gesellsch. 17 Vers.
1904: Die Entwicklung und Bedeutung der Glaskörpers, *Zeitschr. f. wiss. Zool.*, Bd. 76, S. 1-25.

Kolmer, W., 1904: Ueber ein Strukturelement der Stäbchen und Zapfen der Froschretina, *Anat. Anz.*, Bd. 25, S. 102.

Krueckmann, E., 1906: Ueber die Entwicklung und Ausbildung der Stützsubstanz im Sehnerven und in der Netzhaut, *Klin. Monatsbl. f. Augenheilk.*, Bd. 44, N.S., Bd. I, S. 162-191.

Lange, O., 1901: Zur Anatomie des Auges des Neugeborenen. 1. Zur Anatomie des Ciliarmuskels der Neugeborenen, *Klin. Monatsbl. f. Augenheilk.*, Bd. 39, S. 1-6.
2. Suprachorioidealraum, zonula Zinnii, ora serrata und sogenannte physiologische Exkavation der Sehnervenpapille, *Klin. Monatsbl. f. Augenheilk*, Bd. 39, S. 202-213.
1908: *Einblicke in die embryonale Anatomie und Entwicklung des Menschenauges. Nach eigenen Präparaten dargestellt*, Wiesbaden.

Laurens, H. and Williams, J. W., 1917: Photomechanical changes in the retina of normal and transplanted eyes of Amblystoma larvae, *Jour. Exp. Zool.*, vol. 23, pp. 71-84.

Laurens, H. and Detwiler, S. R., 1921: Studies on the retina; The structure of the retina of Alligator mississippiensis and its photomechanical changes, *Jour. Exp. Zool.*, vol. 32, p. 207-234.

Leber, Th., 1908: Die Zirkulations und Ernährungsverhältnisse des Auges; in Graefe-Sämischs *Handbuch der gesamten Augenheilkunde*, 2d ed. Bd 2, 2. (Published 1903).

Leber, Th., and Pilzecker, A., 1906: Neue Untersuchungen ueber den Flüssigkeitswechsel des Auges, *Arch. f. Ophthal.*, Bd. 64, S. 1-127.

LeCron, W. L., 1906–07: Experiments on the origin and differentiation of the lens in the Amblystoma, *Am. Jour. Anat.*, vol. 6, p. 245.

Lenhossek, M., 1903: *Die Entwicklung des Glaskörpers*, Leipzig.

Lewis, W. H., 1903: Wandering pigmented cells arising from the epithelium of the optic cup, with observations on the origin of the m. sphincter pupillae in the chick, *Am. Jour. Anat.*, vol. 2, pp. 405-416.
1904: Experimental studies on the development of the eye in Amphibia; I. On the origin of the lens. Rana palustris, *Am. Jour. Anat.*, vol. 3, pp. 505-536.
1905: Experimental studies on the development of the eye in Amphibia; II. On the cornea, *Jour. Exp. Zool.*, vol. 2, pp. 431-446.
1907: Experimental studies on the development of the eye in Amphibia; III. On the origin and differentiation of the lens, *Am. Jour. Anat.*, vol. 6, pp. 473-509.
1907–08: Lens formation from strange ectoderm in Rana sylvatica, *Am. Jour. Anat.*, vol. 7, p. 45.
1907: Experiments on the origin and differentiation of the optic vesicle in Amphibia, *Am. Jour. Anat.*, vol. 7, p. 259.

Lubosch, W., 1909: Besprechung einen neuen Theorie der Licht- und Farbenempfindung nebst einem Exkurs über die stammesgeschichtliche Entstehung des Wirbeltierauges, *Morphol. Jahrb.*, Bd. 39, S. 146-153.

Magitot, A., 1910: Étude sur le développement de la rétine humaine, *Ann. d'Ocultistique*, T. 143, p. 241.

Mall, F. P., 1893: Histogenesis of the retina in Amblystoma and Necturus, *Jour. Morph.*, vol. 8, pp. 415-432.

Mencl, E., 1907: Neue Tatsachen zur Selbstdifferenzierung der Augenlinse, *Arch. f. Entw.-Mech.*, Bd. 25, S. 431-450.

Merkel, Fr., 1873, *Die Muskulatur der menschlichen Iris*, Rostock.

Merkel, Fr., u. Kallius, E., 1910: Makroskopische Anatomie des Auges, Graefe-Sämisch, *Handb. d. ges. Augenheilk*, 2. Auf., Bd. I, 1 Abt., Kap. I.

Mörner, C. T., 1893: Untersuchung der Proteinsubstanzen in den lichtbrechenden Medien des Auges; II. Die Hornhaut. III. Die Glasmembranen der lichtbrechenden Medien, *Zft. f. physiol. Chemie*, Bd. 18, S. 213-256.

Muench, K., 1906: Ueber die Mechanik der Irisbewegung, *Arch. f. Ophthal.*, Bd. 44, p. 339.

Neal, H. V., 1909: The morphology of the eye muscle nerves, *Proc. 7th Int. Zool. Congr.*, pp. 204-214.

1918: The history of the eye muscles, *Jour. Morph.*, vol. 30, pp. 433-453.

Nussbaum, M., 1892: Vergleichende Studien über die Orbita des Menschen und der Tiere, *Sitzb. Niederrh. Ges.*

1893: Vergleichendanatomische Beiträge zur Kenntniss der Augenmuskeln, *Anat. Anz.*, Bd. 8, S. 208.

1899: Ueber Entwicklung der Augenmuskeln bei den Wirbeltieren, *Sitzb. Niederrh. Ges.*

1901: Die Pars ciliaris retinae des Vogelauges, *Arch. f. mikr. Anat.*, Bd. 57, S. 346-353.

1901: Entwicklung der Binnenmuskeln des Auges der Wirbeltiere, *Arch. f. mikr. Anat.*, Bd. 58, S. 199-230.

1902: Zur Anatomie der Orbita, *Anat. Anz.*, Bd. 21, Erg. heft. S. 137-143, Anat. Vers.

1912: Entwicklungsgeschichte des menschlichen Auges; in Graefe-Sämischs *Handbuch der gesamten Augenheilk*, (2d ed., Part 15, pp. 35-59. Published in Part I of Bd. 2 in 1908) III. Aufl., I. Teil, VIII. Kap.

Ogawa, C., 1921: Experiments on the regeneration of the lens in Diemyctylus, *Jour. Exp. Zool.*, vol. 33, pp. 395-407.

Opin, 1906: Contribution a l'histologie du chiasma chez l'homme. La commissure de Hannover, *Arch. d'Ophthal.*, T. 26, p. 545.

Parker, G. H., 1908: The origin of the lateral eyes of vertebrates, *Am. Nat.*, vol. 42.

1909: The Integumentary nerves of fishes as photoreceptors and their significance for the origin of the vertebrate eyes, *Am. Jour. Physiol.*, vol. 25, pp. 77-80.

Van Pee, P., 1903: Récherches sur l'origine du corps vitré, *Arch. de Biol.*, T. 19, pp. 317-385.

Piersol, G. A., 1897: The Microscopic Anatomy of the Eyeball, Norris and Oliver, *System of Diseases of the Eye*, vol. 1. pp. 217-382.

Piper, H., 1905: Ueber die Funktionen der Stäbchen und Zapfen und über die physiologische Bedeutung des Sehpurpur, *Med. Klinik.*, Jg. I., Nr. 25, S. 629; Nr. 26, S. 658.

Poole, F. S., 1905: The relations of the superior oblique muscle of the eye in the mammals, *Jour. of Anat. and Physiol.*, vol. 39, p. 154.

Prokopenko, P., 1903: Ueber die Vertheilung der elastischen Fasern im menschlichen Auge, *Arch. f. Ophthal.*, Bd. 55, S. 94-120.

Rabl, C., 1898: Ueber den Bau und die Entwicklung der Linse; I. Selachier und Amphibien, *Zft. f. wiss. Zool.*, Bd. 63.

1899: 2. Reptilien und Vögel, *Zft. f. wiss. Zool.*, Bd. 65.

1900-03: 3. Säugetiere. Rückblick und Schluss, *Zft. f. wiss. Zool.*, Bd. 67.

1900: *Ueber den Bau und Entwicklung der Linse*; Leipzig.

1903: Zur Frage nach der Entwicklung des Glaskörpers, *Anat. Anz.*, Bd. 22, S. 573.

1918: Ueber die bilaterale oder nasotemporale Symmetrie des Wirbeltierauges, *Arch. f. mikr. Anat.*, Bd. 90, S. 261-444.

Recklinghausen, von, 1861: Eine Methode, mikroskopische Hohle und solide Gebilde voneinander zu unterscheiden, *Arch. f. path. Anat.*, Bd. 19, S. 451.

Retzius, G., 1894: Ueber den Bau des Glaskörpers und der zonula Zinnii in dem Auge des Menschen und einiger Tiere, *Biolog. Untersuch.*, N.S., Bd. 6, S. 66-87.

Reuter, K., 1897: Ueber die Entwicklung der Augenmuskulatur beim Schwein, *Anat. Hefte*, Bd. 9, S. 365, (Also as *Inaug. Diss. Göttingen*, pp. 21).

Rochon-Duvigneaud, 1920: La vision et l'oeil de l'homme à point de vue de l'anatomie et de physiologie comparées, *Bull. et. Mem. Soc. d'anthrop. de Paris*, 7 S, p. 1-29.

Sasse, 1879: Zur Chemie der Descemet'schen Membran, *Untersuch. d. physiol. Inst. d. Univ. Heidelberg*, Bd. 2.

Schaper, A., 1904: Ueber einige Fälle atypischer Linsenentwickelung unter abnormen Bedingungen, *Anat. Anz.*, Bd. 24, S. 305.

1899: Bemerkung zur Struktur der Kerne der Stäbchen-Sehzellen der Retina, *Anat. Anz.*, Bd. 15, S. 534-538.

Schmidt-Rimpler, H., 1903: Die Farbe der Macula lutea, *Arch. f. Ophthal.*, Bd. 57, p. 24.

Schön, W., 1895: Zonula und ora serrata, *Anat. Anz.*, Bd. 10, S. 360-364.

1895: Der Uebergangsraum der Netzhaut oder die sogenannte ora serrata, *Arch. f. Anat. u. Entwickl.*, S. 417-422.

Schreiber, L., 1904: Ueber vitale Indigkarminfärbung der Hornhaut, nebst Bemerkungen ueber das Verhalten des Indigkarmins im Blute und im Auge, *Arch. f. Ophthal.*, Bd. 58, S. 343-367.

Schultze, O., 1892: Zur Entwicklungsgeschichte des Gefässsystems in Säugetierauge, Kölliker's *Festschrift*; Leipzig.

1902: Ueber die Entwicklung und Bedeutung der ora serrata des menschlichen Auges, *Verh. Phys.-med. Gesel.*, N. S., Bd. 34; Wurzburg.

Seefelder u. Wolfrum, 1906: Zur Entwicklung der vorderen Kammer und des Kammerwinkels beim Menschen, nebst Bemerkungen ueber ihre Entstehung bei Tieren, *Arch. f. Ophthal.*, Bd. 63, S. 430-451.

Seidel, E., 1920: Weitere experimentelle Untersuchungen ueber die Quelle und den Verlauf der intraokulären Saftströmung, *Arch. f. Ophthal.*, Bd. 101, S. 189; Bd. 102, S. 366-382; Bd. 102, S. 383-414.

1921: Weitere experimentelle Untersuchungen ueber die Quelle und den Verlauf der intraokulären Saftströmung, *Arch. f. Ophthal.*, Bd. 105, S. 162-169.

1921: Weitere experimentelle Untersuchungen ueber die Quelle und den Verlauf der intraokulären Saftströmung, *Arch. f. Ophthal.*, Bd. 106, S. 176-186.

Slonaker, J. R., 1897: A comparative study of the area of acute vision in vertebrates, *Jour. Morph.*, vol. 13, p. 445-502.

1902: The eye of the common mole, Scalopo aquaticus machrinus, *Jour. Comp. Neur.*, vol. 12, p. 335-366.

1918: A physiological study of the anatomy of the eye and its accessory parts of the English sparrow (Passer domesticus), *Jour. Morph.*, vol. 31, p. 351-459.

1921: The development of the eye and its accessory parts of the English sparrow (Passer domesticus), *Jour. Morph.*, vol. 35, p. 263-357.

Spampani, G., 1901: Alcune richerche sull'origine et la nature del vitreo, *Monit. Zool. Ital.*, Anno 12, p. 145-153.

Spemann, H., 1901: Ueber Correlationen in der Entwicklung des Auges, *Anat. Anz.*, Bd. 19, Erg.-heft, S. 61-79, Verh. Anat. Gesellsch., 15 Vers.

1903: Ueber Linsenbildung bei defekter Augenblase, *Anat. Anz.*, Bd. 23, S. 457.

1904: Ueber Linsenbildung nach experimentelle Entfernung der primären Linsenbildungszellen, *Comptes rendus 6ieme. Cong. intern. de Zoologie.*

1905: Neue Tatsachenzum Linsenproblem, *Zool. Anz.*, Bd. 28.

1907: Ueber Linsenbildung nach experimenteller Entfernung der primären Linsenbildungszellen, *Zool. Anz.*, Bd. 31.

1912: Zur Entwicklung des Wirbeltierauges, *Zool. Jahrb.*, Bd. 32.

1912: Ueber die Entwicklung umgedrehter Hirnteile bei Amphibienembryonen, *Zool. Jahrb.*, Suppl. 15, Bd. 3, S. 1-48. (*Festschrift für* J. W. Spengel).

Stockard, C. R., 1907: The artificial production of a single median cyclopean eye in the fish embryo by means of sea water solutions of $MgCl_2$, *Arch. f. Ent.-Mech.*, Bd. 23, S. 249-258.

1906–07: The embryonic history of the lens in Bdellostoma stouti in relation to recent experiments, *Am. Jour. Anat.*, vol. 6, p. 511-515.

1910: The independent origin and development of the crystalline lens, *Am. Jour. Anat.*, vol. 10, p. 393-423.

1910: The experimental production of various eye abnormalities and an analysis of the development of the primary parts of the eye, *Arch. f. vergl. Ophthal.*, Bd. 1, S. 473-480.

1913: An experimental study of the position of the optic anlage in Amblystoma punctatum, with a discussion of certain eye defects, *Am. Jour. Anat.*, vol. 15, p. 253-289.

Strahl, H., Zur Entwicklung des menschlichen Auges, *Anat. Anz.*, Bd. 14, S. 298.

Sutton, J. E., 1920: The fascia of the human orbit, *Anat. Rec.*, vol. 18, p. 141-157.

Szent-Györgyi, A., 1917: Untersuchungen ueber den Bau des Glaskörpers des Menschen, *Arch. f. mikr. Anat.*, Bd. 89, S. 324-386.

Szily, A., 1902: Zur Anatomie und Entwicklungsgeschichte der hinteren Irisschichten, mit besonderer Berücksichtigung des musculus sphincter iridis beim Menschen, *Anat. Anz.*, Bd. 20, S. 161-175.

1902: Beitrag zur Kenntnis der Anatomie und Entwicklungsgeschichte der hinteren Irisschichten, mit besonderer Berücksichtigung des musculus sphincter pupillae des Menschen, *Arch. f. Ophthal.*, Bd. 53, S. 459-498.

1904: Zur Glaskörperfrage, *Anat. Anz.*, Bd. 24, S. 417-428.

1908: Ueber das Entstehen eines fibrillären Stützgewebes im Embryo und dessen Verhältnis zur Glaskörperfrage, *Anat. Hefte*, Bd. 35, Heft 107, S. 651-757.

Tartuferi, F., 1903: Ueber das elastische Hornhautgewebe und ueber eine besondere Metallimprägnationsmethode, *Arch. f. Ophthal.*, Bd. 56, S. 419-438.

Vermes, L., 1905: Ueber die Neurofibrillen der Retina, *Anat. Anz.*, Bd. 26, p. 601.

Virchow, H., 1901: Fächer, Zapfen, Leiste, Polster, Gefässe im Glaskörperraum von Wirbeltieren, sowie damit in Verbindungstehende Fragen, *Ergeb. Anat. u. Entwickl.*, Bd. 10, S. 720-844.

1910: Mikroscopische Anatomie der aüsseren Augenhaut und des Lidapparates, Graefe-Sämisch, *Handb. d. ges. Augenheilkunde*, 2 Aufl., Bd. I, II Kap.

Voll, A., 1892: Ueber die Entwicklung der membrana vasculosa retinae, Kölliker's *Festschrift.*

Vossius, A., 1883: Beiträge zur Anatomie des nervus opticus, *Arch. f. Ophthal.*, Bd. 29, Pt. 4, S. 119-150.

Vossius, A., 1921: Persistierendes Blutführendes Pupillarmembrangefäss. Ein Beitrag zur Frage des Blutdruckes in den intraocularen Gefässen, *Arch. f. Ophthal.*, Bd. 104, S. 320-324.

Wachs, H., 1920: Restitution des Auges nach Extirpation von Retina und Linse bei Tritonen, (Neue Versuche zur Wolff'schen Linsenregeneration) II, *Arch. f. Entw.-Mech.*, Bd. 46, S. 328-390.

Weiss, L., 1897: Ueber das Wachstum der menschlichen Auges und ueber die Veränderung der Muskelinsertionen am wachsenden Auge, *Anat. Hefte*, Bd. 8, Heft 25, S. 191-248.

Werber, E. J., 1916: On the blastolytic origin of the 'Independent' lenses of some teratophthalmic embryos and its significance for the normal development of the lens in vertebrates, *Jour. Exp. Zool.*, vol. 21, p. 347-367.

Whitnall, S. E., 1922: *The anatomy of the human orbit and accessory organs of vision*, London.

Wilckens, M., 1861: Ueber die Entwicklung der Hornhaut des Wirbeltierauges, *Zft. f. rat. Med.*, s. Reihe, Bd. 11, S. 167-174.

Wolff, G., 1895: Entwickelungsphysiologische Studien. I. Die Regeneration der Urodelenlinse, *Arch. f. Entw.-Mech.*, Bd. 1, S. 380-390.

Wood, C. A., 1917: *The fundus oculi of birds*, Chicago.

Zuern, J., 1902: Vergleichend histologische Untersuchungen ueber die Retina und die Area Centralis retinae der Haussäugetiere, *Arch. f. Anat. u. Entw.-Suppl. Bd.*

Physiological Optics

§8. Subdivisions of the Subject

Physiological Optics is the science of the visual perceptions by the sense of sight. The objects around us are made visible through the agency of light proceeding thence and falling on our eyes. This light, reaching the retina, which is a sensitive portion of the nervous system, stimulates certain sensations therein. These are conveyed to the brain by the optic nerve, the result being that the mind becomes conscious of the perception of certain objects disposed in space.

Accordingly, the theory of the visual perceptions may be divided into three parts:

1. *The theory of the path of the light in the eye.* Since we are here chiefly concerned with the refraction of the rays, and only incidentally with regular or diffuse reflection, this subdivision of the subject may be entitled *the dioptrics of the eye.*

2. *The theory of the sensations of the nervous mechanism of vision;* in which the sensations are considered by themselves without taking account of the possibility which they afford of recognizing external objects.

3. *The theory of the interpretation of the visual sensations,* dealing with the impressions which these sensations enable us to form of the objects around us.

Thus the difference between *physiological optics* and *physical optics* is that, whereas the former is concerned with the properties and behaviour of light only as they pertain to visual perceptions, the latter investigates optical phenomena and laws independently of the human eye. Physical optics is intimately connected with visual phenomena partly because the eye itself is an optical instrument and is employed for purposes of observation and partly also because it affords the most convenient means of recognizing the existence and propagation of light and of distinguishing one kind of light from another.

For the benefit of such readers as are not thoroughly familiar with the principles of physical optics, the following brief summary of the characteristic properties of light which are of most importance in physiological optics, together with definitions of some of the physical terms that are to be employed later, is inserted at this place.

Nearly all physicists nowadays regard light as a form of motion

in an hypothetical medium known as the luminiferous aether. The *wave theory* of light, as it is called, gives a satisfactory explanation of all the phenomena, and is the basis of this explanation.[1]

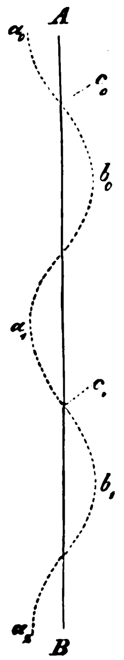

Fig. 23.

A good idea of the mode of motion of the aether particles along a ray of light, according to the fundamental assumption of the wave theory, may be obtained by holding in the hand the upper end of a moist string or a slender chain AB, Fig. 23, and, as it hangs vertically, giving it a rapid lateral motion to and fro. The string will then assume a wave form, as indicated by the dotted line in the figure, and this wave form will travel continuously from the upper to the lower end. As the waves travel down the string, each particle of the string must remain at the same height above the floor, but is free to vibrate to right and left or to and fro in a straight line, or to move in a horizontal circle or ellipse about its central equilibrium position, in accordance with the similar movements of the hand.

The vibrations of a row of aether particles, along which a ray of light is being propagated, would be exactly like the motions of the single parts of the string. Each element of the aether remains always in the vicinity of its original normal position, and merely oscillates in a straight or curved path about this point. It is not the aether particles themselves that are propagated as light, but only the wave form into which they continue to arrange themselves in their movements and changing *phases* of displacement and velocity. The paths of the aether particles lie in planes which are *transverse* or perpendicular to the directions of propagation of the waves of light; just as, in the case of our string, the waves travel vertically towards the floor while each particle of the vibrating cord continues to describe a horizontal path at a constant level. In this respect waves of light are different from the waves in elastic fluids, such as sound waves in air, for example, in which the particles perform *longitudinal* vibrations *parallel* to the direction of propagation.

If the path of the vibrating particles of aether in a train of light waves is rectilinear, the light is said to be *plane-polarised*;[2] whereas if

[1] ¶ HELMHOLTZ explains here the wave theory in terms of the ideas that were prevalent in the middle of the last century, long before the modern electromagnetic theory of light, as developed by MAXWELL and HERTZ, had superseded the earlier notions of the mode of propagation of light. (J. P. C. S.)

[2] ¶ As to the beginnings of the theory of the polarisation of light, see:

NEWTON'S *Optics* (1719), HUYGENS' *Traité de la lumière* (1690), MALUS in *Mémoires de*

the path is circular or elliptical, the rotation taking place in either direction around the orbit, the light is said to be *circularly* or *elliptically polarised.* Two plane-polarised rays whose directions of vibration are perpendicular to each other are said to have *mutually perpendicular polarisation.* Natural light, as it issues from luminous bodies, behaves usually as a uniform mixture of all kinds of differently polarised light; such light is said to be *unpolarised.* It is primarily through the agency of refraction and reflection that we obtain light with an excess of one kind of polarisation, or with one kind only.

If each aether particle concerned in the propagation of light traverses exactly the same path always in the same time and with the same velocity, the light itself is said to be *simple, monochromatic,*[1] or *homogeneous,* and the interval of time required for it to traverse its path once is known as the *period of vibration.* The most striking characteristic whereby light of different vibration periods may be distinguished by the eye is that of colour. Natural light from luminous bodies is usually not simple light of constant period, but consists of trains of waves of almost innumerable frequencies of vibration continually changing in value. Light of this kind is said to be *mixed* or *composite.* The white light of the sun is mixed light. Mixed light may be most readily separated into its monochromatic components by being refracted through a transparent prism, due to the fact that wave-trains of different periods will proceed in different directions after refraction. We may compare the vibrations in a beam of natural light to those which our string would make, if the hand holding it were to execute horizontal movements in various periods and directions, without, however, ever getting very far from its mean position.

Light is propagated with prodigious speed. Its velocity in interstellar space, as determined by astronomical observations, is found to be not less than 310,177.5 Km per second.[2] In all other transparent media the velocity is less than *in vacuo* and depends also on the frequency of vibration.

In crystalline bodies and other media whose molecular structure is different in different directions (*double refracting substances*), the

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physiques et de chimie de la Société d'Arcueil, tome II (1809), and FRESNEL'S *Oeuvres complètes,* tomes I, II, published in Paris in 1868. (J. P. C. S.)

[1] ¶A thing may be "simple" from one point of view and at the same time very complex in other ways. The word "monochromatic" (*einfarbig*) is used in physics to describe light of a definite wave-length or frequency, and this usage is so general there that it can perhaps now never be changed. The expression "single frequency light" is awkward, but it is more accurate and less misleading. (J. P. C. S.)

[2] ¶The mean value of the speed of light *in vacuo* is usually given nowadays as 300,000 Km per second. MICHELSON'S measurements in 1879 gave 186,380 miles per second. (J. P. C. S.)

velocity is also not the same for different directions of propagation and of polarisation.

If a ray of monochromatic, plane-polarised light travels along the line AB, Fig. 23, the aether particles which were originally in that line will now take the wave-form $a_0b_0a_1b_1a_2$, which moves forward with uniform velocity. This results in the formation of alternate loops to right and left, which are of equal length. The length of two such loops, c_0c_1, or more generally, the distance at a given instant from any point on one loop to the corresponding point of the next loop which has the same displacement, is called the *wave-length*. While the crest of the wave is travelling from a_0 to a_1, another crest must be formed at a_0, and the aether particle A must have performed a complete vibration; so that the light travels forward through one wave-length during the time of one vibration period. That is, the wave-length is equal to the vibration period multiplied by the velocity of propagation. Hence it follows that, for light of given vibration period traversing different transparent media, the wave-length must always be proportional to the velocity of propagation; and also that the wave-length in dense transparent substances is in general less than in empty space.

Wave-lengths may be measured by means of interference phenomena, and the corresponding vibration periods then calculated. Interference depends upon the fact that two rays of light will reinforce each other if they give rise to aether movements in the same direction, while if these movements are in opposite directions, they tend to neutralize each other. If two parts of one original ray are made to follow different paths and are then re-united, they will reinforce each other, provided the lengths of the paths are equal or differ by exactly one, or two, or more whole wave-lengths. From observations upon such interference effects it has been found that the wave-lengths of light corresponding to the so-called visible spectrum vary from 0.00039 to 0.00069 mm, and hence that the number of vibrations within the limits of visibility is from 451 to 789 million millions per second.[1]

The disturbances imparted to the surrounding aether by a luminous point-source in an isotropic medium proceed from it uniformly with the same speed in all directions. The result is that the wave expands in spherical form, while the amplitudes of the aether vibrations diminish in proportion as the radius of the sphere increases. It is to be noted, however, that the intensity of the light is proportional to the

[1] ¶The visible spectrum extends from about 723 $\mu\mu$ at the red end to 397 $\mu\mu$ at the violet end, these being the wave-lengths in vacuo. (1 $\mu\mu$ is one millionth of a millimetre or one-tenth of an Ångstrom unit.) Taking the velocity of light as 300 million metres per second, the limiting frequencies of the visible spectrum will be found to be comprised between about 415 and 756 million millions of vibrations per second, corresponding roughly to a range of somewhat less than one octave. (J. P. C. S.)

square of the amplitude of vibration, and that it therefore varies *inversely as the square* of the distance from the source. With respect to the propagation of light, any surface on which the particles of aether are all exactly in the same phase of vibration is called a *wave-surface* or wave-front.

The term *ray of light* requires a special explanation. Mathematically it may be defined as a line perpendicular to the wave-front. Thus, so far as spherical waves are concerned, the rays of light may be regarded as radii of all the concentric spherical surfaces, maintaining their same directions as long as the light continues to travel unhindered in the same transparent medium. As a matter of fact, the vibrations of the aether particles lying along a ray of light are not strictly independent of the movements of the particles along adjacent rays. However, interruptions of these neighbouring vibrations, produced, for example, by the interposition of an opaque body, are found to have little effect on the motions of the aether particles of the original ray under ordinary conditions, especially so far as the eye is concerned. In such circumstances, therefore, the vibrations of the aether particles belonging to a ray may be regarded as practically constituting an isolated mechanism, and as taking place independently of the vibrations in the adjacent rays. The theory of the propagation of light is thereby very greatly simplified. So we are accustomed, in ordinary practice, to assume that each ray of light travels in a straight line, unaffected by the rays adjacent to it; and indeed the inaccuracy arising from such assumption is generally altogether negligible. However, this method of explanation of the expansion of spherical light waves along rays or lines of propagation leads us astray when the light passes through an opening whose dimensions are small enough to be comparable with the wave-length. We find then that appreciable quantities of light are propagated laterally beyond the opening. And in general, wherever light travels past the edge of an opaque obstacle, a small part of it is deflected or, as we say, *diffracted*, from the direct path. In such cases, the phenomena can be explained only by taking into account the action of the entire wave. So far as the physics of the eye is concerned, however, we may tacitly assume that the propagation of light in an isotropic medium is rectilinear.

There is a very marked difference between light and sound in this connection, though it is really only one of degree. The dimensions of most natural objects are so large in comparison with a wave-length of light that the latter magnitude is practically negligible. Nearly all the light goes right on past the side of the obstacle in straight lines, and it is only by the use of special apparatus that we can detect the slight diffraction effects at the edges. Sound waves, on the other hand, are

several inches or feet in length, and therefore exhibit a great deal of diffraction when passing among obstacles. Common experience shows us that we can see only in straight lines, although we can hear around corners. We cannot, therefore, explain sound propagation in terms of "sound rays," as it would lead us too far from the actual facts. It is on this account that the theory of sound has always been so much more difficult to present than that of light. To this same peculiarity of light the eye owes its ability of judging correctly the precise position of a luminous body from the direction of the rays of light that come from it into the eye, whereas the ear possesses a similar faculty only to a very limited extent. On the other hand, any intervening opaque object prevents the eye from seeing what is behind it, whereas the ear can easily hear sounds proceeding from behind an obstacle. It thus happens that the phenomenon of wave diffraction involves certain distinct advantages and disadvantages in the case of both these senses.

When a ray of light is incident upon the boundary surface separating two different transparent media, as a rule a portion of it is turned back (*reflected*) and continues in the first medium, while another portion passes on into the second medium, but generally not without being bent aside or *refracted* from its original direction. If the surface of separation is perfectly smooth and *polished*, and the media are not double refracting, there will be only one reflected ray (*regular reflection*) and only one refracted ray. But if the surface is rough, the light will be scattered by reflection and refraction in many or all directions, even if the incident rays were all parallel (*diffuse reflection* and *refraction*).

As light proceeds through a material medium, its intensity may continue undiminished, no matter how far it travels, in which case the medium is said to be *transparent*. Empty space itself is probably the only example of an absolutely transparent medium. On the other hand, the intensity of the light may gradually diminish, which indeed can happen in two ways. Thus there may be in the medium itself innumerable minute foreign particles, tiny fractures, slight changes of structure, etc., which produce an internal scattering or refraction of the light (*false internal dispersion*), causing the medium to appear cloudy and self-luminous throughout; or the light may simply vanish without having been scattered (*absorption*). As absorption usually is different for light of different wave-lengths, white light is generally coloured after passing through a medium of this nature, and the medium itself appears coloured. Colourless transparent media are those which transmit all rays of visible light without appreciable absorption. They may, however, absorb certain invisible rays, as the infra-red or the

ultra-violet of solar radiation; that is, they may behave towards these rays as coloured media do towards visible light.

The absorption of light is frequently accompanied by chemical actions, probably always by heat, and sometimes by new light. In the last case, each part of the illuminated medium radiates light in all directions which differs, however, in colour and composition from that which was absorbed. The substance becomes self-luminous. This phenomenon is called *phosphorescence* if it persists after the illumination is cut off, and *fluorescence* (or *true internal dispersion*) if it lasts only during illumination. Light derived from a fluorescent substance is in general of longer wave-length than that of the incident light, and its colour and composition are independent of the latter. In one sense, therefore, fluorescence results in an alteration of the wave-length or refrangibility of the light, and it often happens that light which is invisible to the eye or almost so on account of its short wave-length may be detected by allowing it to fall on a fluorescent material such as the acid sulphate of quinine, uranium glass, extract of horse-chestnut bark, amber, etc.

Following is a list of works which deal in a general way with physiological optics:

1600. Fabricius ab Aquapendente, *de visione*. Ven. Fol.
1604. J. Kepler, *Paralipomena ad Vitellionem*. Frankf. Cap. 5.
1613. Francisci Aquilonii, *opticorum libri sex*. Antwerpiae.
1619. Scheiner, *Oculus sive fundamentum opticum, in quo radius visualis eruitur, sive visionis in oculo sedes cernitur et anguli visorii ingenium reperitur*. Oenip.
1738. R. Smith, *A compleat system of optics* with J. Jurin's *Essay upon distinct and indistinct vision*. Cambridge 1738.—Translated in German by Kästner. Altenb. 1755.
1740. Le Cat, *Traité des sens*. Rouen.
1746. P. Camper, *dissert. de visu*. Lugd. Batav.
1759. Porterfield, *Treatise on the eyes, the manner and phaenomena of vision*. Edinb.
1766. Haller, *Elementa physiologiae hum*. Lausanne 1757. Bern 1766.
1819. J. Purkinje, *Beiträge zur Kenntnis des Sehens in subjektiver Hinsicht*. Prag.
1825. J. Purkinje, *Beobachtungen und Versuche zur Physiologie der Sinne*. Bd. II. *Neue Beiträge zur Kenntnis des Sehens*. Berlin.
Lehot, *Nouvelle théorie de la vision*. Paris.
1826. J. Müller, *Zur vergleichenden Physiologie des Gesichtssinnes*. Leipzig.
1828. Muncke, Article: Gesicht und Sehen in Gehlers *physikalischem Wörterbuche*. Leipzig.
1830. A. Hueck, *Das Sehen seinem äusseren Prozesse nach*. Dorpat u. Göttingen.
1831. D. Brewster, *A treatise on optics*.
1834. C. M. N. Bartels, *Beiträge zur Physiologie des Gesichtssinnes*. Berlin.
1836. A. W. Volckmann, *Neue Beiträge zur Physiologie des Gesichtssinnes*.
1837. J. Müller, *Lehrbuch der Physiologie des Menschen*. Coblentz. Bd. II. S. 276-393.
1839. F. W. G. Radicke, *Handbuch der Optik*. Bd. II. S. 211-281.
1842. Burow, *Beiträge zur Physiologie und Physik des menschlichen Auges*. Berlin.
1844. Moser über das Auge in Dove's *Repertorium der Physik*. Berlin. Bd. V.

1845. TH. RUETE, *Lehrbuch der Ophthalmologie.*

1846. VOLCKMANN, Article: Sehen in R. WAGNERS *Handwörterbuch d. Physiologie.* Braunschweig.

1852. C. LUDWIG, *Lehrbuch der Physiologie des Menschen.* Heidelberg. Bd. I. S. 192-263.

1847–53. BRÜCKE, *Berichte über physiologische Optik in Fortschritte der Physik.* Bd. I to V.

Part First

The Dioptrics of the Eye

§9. Laws of Optical Imagery for a System of Spherical Refracting Surfaces.[1]

It is mainly through the agency of refraction that the course of the rays of light is altered in the human eye. Here, however, there is not just one refracting surface, but a whole series of them. Accordingly, let us proceed to consider the general laws of refraction of light in isotropic media, especially of successive refractions at each of a series of curved surfaces, since these laws form the theoretical basis of the first division of this work.

When a ray of light is incident on a surface separating two isotropic media, the directions of the corresponding reflected and refracted rays may be found as follows. Let *ab*, Fig. 24, represent the interface between the two media, which we shall call the *refracting surface*. The straight line *fc* indicates the path of an incident ray; *dc* is a line perpendicular to *ab* at *c*, called the *incidence normal*; and the straight lines *ch* and *cg* represent the paths of the reflected and refracted rays, respectively. The plane containing the normal and the incident ray is called the *plane of incidence*, the angle between these lines being the *angle of incidence* (angle *dcf* or α in the figure); the angle between the normal and the reflected ray (angle *hcd* in the figure) being the *angle of reflection*; and that between the normal and the refracted ray (angle *gce* or β) being the *angle of refraction*.

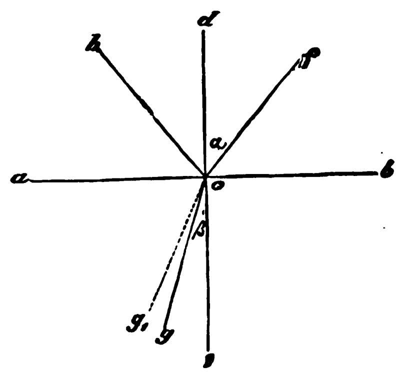

Fig. 24.

When the media are isotropic, (1) the reflected and refracted rays both lie in the plane of incidence; (2) the angle of reflection is equal to the angle of incidence; and (3) the angle of refraction and the angle of incidence are so related that the sines of these angles are in the same ratio as the velocities of propagation of light in the two media.

The ratio of the velocity of light in vacuo to that in a given medium is called the *absolute index of refraction* of that medium. Thus if c is the velocity in vacuo and c_1, c_2 the velocities in the first and second media, whose indices of refraction are denoted by n_1, n_2, respectively, then:

[1] See the first of the Appendices to Part I. G.

$$n_1 = \frac{c}{c_1}, \quad n_2 = \frac{c}{c_2}$$

$$\frac{\sin \alpha}{c_1} = \frac{\sin \beta}{c_2}$$

or

$$n_1 \sin \alpha = n_2 \sin \beta.$$

The law of refraction is usually expressed in this last form. By definition the absolute index of vacuum is unity. For air at ordinary pressures it differs so little from unity (its value being 1.00029 at 0° C and 760 mm pressure), that for most purposes the difference is negligible.

The velocities of different kinds of monochromatic light are all the same in vacuo, but in transparent liquids and solids light of different colours is propagated at different speeds. As a rule, the shorter waves (blue and violet) travel more slowly than the longer waves (red and yellow) in the same medium, and hence the index of refraction of the medium is greater for the former than for the latter. Accordingly, the violet rays are said to be *more refrangible* than the red. Owing to this difference of refrangibility, the different coloured components of white light after refraction generally proceed in different directions through a liquid or solid; so that this affords a means of separating them. In Fig. 24, the upper medium is supposed to be less dense than the lower one. The refracted ray *cg* is bent towards the incidence normal *ce*. The deviation is greater for the violet rays than for the red rays. Thus, if the violet light takes the route *cg*, the red component of the beam *fc* will proceed in a direction cg_1, and be separated from the more refrangible colours.

In the eye, we are concerned with the refraction of light at spherical or approximately spherical surfaces. Now in any case of this kind the laws of refraction are greatly simplified when the light falls on the surface nearly normally, so that the angles of incidence are all very small. Moreover, another simplification is introduced in the case of a system of spherical surfaces if they are so adjusted that their centres of curvature all lie on one straight line, called the *optical axis* of the system. A system of spherical surfaces which fulfills this last condition is called a *centered optical system*. A bundle of rays composed of straight lines which all intersect in a single point is said to be *homocentric*.[1] When a homocentric bundle of rays is refracted through a centered system of spherical refracting surfaces, all the angles of incidence being small, the emergent rays will likewise all meet again at a single point, or proceed as if they were diverging from a luminous point, that is, the bundle of emergent rays will be homocentric also. This point at which the rays are again united is called the *optical image*

[1] ¶A better term is "monocentric." (J. P. C. S.)

of the original point-source. If now the light were sent back through the system from the point where the image is formed, retracing the same paths the rays would be converged at the original luminous source; and hence these two points, where the source and its image are located, are called *conjugate foci* of the rays.

An optical image is said to be *real*, provided the rays, after their various refractions, actually converge to a focus there. A real image, therefore, must be beyond the refracting surfaces. If the emergent rays apparently diverge from a point nearer the source than the furthermost refracting surface, the image is said to be *virtual*. We say "*apparently*," because in this case the rays of light themselves do not actually intersect, but only their geometrical prolongations.

Convex glass lenses (burning glasses, convergent lenses) produce real images of distant objects, as shown in Fig. 25, where the lens is represented by *cd* and the luminous point by *a*. The incident rays *ac* and *ad* are refracted through the lens along *cf* and *de*, and actually intersect at *b*, so as to form a real image there. Proceeding on their way, they diverge from *b* as if it were itself an original luminous source.[1]

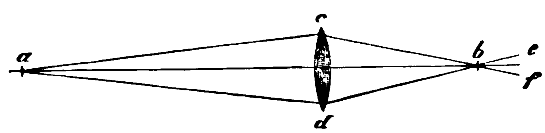

Fig. 25.

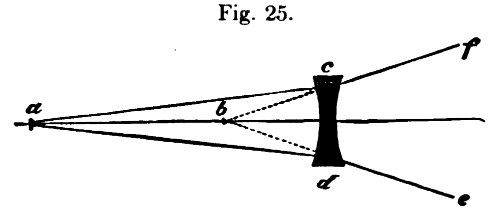

Fig. 26.

Concave glass lenses (divergent lenses), on the other hand, give virtual images of real objects, as shown in Fig. 26, which is lettered in the same way as Fig .25. In this case the emergent rays themselves do not actually intersect, but when they are traced back, they intersect "virtually" at *b*; so that to an eye placed between *f* and *e* it would appear as if the source were actually at *b*.

When several luminous points lie in a plane perpendicular to the axis and so near it that their rays all meet each of the surfaces in turn at very small angles of incidence, the images of these points, whether real or virtual, will also lie in a transversal plane perpendicular to the axis, and will have a relative geometrical arrangement similar to the distribution of the luminous points themselves. If these are the luminous points of an object, the optical image of the object will be similar to it.

[1] ¶This statement is not strictly accurate, because, whereas the luminous point *a* emits rays in all directions, the rays proceeding beyond *b* are confined to a definite cone of rays. (J. P. C. S.)

The *camera obscura* affords a good illustration of the formation of real images of objects which at the same time is very similar to the way they are produced in the eye. It consists of a box A (Fig. 27), blackened on the inside, with a focusing tube mounted in the front side which contains one or more glass lenses l. The back of the box is a ground glass plate g. If the camera is pointed towards a distant luminous object, and care taken to shade the ground glass, there may be seen upon it an inverted image of the object in its natural colours. If the lens is properly focused, the image will be sharply defined. This means that with a suitable lens situated at the proper distance from the ground glass screen, the rays from any given point of the object will be brought to a focus at some point on the screen. This point will receive all the light which enters the instrument from the corresponding point of the object, and will therefore have the same colour and proportional brilliancy. The light from any other part of the object will likewise be brought to its own proper focus, and will not overlap upon the image of the first point.

One of the first things to be noticed here is that the images of objects at different distances from the instrument are not sharply in focus on the screen at the same time; and in order to get an image of a near object, the lens tube must be pulled out more; and pushed in for a distant object. The reason is, of course, that the images of unequally distant objects are themselves unequally distant from the lens, so that they cannot all be in focus on the screen at once.

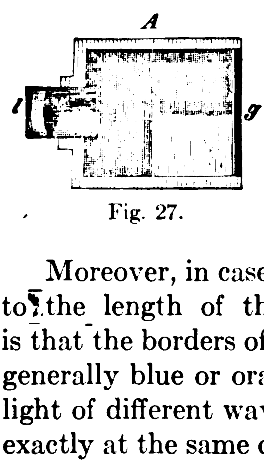

Fig. 27.

Moreover, in case the diameter of the lens is large in proportion to the length of the camera, another thing that will be observed is that the borders of bright parts of the image are fringed with colour, generally blue or orange. Owing to the difference of refrangibility of light of different wave-lengths, the foci for different colours do not lie exactly at the same distance back of the lens, and the coloured images, therefore, do not exactly coincide. This phenomenon is known as *chromatic aberration*. It can be almost completely avoided by the use of a suitable combination of lenses made of different kinds of glass. Optical instruments constructed in this way, so as to be free from chromatic aberration, are said to be *achromatic*.

However, even with monochromatic light it will be found that the images formed in the *camera obscura* and other optical instruments of large aperture are more or less blurred or indistinct. This is because the rays coming from a single point are not refracted by a spherical surface exactly to a focus, but only approximately. In order for the focus

to be exact, the angles of incidence must be infinitely small. This second kind of aberration is called *spherical aberration*. An *aplanatic*[1] instrument is one in which the effect is minimized as far as possible by a suitable combination of the refracting surfaces. Perfect aplanatism, generally speaking, is not attainable with spherical refracting surfaces; and for that purpose other curved surfaces which are in fact surfaces of revolution of the second or fourth order have to be used; but hitherto they have been but little employed in actual optical instruments.

Provided certain points on the axis of a centered system of spherical refracting surfaces, known as the *cardinal points* of the system, have been previously located, it is a comparatively simple matter to find the position and size of the image of an object produced by the so-called "paraxial" rays, and also to determine the procedure of any such ray. There are three pairs of these points, namely, the two *focal points*, the two *principal points*, and the two *nodal points*.

The side of the system from which the light comes will be called the *first side*, and the other side to which it goes, the *second side*. The indices of refraction of the first and last media will be denoted by n' and n'', respectively.

The *first focal point* is the point where an incident ray must cross the optical axis in order to emerge from the system in a direction parallel to the axis; and the *second focal point* is the point where an emergent ray, which was originally parallel to the axis, crosses the axis. The *second principal point* is the image of the *first;* that is, the principal points are a pair of conjugate points on the optical axis, so that a ray which before refraction is aimed at the first principal point will issue from the system along a straight line which passes through the second principal point. The *principal planes* of the system are the pair of conjugate planes perpendicular to the axis at the principal points; and the characteristic property of these planes is that an object lying in the first principal plane is reproduced by an image in the second principal plane which has exactly the same dimensions as the object and is oriented the same way. The principal points are the only pair of conjugate points of the system for which this condition is satisfied. The two nodal points are likewise a pair of conjugate points on the axis, characterized in this case by the fact that an incident ray which is directed to the *first nodal point* will emerge from the system in the same direction along a line which passes through the *second nodal point*.

The distance of the first principal point from the first focal point is called the *first focal length* of the optical system. It is reckoned as

[1] ¶The term "aplanatic," as commonly used nowadays, implies also the fulfillment of ABBE's so-called "sine condition." (J. P. C. S.)

positive when the first principal point lies beyond the first focal point in the direction of the incident light. Thus, in Fig. 28, suppose that the light goes along the axis from A towards B; and let f', f'' designate the focal points; h', h'' the principal points; and k', k'' the nodal points. In this case the first focal length $f'h'$ is a positive step along the axis. The *second focal length*, on the contrary, is defined as the distance

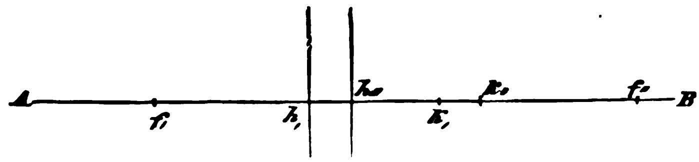

Fig. 28.

of the second focal point from the second principal point, that is the step $h''f''$, which is also a positive step in the adjoining diagram.[1]

The step from the first focal point to the first nodal point is the same as that from the second principal point to the second focal point; and, similarly, the step from the second focal point to the second nodal point is the same as that from the first principal point to the first focal point; that is,

$$\left.\begin{aligned} f'k' &= h''f'' \\ h'f' &= f''k'' \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (\alpha)$$

Hence it follows also that

$$k'h' = k''h'' = f'h' - f''h'' \quad . \quad . \quad . \quad . \quad . \quad . \qquad (\beta)$$

Moreover, the interval between the principal points is equal to that between the nodal points, that is,

$$h'h'' = k'k'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (\gamma)$$

Finally, the ratio of the focal lengths is equal to that of the indices of refraction of the first and last media: therefore

$$\frac{f'h'}{n'} = \frac{h''f''}{n''}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (\delta)$$

Accordingly, if the first and last media are the same ($n' = n''$), as is the case in most optical instruments, but not in the eye, the two focal lengths are equal and the principal points and nodal points are one and the same pair of conjugate points on the axis.

[1] ¶According to the author's definitions, the focal lengths ordinarily, therefore, have the same sign; but nowadays they are both usually defined in the same way, so that generally the two focal lengths are opposite in sign. (J. P. C. S.)

The first focal point, principal point and nodal point are considered as being in the first medium and are concerned with the incident rays; whereas the second focal point, principal point and nodal point are to be regarded as being in the last medium and having to do with the emergent rays.

The focal planes are perpendicular to the optical axis at the focal points. Incident rays which meet in a point in the first focal plane will emerge from the system as a bundle of parallel rays. According to the definition of the nodal points, the direction of this bundle of parallel rays will be parallel to a straight line drawn from the point of intersection of the incident rays in the first focal plane to the first nodal point.

A bundle of parallel rays in the first medium will be brought to a focus at a point in the second focal plane; and the position of this focus will be found by drawing the straight line which is parallel to the incident rays and passes through the second nodal point.

These rules enable us to find the path of a ray in the last medium when we know how it goes in the first, and to determine the position of the image of a given luminous point.

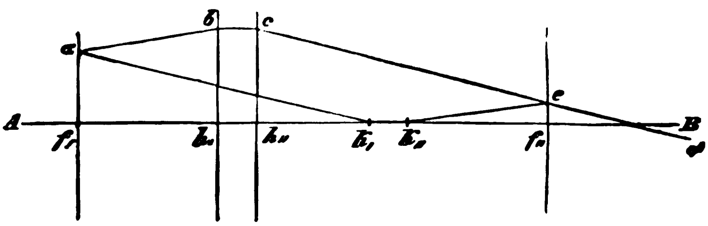

Fig. 29.

Thus, for example, being given the path ab (Fig. 29) of a ray in the first medium, suppose it is required to find its path in the last medium. Let a designate the point where the straight line ab crosses the first focal plane, and let b designate the point where it crosses the first principal plane. The image of the point b lies in the second principal plane, since one principal plane is the image of the other. And since, by the definition of these planes, the image of b must be exactly as far from the axis in the same direction, it must lie at the point c found by drawing a straight line through b parallel to the axis until it meets the second principal plane. Every ray originating at b or passing through this point must, therefore, emerge from the system along a straight line which goes through c.

Thus, the continuation of the ray ab goes through the point c. Now connect the point a with the first nodal point k' by a straight line and draw cd parallel to ak'. Then cd is the continuation of ab in the last medium; because, since a is a point in the first focal plane, all

incident rays that go through it will emerge in parallel directions, and since ak' may be considered as one such ray and since the continuation of it must be parallel to ak', it follows that the continuation of the ray ab in the last medium must also be parallel to ak'.

Or we may also use the property of the second focal plane to make this construction, as follows: Draw the straight line bc parallel to the optical axis meeting the second principal plane at c; and through the second nodal point draw $k''e$ parallel to ab meeting the second focal plane at e. The straight line ce is the path of the emergent ray.

Or suppose it is required to construct the image of a given luminous point a (Fig. 30). All that is necessary here is to construct the paths of two rays coming from a and to determine their ultimate point of intersection. When the given point a is not on the axis, the rays most convenient to use for this purpose are ac, parallel to the axis, and ak', directed towards the first nodal point. From the point c where ac crosses the second principal plane, draw the straight line cf'' through the second focal point, and determine the point e where this line intersects a straight line $k''e$ drawn through the second nodal point parallel to ak'. The point e thus found is the image of the point a.

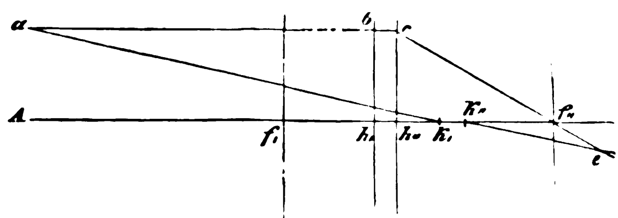

Fig. 30.

That the rays ac and ak' will, after refraction, follow the paths ce and $k''e$, respectively, follows immediately from the foregoing rules and definitions.

If a lies on the axis, one of the rays will follow the axis itself, without deviation. It will then be necessary to construct just one other ray, and the image will be its intersection with the axis.

Having thus briefly outlined the results of the theory for the benefit of those who wish only to know the facts, let us proceed now to the complete mathematical treatment of the subject.

Refraction at a Spherical Surface

In Fig. 31, let a designate the centre of the spherical surface cb, and p the position of a luminous point outside it. A ray proceeding from p towards the centre meets the surface normally and continues along the same straight line aq. Another ray, proceeding along pc,

meets the surface at c and is refracted there into the second medium. The first question is, what is its new path? By the law of refraction given above, it must remain in the plane of incidence determined by the incident ray and the normal at the point of incidence. The radius of

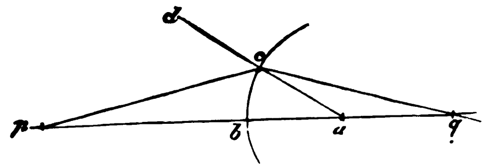

Fig. 31.

a sphere being everywhere perpendicular to its surface, the incidence normal in this case is simply the prolongation of the radius ac, and the plane of incidence is the plane containing the lines pc and ad. The line pq also lies in this plane, since it contains the points p and a that are on pq. The refracted ray, therefore, must intersect the straight line pa in some point q. Where is this point with reference to the point b? Should the ray happen to be parallel to pa, the intersection q may be regarded as at an infinite distance. Its position depends on the fact that

$$n' \sin(pcd) = n'' \sin(qca) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where n' and n'' denote the indices of refraction of the first and second media, respectively.

By the law of sines in trigonometry, we derive the following relations from the triangles apc and aqc:

$$\frac{\sin(pca)}{\sin(cpa)} = \frac{ap}{ac}.$$

$$\frac{\sin(qca)}{\sin(cqa)} = \frac{aq}{ac}.$$

Dividing the second of these equations by the first, and noting that the sine of the angle pca is equal to the sine of its supplement pcd, we obtain:

$$\frac{\sin(pcd)}{\sin(qca)} \cdot \frac{\sin(cqa)}{\sin(cpa)} = \frac{ap}{aq}.$$

Equation (1) may be written:

$$\frac{\sin(pcd)}{\sin(qca)} = \frac{n''}{n'}$$

and from the triangle pcq we have:

$$\frac{\sin(cqa)}{\sin(cpa)} = \frac{cp}{cq}.$$

Combining the last three equations, we find:

$$\frac{n'' \cdot cp}{n' \cdot cq} = \frac{ap}{aq} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

When the point p is infinitely far away, cp and ap may be considered as being ultimately equal, and hence in this case:

$$n_{\prime}.cq = n_{\prime\prime}.aq \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2a)$$

With the aid of equation (2), the path of the refracted ray may easily be constructed. If this is performed, not only for one single ray, but, by varying the position of the point c, for a number of other rays also, it will be found that the refracted rays as thus determined do not all intersect each other exactly at one point, and that their successive points of intersection lie along a curved (*caustic*) line, as shown in Fig. 32 for the case when the incident rays are parallel. In this diagram BB represents the spherical refracting surface and CC the incident rays; GFG is the caustic line, which is the enveloping curve of the bundle of refracted rays. The rays that pass closest to the centre meet at the cuspidal point F.

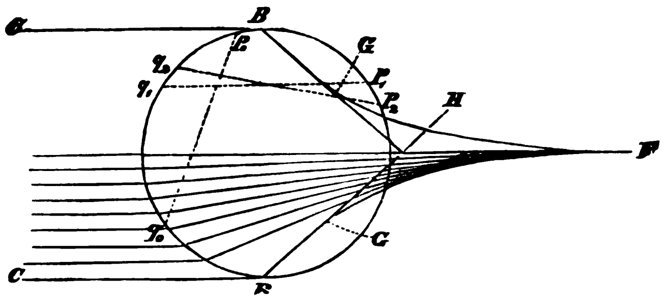

Fig. 32.

If the discussion is confined to the case of such rays as are incident on the refracting surface nearly normally, it is obvious from Fig. 31 that the ratio $\frac{cp}{cq}$ becomes more and more nearly equal to the ratio $\frac{bp}{bq}$, when the point c is taken nearer and nearer to b. In this case equation (2) becomes:

$$\frac{n_{\prime\prime}.bp}{n_{\prime}.bq} = \frac{ap}{aq} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2b)$$

Putting the radius $ab = r$, and employing also the following symbols: $bp = f_{\prime}$, $bq = f_{\prime\prime}$, $ap = g_{\prime}$, $aq = g_{\prime\prime}$, we have therefore

$$\left.\begin{aligned} f_{\prime} + r &= g_{\prime} \\ f_{\prime\prime} \quad &= g_{\prime\prime} + r. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2c)$$

Equation (2b) becomes:

$$\frac{n_{\prime\prime} f_{\prime}}{n_{\prime} f_{\prime\prime}} = \frac{f_{\prime} + r}{f_{\prime\prime} - r}$$

or

$$\frac{n_{\prime\prime}(g_{\prime} - r)}{n_{\prime}(g_{\prime\prime} + r)} = \frac{g_{\prime}}{g_{\prime\prime}}.$$

Thence, by a simple transformation, we may write:

$$\left.\begin{aligned} &\frac{n_{\prime}}{f_{\prime}}+\frac{n_{\prime\prime}}{f_{\prime\prime}}=\frac{n_{\prime\prime}-n_{\prime}}{r} \\ \text{or}\quad &\frac{n_{\prime\prime}}{g_{\prime}}+\frac{n_{\prime}}{g_{\prime\prime}}=\frac{n_{\prime\prime}-n_{\prime}}{r}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

from which the required distances $f_{\prime\prime}$ or $g_{\prime\prime}$ may be found.

If $F_{\prime\prime}$ and $G_{\prime\prime}$ denote the values of $f_{\prime\prime}$ and $g_{\prime\prime}$ when the luminous source p is infinitely far away, that is, when $f_{\prime}=g_{\prime}=\infty$, we find:

$$\left.\begin{aligned} F_{\prime\prime}&=\frac{n_{\prime\prime}r}{n_{\prime\prime}-n_{\prime}} \\ G_{\prime\prime}&=\frac{n_{\prime}r}{n_{\prime\prime}-n_{\prime}}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3a)$$

Similary, if $f_{\prime\prime}$ and $g_{\prime\prime}$ are made infinite, the values of $f_{\prime}$ and $g_{\prime}$ are:

$$\left.\begin{aligned} F_{\prime}&=\frac{n_{\prime}r}{n_{\prime\prime}-n_{\prime}}=G_{\prime\prime} \\ G_{\prime}&=\frac{n_{\prime\prime}r}{n_{\prime\prime}-n_{\prime}}=F_{\prime\prime} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3b)$$

Equations (3) may now be written simply as follows:

$$\left.\begin{aligned} &\frac{F_{\prime}}{f_{\prime}}+\frac{F_{\prime\prime}}{f_{\prime\prime}}=1 \\ &\frac{G_{\prime}}{g_{\prime}}+\frac{G_{\prime\prime}}{g_{\prime\prime}}=1. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3c)$$

Solving the first of these equations for $f_{\prime}$, and then for $f_{\prime\prime}$, we obtain:

$$\left.\begin{aligned} f_{\prime}&=\frac{F_{\prime}f_{\prime\prime}}{f_{\prime\prime}-F_{\prime\prime}} \\ f_{\prime\prime}&=\frac{F_{\prime\prime}f_{\prime}}{f_{\prime}-F_{\prime}}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3d)$$

Whenever either of these values turns out to be negative, it means that the corresponding point, p or q, lies on the opposite side of the refracting surface from that indicated in Fig. 31.

Remarks. 1. If the light, instead of originating at p in the first medium were to come from q in the second medium, qc would be the incident ray and cp the corresponding refracted ray. Thus rays diverging from q and meeting the surface nearly normally, would be brought to a focus at p. This leads at once to the formulae for refraction when the light falls on the concave side of the spherical surface. All that is necessary is to interchange the terms *first medium* and *second medium* and the subscripts in the symbols; whereupon the original equations (3) become

$$\begin{aligned} &\frac{n_{\prime}}{f_{\prime}}+\frac{n_{\prime\prime}}{f_{\prime\prime}}=\frac{n_{\prime}-n_{\prime\prime}}{r} \\ &\frac{n_{\prime\prime}}{g_{\prime}}+\frac{n_{\prime}}{g_{\prime\prime}}=\frac{n_{\prime}-n_{\prime\prime}}{r}. \end{aligned}$$

But these differ from (3) only in the sign of the right-hand term in each case; so that all we have to do is to consider r negative in order that equations (3) may apply also to the case of a concave surface. The same thing is true with respect to equations (3a), (3b), (3c) and (3d), which are all derived from (3).

2. If q is the image of p, then p is also the image of q. This mutual relation is expressed by saying that a pair of such points are *conjugate foci*, without implying which one of them is the source of light. Moreover, as far as the law of refraction is concerned, it is immaterial whether the point from which the light comes belongs to an actual material body, and is self-luminous or not, or is merely a point of intersection of previously refracted rays. Thus, the source may be only a virtual focus of such rays, which has to be constructed by producing the ray paths backwards until these lines intersect at a point on the other side of the surface at which the rays were refracted.

3. Another thing to be noted is that the laws of the *reflection* of rays at a curved mirror may also be deduced from equations (3) by simply putting $n_{,,} = -n_{,}$. Hereafter we shall need these formulae to investigate the reflex images in the refracting surfaces in the eye. Some writers prefer, however, to use special symbols for the mirror formulae. When $-n_{,}$ is substituted for $n_{,,}$ in equations (3), we obtain:

$$\frac{1}{f_{,}} - \frac{1}{f_{,,}} = -\frac{2}{r}.$$

When r is positive, that is, according to the above, when the mirror is convex, then for $f_{,} = \infty$, the value of $f_{,,}$ is equal to $\frac{r}{2}$ and is positive, so that the focus of the mirror lies behind it and is therefore virtual. For a concave mirror, r is negative and $f_{,,}$ is found to be negative also; that is, the focus of a concave mirror is in front of it and is therefore real. Usually, however, it is convenient to reckon the distance of the image from the mirror as positive when the image is real. Therefore, the signs of $f_{,,}$ and of the radius (r) of the mirror must be taken opposite to those used in case of refraction; so that the fundamental equation has to be written:

$$\frac{1}{f_{,}} + \frac{1}{f_{,,}} = \frac{2}{r}.$$

4. If r is infinite, the refracting surface is plane; in which case, according to (3a), the focal lengths become infinite also, and the first of equations (3) is transformed into

$$\frac{n_{,}}{f_{,}} + \frac{n_{,,}}{f_{,,}} = 0$$

or

$$f_{\prime\prime} = -\frac{n_{\prime\prime}}{n_{\prime}} f_{\prime} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3c)$$

In a plane refracting surface, therefore, the image lies on the same side of the surface as the source, but at a different distance.

Image of an Object in a Spherical Refracting Surface

Hereafter in speaking of an object whose image is formed in a curved refracting surface, it is to be understood that the term refers to a plane object placed perpendicular to the axis of the optical system; and furthermore that it is of very limited size and, for the purposes of the image, emits only such rays as meet the refracting surface nearly normally and therefore at very small inclinations to the optical axis.

When the image of a luminous point is produced by a spherical refracting surface, the straight line joining this point with the centre of the surface may be considered as the axis; but for an object as above defined the axis is the straight line drawn through the centre of the surface perpendicular to the plane of the object.

In Fig. 33, the straight line *sp* represents the object and the straight line *pr* drawn through the centre (a) of the spherical refracting surface perpendicular to *sp* is the optical axis. The image of a point *s* near the axis is formed at *t*, whose position is given by two rectangular coördinates, *ar* and *rt*, parallel and perpendicular to the axis, respectively.

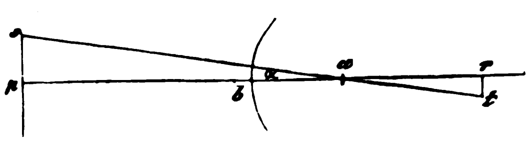

Fig. 33.

Considering the luminous point *s* by itself, its image, as we have seen, must lie somewhere on the straight line *sa*. Put $sa = y_{\prime}$, and $at = y_{\prime\prime}$; then according to equation (3c):

$$\frac{G_{\prime}}{\gamma_{\prime}} + \frac{G_{\prime\prime}}{\gamma_{\prime\prime}} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now put $pa = g_{\prime}$, $ar = x$ and angle $sap = \alpha$; then

$$\gamma_{\prime} = \frac{g_{\prime}}{\cos \alpha}$$

$$\gamma_{\prime\prime} = \frac{x}{\cos \alpha}.$$

Substituting these values in (4), we find:

$$\frac{G_{\prime}}{g_{\prime}} + \frac{G_{\prime\prime}}{x} = \frac{1}{\cos \alpha}.$$

Now by the conditions as to the size of the object, the angle α must be very small, and by putting $\cos\alpha = 1$, we shall neglect only a small quantity of the second order of smallness; and then we may write:

$$\frac{G'}{g'} + \frac{G''}{x} = 1.$$

If g'' denotes the distance from a of the image of the axial point p, then

$$\frac{G'}{g'} + \frac{G''}{g''} = 1,$$

and hence

$$x = g'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

That is, the point r at the foot of the perpendicular tr is the image of p.

Accordingly, the images of all points lying in a plane perpendicular to the axis at p are found to be approximately in a plane perpendicular to the axis through the image of p. Thus having first located the point r which is the image of the axial point p, the image of any other point in the object may be found by simply drawing a line from the given point through the centre of the spherical refracting surface, and the point where it crosses the transversal image-plane at r will be the required image. It follows from the principles of elementary geometry that the image and the object are similar.

It is easy to find the ratio of the corresponding linear dimensions of the object and its image. For example, let the height of the object ps be denoted by β', and the corresponding length in the image rt by β''; then

$$-\frac{\beta'}{\beta''} = \frac{g'}{g''}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where the minus sign has to be inserted in the formula so that it will apply to the cases of both erect and inverted images. Combining this equation with equations (2c), (3a), (3b) and (3c), we find:

$$\frac{\beta''}{\beta'} = \frac{G''}{G' - g'} = \frac{G'' - g''}{G'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6a)$$

or

$$\frac{\beta''}{\beta'} = \frac{F'}{F' - f'} = \frac{F'' - f''}{F''} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6b)$$

When the refracting surface is plane, the focal lengths become infinite, and for this case equation (6b) becomes

$$\frac{\beta''}{\beta'} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6c)$$

Accordingly, the image in a plane refracting surface is exactly as large as the object.

Generalization of the preceding results. Now let us inquire how the focal points, principal points and nodal points, as above defined, are applicable in the case of a single spherical refracting surface.

The *focal points* are the points where rays intersect in one medium that proceed parallel to the axis in the other medium. The positions of the focal points can be found from equations (3a) and (3b), because the magnitudes denoted by $F_{\prime}$, $F_{\prime\prime}$ and $G_{\prime}$, $G_{\prime\prime}$ are the distances of the focal points from the vertex and centre of the refracting surface, respectively.

The *focal planes* are the transversal planes perpendicular to the axis at the focal points. Since the conjugate point to either focal point is infinitely far away, the same statement must be true with respect to any point in one of the focal planes, provided it is not too far from the axis. In other words, rays which meet in a point in either focal plane will be parallel to each other in the other medium.

The characteristic property of the *principal points* is that object and image lying in the *principal planes*, which are perpendicular to the axis at these points, have precisely the same dimensions in all respects. For these planes, therefore, $\beta_{\prime} = \beta_{\prime\prime}$. According to equations (6b), this means that here $f_{\prime} = f_{\prime\prime} = 0$, which conditions must both be satisfied, as shown by equations (3d). Consequently, the two principal points of a spherical refracting surface coincide with each other at the vertex, so that object and image are here coincident.

The peculiar property of the *nodal points* is that a ray which is directed towards the first nodal point before refraction will proceed through the second nodal point after refraction in the same direction as before. The two nodal points of a spherical refracting surface also coincide, both being at the centre of the surface; because any ray whose path in the first medium is directed towards the centre of curvature will proceed into the second medium not only in the same direction but along the same straight line.

The constructions given above which were based on the properties of these special points and planes are, therefore, applicable also to the case of a single refracting surface. The process is simpler here because every point in the first principal plane is its own image, and also because a ray directed to the first nodal point proceeds straight on along the same line.

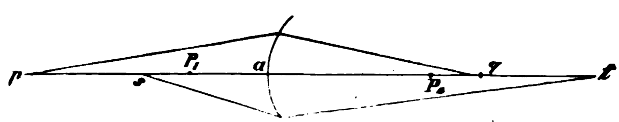

Fig. 34.

The two equations (3c), similar in form, enable us to determine the abscissa of the axial image-point, with reference, however, to two

different points as origins of measurement. More general equations of the same simple type are obtained by taking as origins in the two media any pair of conjugate points on the axis. For example, suppose the points designated by s, t and p, q in Fig. 34 are two pairs of conjugate points on the axis of a spherical refracting surface, whose focal points are P_1 and P_2; and let us introduce here the following symbols:

$$
\begin{array}{ll}
sa=f_1, & P_1a=F_1, \\
ta=f_2, & P_2a=F_2, \\
pa=\varphi_1, & \\
qa=\varphi_2, & \\
ps=h_1, & qt=-h_2, \\
P_1s=-H_1, & tP_2=-H_2,
\end{array}
$$

then

$$
\begin{array}{ll}
(\alpha) & \frac{F_1}{f_1}+\frac{F_2}{f_2}=1 \\
(\beta) & \frac{F_1}{\varphi_1}+\frac{F_2}{\varphi_2}=1 \\
(\gamma) & \varphi_1-f_1=h_1, \\
(\delta) & \varphi_2-f_2=h_2, \\
(\epsilon) & F_1-f_1=H_1, \\
(\zeta) & F_2-f_2=H_2.
\end{array}
$$

Substituting in (β) the values of φ_1 and φ_2 as obtained from (γ) and (δ), we obtain

$$\frac{F_1}{h_1+f_1}+\frac{F_2}{h_2+f_2}=1,$$

or

$$F_1(h_2+f_2)+F_2(h_1+f_1)=(h_1+f_1)(h_2+f_2).$$

Writing (α) in the form,

$$F_1f_2+F_2f_1=f_1f_2,$$

and subtracting this equation from the one above, we get:

$$F_1h_2+F_2h_1=h_1h_2+h_1f_2+h_2f_1$$

or

$$(F_1-f_1)h_2+(F_2-f_2)h_1=h_1h_2,$$

which by virtue of equations (ϵ) and (ζ) may be put in the following form:

$$H_1h_2+H_2h_1=h_1h_2,$$

or

$$\frac{H_1}{h_1}+\frac{H_2}{h_2}=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Thus if distances along the axis are reckoned from any given pair of conjugate foci a formula of this same simple type is always obtained. Since the centre of curvature of the refracting surface and likewise any point on the surface coincides with its conjugate point, these

points are their own images, and equations (3c) prove to be only special examples of the general formula (7).

If the point s coincides with the first focal point, equation (7) is confusing, because now both H_2 and h_2 become infinite. But in this case the corresponding equation can be obtained easily from the first of equations (3d), as follows:

$$f' = \frac{F'f''}{f'' - F''}.$$

By subtracting F' from both sides the preceding equation may be written:

$$f' - F' = \frac{F'F''}{f'' - F''} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7a)$$

If in this equation we put $f' - F' = l'$, $f'' - F'' = l''$, where l', l'' denote the distances of axial object-point and image-point measured, in opposite directions, from the first focal point and the second focal point, respectively, the connection between a pair of conjugate points is given in its simplest form, as follows:

$$l'l'' = F'F'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7b)$$

In terms of the same magnitudes, formula (6b) for the magnification-ratio becomes

$$\left.\begin{array}{l} \dfrac{\beta'}{\beta''} = -\dfrac{l'}{F'} \\ \text{or} \\ \dfrac{\beta''}{\beta'} = -\dfrac{l''}{F''}. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7c)$$

Connection between the linear magnification and the angular magnification. In Fig. 35 let sp and qr represent an object and its image in a spherical refracting surface whose axis is pq. Let α_1, α_2 denote the angles which a ray coming from p makes with the axis before and after refraction, respectively; each of these angles being reckoned as positive or

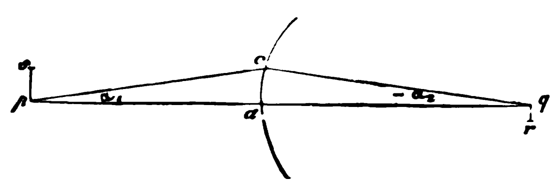

Fig. 35.

negative according as the ray in question leaves the axis above or below it, respectively. Thus $\angle cpa = \alpha_1$, $\angle cqa = -\alpha_2$. Also, as before, let $ps = \beta_1$, $qr = \beta_2$, $ap = f_1$ and $aq = f_2$. Since by hypothesis the slopes of the rays are all small, the arc ac may be considered here as a short straight line perpendicular to the axis at a; and so we may then write:

$$ac = f_1 \tan\alpha_1,$$
$$ac = -f_2 \tan\alpha_2,$$

and therefore

$$f_1\tan\alpha_1 = -f_2\tan\alpha_2 \qquad \text{(A)}$$

Moreover, from equations (3d) and (6b):

$$\frac{f_2}{f_1} = \frac{F_2}{f_1 - F_1} = \frac{f_2 - F_2}{F_1},$$

$$\frac{\beta_2}{\beta_1} = \frac{F_1}{F_1 - f_1} = \frac{F_2 - f_2}{F_2}.$$

From (3a) and (3b), $\frac{F_2}{F_1} = \frac{n_2}{n_1}$; and hence

$$\frac{f_2}{f_1} = -\frac{n_2}{n_1} \cdot \frac{\beta_2}{\beta_1}.$$

Combining this equation with equation (A), we obtain the relation sought, namely:

$$n_1\beta_1\tan\alpha_1 = n_2\beta_2\tan\alpha_2 \qquad \text{(7d}$$

This important result expresses the relation between the linear magnification and the angular magnification without involving directly the distance of the object or the focal lengths of the refracting surface.[1]

Centered System of Spherical Refracting Surfaces

Let us proceed now to investigate the imagery in the case of a symmetrical optical instrument consisting of a system of spherical refracting surfaces whose centres all lie on a straight line which is therefore the axis of symmetry or *optical axis* of the system.

The light is incident in the *first medium*, whose index of refraction is denoted by n_1, on the *first surface*; the *second medium*, of index n_2, is the medium comprised between the first surface and the *second surface*, etc.; and the *last medium* of all is the medium beyond the last or mth surface, whose index of refraction is denoted therefore by n_{m+1}, since the number of media is always one more than the number of surfaces. As above, the radius of a surface is counted positive or negative according as the surface is convex or concave towards the side of the instrument from which the light comes. Moreover, it should be stated here once for all that when an image is said to lie in a certain medium, this means that it is the focus of the rays in their passage

[1] ¶This is a famous law in geometrical optics, sometimes called the "Helmholtz equation" or the "Lagrange-Helmholtz equation," because, although Helmholtz discovered it independently and recognized its peculiar importance, he himself attributed it to Lagrange who had published a special case of the general law in 1803. But Robert Smith in his *Compleat System of Opticks* (Cambridge 1738) had enunciated this law for a system of infinitely thin lenses long before Lagrange, so that nowadays it is sometimes referred to as the "Smith-Helmholtz equation." (J. P. C. S.)

through this medium, no matter where the image may be situated in space and no matter whether it is real or virtual.

It has been shown that a narrow bundle of rays that all meet a spherical refracting surface nearly normally will be homocentric after refraction, if it were homocentric before refraction. And this remains true no matter how many times the rays are refracted in traversing a centered system of spherical refracting surfaces. If the object is composed of a number of luminous points all lying in a transversal plane perpendicular to the optical axis, the new foci after the first refraction will also lie in a parallel plane in the second medium in similar geometrical arrangement; and the same thing is true after each successive refraction from one medium to the next; so that the image formed in the last medium, that is, the resultant image in the entire system, will lie in a certain transversal plane perpendicular to the optical axis and will be geometrically similar to the object.

Since the image formed in each medium may be regarded as an object with respect to the next spherical refracting surface, the position and size of the final image may be computed without any particular difficulty, although even for a moderate number of refractions, the formulae become very complicated.

Our problem therefore is to derive, if possible, some general laws applicable to any number of refracting surfaces; which is particularly necessary in dealing with the optical system of the human eye, because the crystalline lens has a variable index of refraction and is to be regarded therefore as composed of innumerable surfaces or layers; so that it is no easy matter to trace the path of a ray of light in the eye.

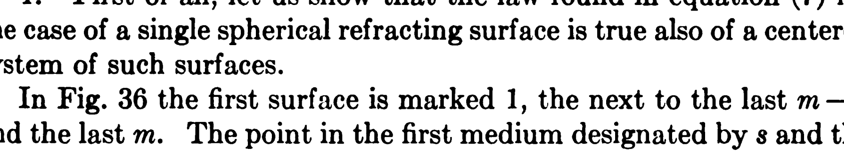

Fig. 36.

1. First of all, let us show that the law found in equation (7) for the case of a single spherical refracting surface is true also of a centered system of such surfaces.

In Fig. 36 the first surface is marked 1, the next to the last $m-1$, and the last m. The point in the first medium designated by s and the point in the last medium designated by u are a pair of conjugate points on the optical axis with respect to the entire system; and the points p and r are another pair of conjugate axial points. Putting $ps = h_1$, and $ur = h_{m+1}$, what we have to prove is that

$$\frac{H_1}{h_1}+\frac{H_2}{h_{m+1}}=1,$$

where H_1, H_2 denote the distances of the first and second focal points of the system from the points s and u, respectively. In order to show that this is a general law, it will first be proved that if it is true for a system of $(m-1)$ surfaces, it is true also for a system of m surfaces. Then, since we know it is true for one surface, it holds for two; and therefore for three; and so on indefinitely.

The images of the points s and p in the first $(m-1)$ surfaces are formed in the mth medium at the points t and q respectively; put $tq=h_m$. Let L_1, L_2 denote the distances of the two focal points of the system of $(m-1)$ surfaces from the points s, t, respectively; and let M_1, M_2 denote the distances of the two focal points of the mth or last surface from the points t, u, respectively. These distances must be measured from the points s, t and u, and considered as positive when they extend in the same direction as that in which the light traverses the axis. Now by hypothesis:

$$\frac{L_1}{h_1}+\frac{L_2}{h_m}=1,$$

and for the refraction at the last surface:

$$-\frac{M_1}{h_m}+\frac{M_2}{h_{m+1}}=1.$$

Dividing the first of these equations by L_2 and the second by M_1, and then adding them, we get:

$$\frac{L_1}{L_2}\cdot\frac{1}{h_1}+\frac{M_2}{M_1}\frac{1}{h_{m+1}}=\frac{1}{L_2}+\frac{1}{M_1}$$

or

$$\frac{M_1L_1}{M_1+L_2}\cdot\frac{1}{h_1}+\frac{M_2L_2}{M_1+L_2}\cdot\frac{1}{h_{m+1}}=1.$$

If now we put $h_1=\infty$ in which case $h_{m+1}=H_2$, this equation gives:

$$H_2=\frac{M_2L_2}{M_1+L_2},$$

and if we put $h_{m+1}=\infty$, so that $h_1=H_1$, then

$$H_1=\frac{M_1L_1}{M_1+L_2}.$$

Hence, finally

$$\frac{H_1}{h_1}+\frac{H_2}{h_{m+1}}=1 \quad \ldots \ldots \ldots \quad (8)$$

as we set out to prove.

For every real value of h_1, positive or negative, according to this equation, there is one corresponding value of h_{m+1}, and only one such value; and, conversely, to each value of h_{m+1} corresponds one value of h_1. Therefore either of the two conjugate foci may be taken anywhere on the optical axis; and whenever the position of one of these points is given, that of the other is uniquely determined.

2. In every optical system there is one pair of conjugate foci, and only one pair, for which object and image in the corresponding conjugate planes at right angles to the axis are equal in every respect. This pair of conjugate planes is the pair of *principal planes* of the optical system; and the corresponding points on the axis are the *principal points*. The focal lengths of the optical system are to each other in the same ratio as the indices of refraction of the first and last media.

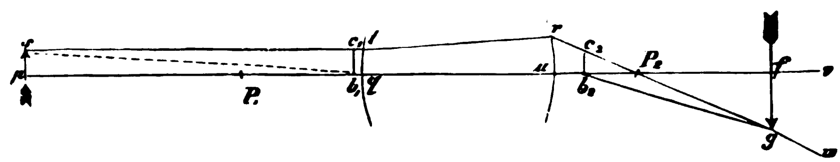

Fig. 37.

The construction of the image of any object is very simple. Suppose that p (Fig. 37) is the point where the optical axis meets the object ps, where s is any other point of the object near p. Think of the object as moving along the axis, so that the point s traces out a straight line st parallel to the axis. If the straight line st is the path of a ray of light, this ray will always go through the point s, no matter what the distance pq is. Now all rays parallel to the axis will emerge from the system so as to cross the axis in the last medium at the second focal point P_2. If the straight line ru which crosses the axis at P_2 is the path of the emergent ray corresponding to the incident ray st, the image of the point s will always remain on this line. Let the straight line fg perpendicular to the axis be the image of the object ps. As p moves along the axis, its conjugate point f must move also along the axis while the point g conjugate to s will move along the straight line ru. Evidently the size of the image fg will vary directly as the distance P_2f; as we know already from equations (6a) and (6b) is the case with a single refracting surface. Also since, according to equation (8), P_2f can have any value whatsoever, positive or negative, the same is true as to the size of the image, provided the length of the image is reckoned as negative when the image is inverted. Thus there is only one position of the object ps for which its image will be precisely equal to it in both size and direction; suppose that this position is b_1c_1 and that the image then is b_2c_2. The pair of conjugate points b_1, b_2 are

therefore the positions of the so-called *principal points* of the system.

Employing the following symbols,

$$sp = c_2b_2 = \beta_1,$$
$$fg = -\beta_2,$$
$$b_1P_1 = F_1, \qquad b_1p = f_1,$$
$$b_2P_2 = F_2, \qquad b_2f = f_2,$$

we have here

$$\frac{c_2b_2}{fg} = \frac{b_2P_2}{P_2f}$$

or

$$-\frac{\beta_1}{\beta_2} = \frac{F_2}{f_2 - F_2},$$

but by equation (8)

$$\frac{F_1}{f_1} + \frac{F_2}{f_2} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8a)$$

and hence we get

$$\frac{\beta_1}{\beta_2} = \frac{F_2}{F_2 - f_2} = \frac{F_1 - f_1}{f_1} \quad ; \quad . \quad . \quad . \quad . \quad . \quad (8b)$$

a formula that corresponds to equation (6b) for a single refracting surface. Let l_1, l_2 denote the distances of object and image from the first and second focal points, respectively, that is, put

$$l_1 = f_1 - F_1,$$
$$l_2 = f_2 - F_2.$$

Accordingly, eliminating f_1, f_2 from equations (8a) and (8b), we find the so-called image-equations for a centered system of spherical refracting surfaces, in their simplest forms, as follows:

$$l_1l_2 = F_1F_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8c)$$

$$\left.\begin{aligned} \frac{\beta_1}{\beta_2} &= -\frac{l_1}{F_1} \\ \frac{\beta_2}{\beta_1} &= -\frac{l_2}{F_2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8d)$$

These results should be compared with equations (7b) and (7c) which were obtained for a single surface.

Finally, in order to find the ratio of the focal lengths F_1 and F_2, let us apply equation (7d) to the case of the incident ray whose path is along the straight line joining s and b_1 and which emerges therefore along a straight line which goes through g and crosses the axis at b_2. Let γ' denote the size of an object in the first principal plane, and let γ'', γ''', etc., and γ_{m+1} denote the sizes of the successive images of this object formed after each refraction. By the characteristic property of the principal planes the last image of all must be of the same size as the object, that is, $\gamma_{m+1} = \gamma'$. Moreover, let α', α'', . . . α_{m+1} denote

the angles which the ray sb_1 makes with the axis in each medium in succession from the first to the last; so that, for example,

$$\angle sb_1p = -\alpha',$$
$$\angle gb_2f = -\alpha_{m+1}.$$

Now according to equation (7d)

$$n'\gamma'\tan\alpha' = n''\gamma''\tan\alpha'',$$
$$n''\gamma''\tan\alpha'' = n'''\gamma'''\tan\alpha''', \text{ etc.,}$$

and, consequently,

$$n'\gamma'\tan\alpha' = n_{m+1}\gamma_{m+1}\tan\alpha_{m+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

or since $\gamma' = \gamma_{m+1}$,

$$n'\tan\alpha' = n_{m+1}\ \tan\alpha_{m+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9a)$$

Moreover, according to the notation above,

$$sp = \beta_1 \quad = -f_1\tan\alpha',$$
$$fg = -\beta_2 = -f_2\tan\alpha_{m+1},$$

and consequently

$$\frac{n'\beta_1}{f_1} = -\frac{n_{m+1}\beta_2}{f_2}.$$

Substituting here the value of f_2 as obtained from equation (8a), we get

$$\frac{n'\beta_1}{f_1 - F_1} = -\frac{n_{m+1}\beta_2}{F_2}$$

and according to equation (8b),

$$\frac{\beta_1}{f_1 - F_1} = -\frac{\beta_2}{F_1}.$$

From these two equations we derive immediately the fundamental relation between the focal lengths of an optical system, namely:

$$\frac{n'}{n_{m+1}} = \frac{F_1}{F_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9c)$$

3. In every optical system there is one, and only one, pair of *nodal points*. If a ray crosses the axis in the first medium at the first nodal point, the corresponding ray in the last medium will cross the axis at the second nodal point along a line which is parallel to the original direction of the ray. The transversal conjugate planes through these points perpendicular to the axis are called the *nodal planes*. The second nodal point is the image of the first, since rays that meet at the first point must meet again at the second point. The distances between the nodal points and focal points of an optical system are inversely proportional to the indices of refraction of the first and last media.

If equation (9) is applied to the nodal points we must put $a_{\prime} = a_{m+1}$, and thus we obtain:

$$n_{\prime}\gamma_{\prime} = n_{m+1}\gamma_{m+1}.$$

Hence, we see that the linear dimensions of object and image lying in the nodal planes are likewise inversely proportional to the indices of refraction of the first and last media.

Since the size of the image is proportional to its distance from the second focal plane, this distance can be found, provided the size of the image is given. If the image lies in the second principal plane, it has the same size as the object and its distance from the second focal point is F_2. On the other hand, if the image is formed in the second nodal plane, its size, as has just been shown, will be

$$\gamma_{m+1} = \frac{n_{\prime}}{n_{m+1}}\gamma_{\prime}.$$

If G_2 denotes its distance in this case from the second focal point, then

$$\frac{\gamma_{\prime}}{\gamma_{m+1}} = \frac{F_2}{G_2},$$

and therefore by equation (9c),

$$G_2 = \frac{n_{\prime}}{n_{m+1}}F_2 = F_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10a)$$

Accordingly, the distance between the second principal plane and the second nodal plane is

$$\begin{aligned} a_2 &= F_2 - G_2 \\ &= F_2 - F_1. \end{aligned}$$

The nodal planes are a pair of conjugate planes; and if a_1 denotes the distance of the first nodal plane from the first principal plane, that is, if

$$a_1 = G_1 - F_1,$$

then by equation (8a), we have:

$$-\frac{F_1}{a_1} + \frac{F_2}{a_2} = 1,$$

hence

$$\begin{aligned} a_1 &= a_2 = F_2 - F_1 \\ G_1 &= F_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10b) \end{aligned}$$

and

$$\frac{G_1}{G_2} = \frac{n_{m+1}}{n_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10c)$$

The Cardinal Points of a Compound Optical System, Composed of Two Systems with Axes in Same Straight Line

The two component systems are designated in Fig. 38 by A and B. The focal points and principal points of system A are designated by $p_\prime$, $p_{\prime\prime}$ and $a_\prime$, $a_{\prime\prime}$, respectively; and the focal points and principal points of system B are designated by $\pi_\prime$, $\pi_{\prime\prime}$ and $\alpha_\prime$, $\alpha_{\prime\prime}$, respectively. Let d denote the distance of the first principal point of system A from the second principal point of system B; this interval being taken as positive when, as in Fig. 38, the point $\alpha_\prime$ lies beyond the point $a_{\prime\prime}$. The focal lengths of the first system are $a_\prime p_\prime = f_\prime$ and $a_{\prime\prime}p_{\prime\prime} = f_{\prime\prime}$, and

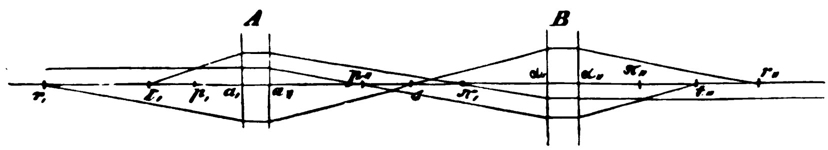

Fig. 38.

those of the second system are $\alpha_\prime\pi_\prime = \varphi_\prime$ and $\alpha_{\prime\prime}\pi_{\prime\prime} = \varphi_{\prime\prime}$. Evidently, the first focal point ($\pi_\prime$) of system B is the image of the *first focal point* ($t_\prime$) of the compound system as formed by system A. Hence, a ray which in the first medium crosses the axis at $t_\prime$ will, after traversing system A, cross the axis at $\pi_\prime$ and emerge finally in the last medium along a straight line parallel to the axis; because by definition this must be the case with a ray that crosses the axis originally at the first focal point of the compound system. Since $a_{\prime\prime}\pi_{\prime\prime} = d - \varphi_\prime$, we have:

$$a_\prime t_\prime = \frac{(d-\varphi_\prime)f_\prime}{d-\varphi_\prime - f_{\prime\prime}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11a)$$

Similarly, the *second focal point* ($t_{\prime\prime}$) of the compound system is the image of the second focal point ($p_{\prime\prime}$) of system A in system B. Hence

$$\alpha_{\prime\prime}t_{\prime\prime} = \frac{(d-f_{\prime\prime})\varphi_{\prime\prime}}{d-\varphi_\prime - f_{\prime\prime}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11b)$$

The *principal points* of the compound system are a pair of conjugate points with respect to it. Accordingly, there must be a point on the axis which is the image of the first principal point in system A, and at the same time whose image in system B is at the second principal point of the compound system. Suppose this point is at s in Fig. 38 and that $r_\prime$, $r_{\prime\prime}$ are the principal points of the compound system. If s is the image of $r_\prime$ in system A, and if $r_{\prime\prime}$ is the image of s in system B, then $r_{\prime\prime}$ is the image of $r_\prime$ in the compound system. This is one of the conditions that these points must satisfy; the other condition being that object and image in the principal planes must be equal in every way. Suppose, therefore, that an object of height $\beta_\prime$ is set up at the

point r', and that the image of this object in system A is formed at s and has the height σ; and that the image of σ in system B is formed at r'' and has the height β''. Putting $x = a''s$, and $y = s a'$, we get from equation (8b):

$$\frac{\beta'}{\sigma} = \frac{f''}{f'' - x}$$

$$\frac{\beta''}{\sigma} = \frac{\varphi'}{\varphi' - y}.$$

Now if we are to have here $\beta' = \beta''$, then

$$\frac{f''}{f'' - x} = \frac{\varphi'}{\varphi' - y}$$

or

$$\frac{x}{f''} = \frac{y}{\varphi'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11c)$$

or

$$\frac{a''s}{a''p''} = \frac{a's}{a'\pi'}.$$

Thus, we have the following rule for finding the point s in the medium common to the two component systems A and B, which is conjugate with respect to both systems to the principal points (r', r'') of the compound system $(A+B)$: Divide the interval between the second principal point (a'') of system A and the first principal point (a') of system B in two parts which are in the same ratio as the second focal length of system A is to the first focal length of system B.

Since $x + y = d$, then according to equation (11c),

$$\frac{x}{f''} = \frac{d - x}{\varphi'}$$

$$\frac{d - y}{f''} = \frac{y}{\varphi'}.$$

Hence,

$$x = \frac{df''}{\varphi' + f''}$$

$$y = \frac{d\varphi'}{\varphi' + f''}.$$

By using the above value of x, the distance $a'r' = h'$ of the first principal point of the compound system from the first principal point of system A may be found, as follows:

$$h' = \frac{xf'}{x - f''}$$

$$h' = \frac{df'}{d - \varphi' - f''} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11d)$$

Similarly, the distance $\mathrm{a}_{\prime\prime}r_{\prime\prime}=h_{\prime\prime}$ of the second principal point of the compound system from the second principal point of system B is given by

$$h_{\prime\prime}=\frac{\varphi_{\prime\prime}y}{y-\varphi_{\prime}}$$

$$h_{\prime\prime}=\frac{d\varphi_{\prime\prime}}{d-\varphi_{\prime}-f_{\prime\prime}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(11e)}$$

Thence, we obtain the following expressions for the focal lengths ($F_{\prime}$, $F_{\prime\prime}$) of the compound system:

$$\left.\begin{aligned} F_{\prime} &= a_{\prime}t_{\prime}-a_{\prime}r_{\prime} \\ &= \frac{\varphi_{\prime}f_{\prime}}{\varphi_{\prime}+f_{\prime\prime}-d} \\ F_{\prime\prime} &= \mathrm{a}_{\prime\prime}t_{\prime\prime}-\mathrm{a}_{\prime\prime}r_{\prime\prime} \\ &= \frac{\varphi_{\prime\prime}f_{\prime\prime}}{\varphi_{\prime}+f_{\prime\prime}-d} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(11f)}$$

Having found the positions of the principal points and focal points, we can easily locate the nodal points; because the distance of the first nodal point from the first focal point is equal to the second focal length, and the distance of the second nodal point from the second focal point is equal to the first focal length.

But if it is required to find the positions of the nodal points without having ascertained those of the principal points, a process similar to the above can be employed, by utilizing the fact that the lengths of conjugate lines in the nodal planes are inversely proportional to the indices of refraction of the respective media.

For example, in Fig. 38 suppose that the points designated by $a_{\prime}$, $a_{\prime\prime}$ and $\mathrm{a}_{\prime}$, $\mathrm{a}_{\prime\prime}$ are the nodal points of systems A and B, respectively; and that the points designated by $r_{\prime}$, $r_{\prime\prime}$ are the nodal points of the compound system; so that now

$$a_{\prime}p_{\prime}=f_{\prime\prime} \qquad \mathrm{a}_{\prime}\pi_{\prime}=\varphi_{\prime\prime}$$
$$a_{\prime\prime}p_{\prime\prime}=f_{\prime} \qquad \mathrm{a}_{\prime\prime}\pi_{\prime\prime}=\varphi_{\prime}$$
$$a_{\prime\prime}s=x \qquad \mathrm{a}_{\prime}s=y.$$

Then

$$a_{\prime}r_{\prime}=\frac{xf_{\prime\prime}}{x-f_{\prime}}$$

$$\mathrm{a}_{\prime\prime}r_{\prime\prime}=\frac{y\varphi_{\prime}}{y-\varphi_{\prime\prime}}.$$

Let $\beta_{\prime}$ denote the height of an object in the first medium placed at $r_{\prime}$, and let σ denote the height of its image in system A, which will be formed at s; and, finally, let $\beta_{\prime\prime}$ denote the height of the image of σ in system B, which will be formed at $r_{\prime\prime}$; then according to the property of the nodal points,

$$\frac{\beta_{\prime}}{\sigma}=\frac{a_{\prime}r_{\prime}}{x}=\frac{f_{\prime\prime}}{x-f_{\prime}}$$

$$\frac{\beta_{\prime\prime}}{\sigma}=\frac{a_{\prime\prime}r_{\prime\prime}}{y}=\frac{\varphi_{\prime}}{y-\varphi_{\prime\prime}}.$$

If $n_{\prime}$, $n_{\prime\prime}$ denote the indices of refraction of the first and last media, and if ν denotes the index of refraction of the intervening medium between the two components of the compound system, we must have for the nodal planes:

$$n_{\prime}\beta_{\prime}=n_{\prime\prime}\beta_{\prime\prime},$$

and hence

$$\frac{n_{\prime}f_{\prime\prime}}{x-f_{\prime}}=\frac{n_{\prime\prime}\varphi_{\prime}}{y-\varphi_{\prime\prime}}.$$

But

$$n_{\prime}f_{\prime\prime}=\nu f_{\prime},$$

therefore

$$n_{\prime\prime}\varphi_{\prime}=\nu\varphi_{\prime\prime},$$

and

$$\frac{f_{\prime}}{x-f_{\prime}}=\frac{\varphi_{\prime\prime}}{y-\varphi_{\prime\prime}}$$

$$\frac{x}{f_{\prime}}=\frac{y}{\varphi_{\prime\prime}}$$

or

$$\frac{a_{\prime\prime}s}{a_{\prime\prime}p_{\prime\prime}}=\frac{a_{\prime}s}{a_{\prime}\pi_{\prime}}.$$

This same equation was obtained above as equation (11c) on the assumption that the points designated by $a_{\prime}$, $a_{\prime\prime}$, $\alpha_{\prime}$, $\alpha_{\prime\prime}$, $r_{\prime}$ and $r_{\prime\prime}$ were the principal points. Accordingly, the positions of the nodal points of the compound system, with respect to those of the component systems, are found by exactly a similar process as was used for finding the principal points.

The simplest case is when the compound system is a combination of two spherical refracting surfaces, and it will be worth while to give the special formulae for this system. If the radii of the first and second surfaces are denoted by r_1 and r_2, respectively, and if d denotes the distance of the vertex of the second surface from that of the first; and, finally, if n_1, n_2, n_3 denote the indices of refraction of the three media in their actual order, then by equations (3a) and (3b):

$$f_1=\frac{n_1r_1}{n_2-n_1} \qquad \varphi_1=\frac{n_2r_2}{n_3-n_2}$$

$$f_2=\frac{n_2r_1}{n_2-n_1} \qquad \varphi_2=\frac{n_3r_2}{n_3-n_2}.$$

By way of abbreviation, put

$$n_2(n_3-n_2)r_1+n_2(n_2-n_1)r_2-(n_3-n_2)(n_2-n_1)d=N.$$

Then the focal lengths are

$$\left.\begin{aligned} F_1 &= \frac{n_1 n_2 r_1 r_2}{N} \\ F_2 &= \frac{n_2 n_3 r_1 r_2}{N} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (12)$$

The distances of the principal points from the two surfaces are

$$\left.\begin{aligned} h_1 &= \frac{n_1(n_2-n_3)dr_1}{N} \\ h_2 &= \frac{n_3(n_1-n_2)dr_2}{N} \end{aligned}\right\} \quad \ldots\ldots \quad (12a)$$

and the distance of these points from each other is

$$H = d \cdot \frac{(n_2-n_1)(n_3-n_2)(r_1-r_2-d)}{N} \quad \ldots\ldots \quad (12b)$$

For $d=0$, we have $h_1 = h_2 = H = 0$, and

$$F_1 = \frac{n_1 r_1 r_2}{(n_3-n_2)r_1 + (n_2-n_1)r_2}$$

$$F_2 = \frac{n_3 r_1 r_2}{(n_3-n_2)r_1 + (n_2-n_1)r_2}.$$

If in these equations we put $r_2 = r_1$, we obtain:

$$F_1 = \frac{n_1 r_1}{n_3 - n_1}$$

$$F_2 = \frac{n_3 r_1}{n_3 - n_1}.$$

In this case the focal points and principal points are exactly the same as if there were only a single refracting surface; the result is independent of n_2. Therefore:

In considering any system of refracting spherical surfaces, we may think of each surface as replaced by an infinitely thin layer of arbitrary index of refraction, bounded by concentric spherical surfaces, without in any way altering the optical properties of the system.

This proposition will serve later to simplify many of our discussions.

Lastly, the formulae will be given for lenses with spherical surfaces surrounded by the same medium on both sides ($n_3 = n_1$):

$$F_1 = F_2 = \frac{n_1 n_2 r_1 r_2}{(n_2-n_1)[n_2(r_2-r_1)+(n_2-n_1)d]} \quad \ldots \quad (13)$$

In this case the nodal points are the same as the principal points, and their distances from the two vertices are:

$$\left.\begin{aligned} h_1 &= \frac{n_1 d r_1}{n_2(r_2-r_1)+(n_2-n_1)d} \\ h_2 &= -\frac{n_1 d r_2}{n_2(r_2-r_1)+(n_2-n_1)d} \end{aligned}\right\} \quad \ldots\ldots \quad (13a)$$

these distances being reckoned positive or negative according as the principal points are outside the lens or inside it, respectively. The interval between the principal points is

$$H = d \cdot \frac{(n_2 - n_1)(d + r_2 - r_1)}{n_2(r_2 - r_1) + (n_2 - n_1)d} \quad . \quad . \quad . \quad . \quad . \quad (13b)$$

The optical centre of the lens is the point on the axis which is conjugate to the first nodal point with respect to the first surface, and to which therefore the second nodal point is conjugate with respect to the second surface; its distances from the two surfaces are proportional to the radii.

Evidently, so far as the position and size of the image is concerned, two optical systems are equivalent in their effects, provided their focal points and principal points have the same positions; and provided also the first and last media of the two systems are the same; because the ratio of the indices of refraction of these media cannot be changed without altering the ratio of the focal lengths. Thus, one optical system can be substituted for another one if they both have the same focal length and the same interval between the pair of principal points, the first and last media being the same for both systems. In a system composed of two refracting surfaces, the two factors above mentioned involve only the four magnitudes denoted by r_1, r_2, n_2 and d and therefore a system with only two refracting spherical surfaces may always be substituted for a centered system of any number of surfaces; in the sense that the images in the two cases will be exactly the same as to both size and position. As a matter of fact, we may even impose two other conditions on the simple system, as, for example, that the interior medium shall have a certain index of refraction, etc.

We shall now consider the several kinds of lenses;[1] that is, systems composed of two refracting surfaces separated by a distance less than either radius of curvature, surrounded on the outside by a single medium, which has a lower index of refraction than that of the interior medium. We shall often have occasion to speak of systems of this kind, which may be classified by their form, as follows:

1. *Double Convex Lens*, in which both surfaces are convex, that is, r_1 is positive and r_2 negative, and hence, according to formula (13), the focal lengths are positive. As the distances of the principal points from the surfaces are both negative, these points lie inside the lens; but since

[1] ¶The author seems to have specially in mind here ordinary glass lenses surrounded by air. It is perfectly possible, however, to think of an air lens surrounded by glass. Nor is it necessary that the thickness of the lens should be limited as stated in the text. Moreover, a perfectly general definition must include astigmatic lenses (cylindrical lenses, toric lenses, etc.) and other aspherical lenses. (J. P. C. S.)

the interval between them is positive, the first principal point is in front of the second. The positions of the focal points (p_1, p_2) and the principal points (h_1, h_2) of a double convex lens are indicated in Fig. 39, where the first and second numerals designate the first and second surfaces. A *plano-convex* lens is a special form of double convex lens in which one of the radii of curvature is infinite, and consequently one of the principal points lies at the vertex of the curved surface.

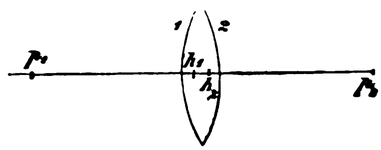

Fig. 39.

2. *Double Concave Lens*, in which both surfaces are concave, that is, r_1 is negative and r_2 positive, and hence the focal lengths are negative. Here also the distances of the principal points from the surfaces are both negative, that is, these points lie inside the lens; and since they are separated by a positive interval, the first principal point lies in front of the second, as shown in Fig. 40, which is a diagram of a double concave lens, the same letters being used here to designate the focal points and principal points as in Fig. 39. A *plano-concave* lens is a special form of double concave lens with the radius of one surface infinite, so that, just as in a plano-convex lens, one of the principal points coincides with the vertex of the curved surface.

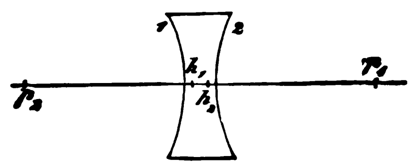

Fig. 40.

3. *Meniscus Lens*, in which both radii are positive or both negative, It is sufficient to consider simply the case when both radii are positive, as all we have to do then is to turn the lens around in order to have the case when both radii are negative. The focal lengths are positive, provided

$$n_2(r_2+d-r_1)>n_1d;$$

but they become infinite when these two expressions are equal, and negative when the left-hand member is less than the right-hand member. The expression within the parentheses is the distance of the centre of curvature of the second surface from that of the first. If the second centre lies beyond the first, the lens will be thicker in the middle than at the edge; but in the opposite case, it will be thinner in the middle. Thus, the focal lengths of a meniscus lens are positive or negative according as the lens is, or is not, thicker in the middle than

it is at the edge. The first principal point of a meniscus lens lies before the convex surface (that is, on its convex side), when the focal length is positive; and may be very far away if the focal length is very great, and infinitely far when the focal length is infinite. When the focal length is negative, the first principal point is situated beyond the convex surface (that is, on its concave side), and likewise recedes to infinity when the focal length becomes infinite. The second principal point of a meniscus lens lies in front of the concave surface of the lens (that is, on its convex side), when the focal length is positive, and beyond this surface when the focal length is negative, being at infinity when the focal length is infinite. For a positive focal length, the second principal point always lies beyond the first, that is, nearer the lens. If the focal length is negative, the second principal point lies beyond the first, that is, farther from the lens provided the lens is thinner at the middle than towards the edge; but it lies in front of the first principal point when the lens is thicker in the middle and yet has a negative focal length. If the two surfaces of the lens are concentric, the two principal points coincide with each other at the common centre of curvature. A meniscus lens with positive focal length is shown in Fig. 41. Both meniscus lenses in the next two diagrams have negative focal lengths, but the one in Fig. 42 is thinner in the middle than out towards the edge, whereas it is just the other way in Fig. 43. In all three lenses the centres of the first and second surfaces are designated by c_1 and c_2, respectively. The focal points are never inside the lens, and are always on opposite sides of it.[1]

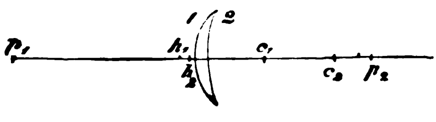

Fig. 41.

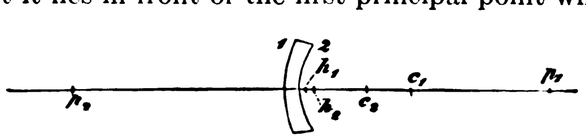

Fig. 42.

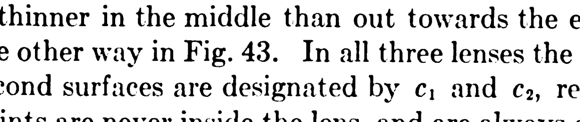

Fig. 43.

In case the focal lengths of the system are equal, the image-equations (8a) and (8b) become:

$$\frac{1}{f_1}+\frac{1}{f_2}=\frac{1}{F} \qquad (14)$$

or

$$f_2=\frac{Ff_1}{f_1-F} \qquad (14a)$$

[1] This remark applies only to the type of lenses last mentioned. G.

and

$$\frac{\beta_2}{\beta_1}=\frac{F}{F-f_1}=\frac{F-f_2}{F} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14b)$$

In the case of lenses with *positive focal length* (*convergent* or *positive* lenses, as they are called), the image of an infinitely distant object ($f_1=\infty$) lies beyond the lens in its second focal plane, is real and inverted, and is infinitely small as compared with the object itself. As the object is brought nearer the lens, the image, which is still real and inverted, recedes from the lens and gets bigger, until finally when the object arrives in the first or anterior focal plane ($f_1=F$), both the distance and size of the image are infinite. This is easily seen from equation (14), which may be written

$$\frac{1}{f_2}=\frac{1}{F}-\frac{1}{f_1},$$

by giving f_1 any values from ∞ to F, in which case the corresponding values of f_2 will be found to increase from F to ∞. The image is inverted, that is, the sign of

$$\beta_2=-\beta_1\frac{F}{f_1-F}$$

continues negative so long as f_1 is greater than F; and as f_1 diminishes from ∞ to F, β_2 assumes all negative values from 0 to $-\infty$.

Similarly, it may be shown that as the object proceeds from the first focal plane to the first principal plane, f_2 assumes all positive values from ∞ to 0; so that the image now is virtual and erect and lies in front of the lens on the same side as the object, proceeding from infinity until it arrives in the second principal plane, where it has the same size as the object.

Finally, for negative values of f_1, the object itself is virtual, and in this case f_2 will always be positive and numerically less than f_1, the image being real and erect and smaller than the virtual object.

Thus a convergent lens converges parallel incident rays to a real focus on the other side of the lens. Convergent incident rays are made still more convergent, and divergent incident rays less divergent or even convergent, according as they proceed from a point on the axis nearer to or farther from the convergent lens than the first focal point.

Lenses with negative focal lengths are called *divergent* or *negative* lenses, because they make parallel incident rays divergent, divergent rays still more divergent, and convergent rays less convergent or even divergent. If the absolute value of the focal length is denoted by P, that is, if $P=-F$, then for a divergent lens

$$\frac{1}{f_2}=-\frac{1}{P}-\frac{1}{f_1}$$

$$\beta_2=\beta_1\frac{P}{f_1+P}.$$

Thus the values of f_2 corresponding to all positive values of f_1 will be found to be negative, so that as f_1 decreases from ∞ to 0, f_2 will vary from $-P$ to 0, and β_2 from 0 to β_1. A divergent lens therefore gives a virtual image of a real object anywhere in front of the first principal plane, the corresponding image being erect, smaller than the object, and nearer the lens and in front of the second principal plane. The values of f_2 corresponding to negative values of f_1 that are numerically less than P are positive, and as f_1 assumes all values between 0 and $-P$, f_2 has all positive values, and β_2 changes from β_1 to ∞. Thus convergent incident rays are made less convergent provided they are directed originally to a point on the axis nearer the lens than the focal point. For negative values of f_1 numerically greater than P, f_2 and β_2 are both negative, and in such cases the image in a divergent lens is virtual and inverted. As f_1 varies from $-P$ to $-\infty$, f_2 varies oppositely from $-\infty$ to $-P$, and β_2 from $-\infty$ to 0. Hence rays originally converging towards a point *beyond* the further focal point are made divergent by a divergent lens.

The distance e between a pair of conjugate points on the axis is equal to f_1+a+f_2, where a denotes the interval between the principal points; e being counted as positive when the image point lies beyond the object-point. Substituting for f_2 its value, we obtain as the expression for this distance

$$e=\frac{f_1^2}{f_1-F}+a.$$

Differentiating this equation with respect to f_1, we have:

$$\frac{de}{df_1}=\frac{f_1^2-2f_1F}{(f_1-F)^2}.$$

Hence, $de=0$, that is, this interval e is a maximum or minimum, provided either $f_1=0$ or $f_1=2F$. As a matter of fact, whether the focal length of the lens is positive or negative, e is a minimum for $f_1=2F$ and a maximum for $f_1=0$, as is easily seen from the expression for e.

The following is a partial list of works which contain a treatment of the theory of refraction by a centered system of spherical surfaces:[1]

[1] ¶This list is by no means complete. For example, H. CODDINGTON'S *A Treatise on the Reflection and Refraction of Light* (London 1829) should certainly be included, as well as Sir W. R. HAMILTON'S papers on "Theory of systems of rays" in *Trans. Roy. Irish Acad.*, xv (1828), 69-174; xvi (1830), 3-62, 93-126, and xvii (1837), 1-44. (J. P. C. S.)

1738. COTES in SMITH, *A compleat system of opticks.* Cambridge. Vol. II. p. 76.
1757, 1761. EULER in *Histoire de l'Acad. roy. de Berlin pour* 1757. p. 283.—Ibid., *pour* 1761. p. 201.
1765. EULER, *Précis d'une théorie générale de la dioptrique* in *Hist. de l'acad. roy. des sc. de Paris,* 1765. p. 555.
1778, 1803. LAGRANGE in *Nouv. Mém. de l'acad. roy. de Berlin pour* 1778. p. 162.—Ibid. 1803. p. 1.
1882. PIOLA in *Effemeridi astron. di Milano per* 1822.
1830. MÖBIUS in CRELLES *Journal für Mathematik.* Bd. V. S. 113.
1841. *BESSEL in *Astronom. Nachrichten.* Bd. XVIII. S. 97.
*GAUSS, *Dioptrische Untersuchungen.* Göttingen.—Reprint from *Abhandl. d. Kön. Ges. d. Wiss. zu Göttingen.* T. 1. for years 1838–43.
1844. ENCKE, *De formulis dioptricis. Ein Programm.* Berlin.
MOSER, Ueber das Auge, in DOVE's *Repert. d. Physik.* Bd. V. S. 289.
1851. LISTING, Art. Dioptrik des Auges, in R. WAGNERS *Handwörterbuch d. Physiologie.* Bd. IV. S. 451.

§10. Optical System of the Eye[1]

In its optical behaviour the eye is essentially like a *camera obscura.*[2] In order for a luminous point to be seen distinctly, the light diverging from it must be refracted by the media of the eye and thereby converged at some point of the retina. On the surface of this membrane a real optical image is projected of the external objects in view, which is inverted and very much reduced in size. By carefully removing a central portion of the sclerotica and choroid coating at the back of a freshly enucleated eye, thus exposing the retina from behind, and then pointing the cornea towards a bright object, a small, inverted image, as above described, can actually be seen on the retina, sharply defined.[3] A still better plan is that of GERLING,[4] in which a small portion of the retina is removed with a needle and a little plate of glass or mica inserted in the opening. It is quite easy to observe the retinal image in the eye of a white rabbit, the choroid of which is devoid of pigment. In this case it is not even necessary to remove the hard coating, as the image can be seen through it, not as distinctly, of course, as when the retina itself is directly exposed, but clearly enough to recognize and locate it. It is even possible to see the image sometimes through the sclerotica of the eye of a live individual, especially if he is blond and has bright blue eyes which usually contain scant pigment in the choroid coating. The patient is taken into a dark room and made to turn his eye towards the temporal side, so as to expose the white eye-

[1] See Appendix II on "Refraction of Rays in the Eye" at end of Part I. G.

[2] ¶This comparison was made by J. B. PORTA (1545–1615), the inventor of the "pinhole camera" and *camera obscura.* (J. P. C. S.)

[3] ¶This experiment was performed by C. SCHEINER (1573–1650), first, with eyes of sheep and oxen and, in 1625, with a human eye. KEPLER had already inferred that the image must be formed on the retina, although PORTA seems to have supposed that it was focused in or on the crystalline lens. (J. P. C. S.)

[4] POGGENDORFF, *Ann.* XLVI. 243.

ball well around towards its rear surface. When a candle is held a little to one side of the averted eye, its image will be visible on the inner side of the retina, and frequently it sparkles so clearly through the translucent coatings of the eye that its inverted position, the tip of the flame, and the wick at its base, can be recognized.[1]

The best method of investigating the retinal image in the living eye is with an ophthalmoscope, which will be described in §16. With the aid of this instrument, it is possible to look directly in the eye from the front and see clearly not only the retina itself and its blood vessels but the optical images that are projected on it. That this is actually the case is proved by the fact that if the eye under examination is focused on an object that is bright enough, a distinct and sharply defined image of it may be seen by the observer on the surface of the retina.

As has already been mentioned in describing this membrane, there is in the fundus of the eye a small area of the retina of peculiar structure known as the yellow spot. At its centre is a small depression, the *fovea centralis*, where the blood vessels which ramify all over the rest of the retina are entirely lacking. Here there is nothing but nerve tissue; and in fact the retinal membrane itself at this particular place appears to be composed entirely of nerve cells and cones. Physiologically, as the place of direct vision, it is of the highest importance. Whenever we look directly at anything and fix it in the eye, its image falls on the fovea.[2] This fact, which had long been inferred from the special structure of the yellow spot, can be verified directly with the ophthalmoscope. When the entire retina is illuminated, the location of the yellow spot is easily recognized by the absence of blood vessels. In the middle of this veinless area, where the fovea is, there is a peculiar bright place, described first by Coccius,[3] which he explains as a reflex from the foveal cavity. Moreover, Donders[4] has shown that this bright reflex appears always at that part of the retinal image that corresponds to the point of fixation in the field of view, and the writer has verified this observation. From the position of the so-called foveal reflex it is possible to tell exactly the point of fixation of the eye in the field of view, and by directing the patient to look first at one place and then at another, the observer can watch the reflex being focused at the corresponding place in the retinal image. The method of procedure will be described in §16.

[1] Volkmann, Article: Sehen in Wagners *Handwörterbuch d. Physiologie.* S. 286-289.

[2] ¶This is called direct vision or foveal vision. (J. P. C. S.)

[3] *Ueber die Anwendung des Augenspiegels.* Leipzig 1853. S. 64.

[4] *Onderzoekingen gedaan in het Physiolog. Laborat. d. Utrechtsche Hoogeschool.* Jaar VI. S. 133.

Usually the optical image on the retina is not perfectly sharp except in the vicinity of the axis of the eye, and farther off it is not so well defined. Ordinarily, therefore, the point of fixation is the only point in the field of view that is seen distinctly at any one time, everything else being more or less vague. The vagueness of indirect vision is apparently due to diminished sensitivity of the retina outside the foveal region, because even a short distance out when the image is still sharply outlined the impression is not very distinct. The eye is an optical contrivance of remarkably wide field of view, but it is only within a very limited part of this field that the images are clear-cut. The entire field is like a drawing which is carefully executed to delineate the most important central part of the picture, while the surroundings are simply sketched in, more and more lightly out towards the borders.

However, in virtue of the mobility of the eye, it can be quickly focused on the various parts of the field in succession. A human being cannot attend to more than one object at a time, and the one point that he can see distinctly is enough to occupy him fully at that moment. Still, the attention is often distracted by the details, and then, in spite of the vagueness of the broad field of view, the eye is capable of taking in at a rapid glance the main features of the whole surroundings, and of noting immediately the sudden appearances of new objects in the remoter parts of the field.

The field of view of a single eye is determined by the diameter of the pupil and its position with respect to the edge of the cornea. By observing his own eye in a mirror in a dark room, with a light placed to one side, the author found that he could perceive the presence of the light so long as rays from it fall on the opposite edge of the pupil and enter the pupil itself. Therefore any light which passes through the cornea and enters the pupil must fall on some sensitive part of the retina. It is true that the pupil lies somewhat farther back than the outer edge of the cornea, but owing to the refraction by the cornea, rays can enter the pupil through the edge of the cornea even when they were originally perpendicular to the optical axis of the eye. Thus the field of view of the eye is approximately a hemisphere, which is wider than that of any artificial optical instrument. There are, of course, individual variations in different eyes, depending on the diameter and position of the pupil. In near vision the pupil is displaced a little towards the cornea, thereby increasing the field somewhat; as the author can readily see in his own eye by holding a bright light at the outer edge of the field.

Part of the field of view of each eye separately is intercepted above and below and on the inside by cheeks, eyebrows, and nose; on the outside, however, the field is entirely unobstructed. But both eyes

together, directed straight ahead into the distance, command an horizontal arc of 180° or more. The extent of the field is greatly increased beyond this by the rotatory movements of the eyes, which will be discussed later.

Rays from a distant point entering the eye are first refracted by the cornea so that, without being again intercepted, they would come to a focus about 10 mm beyond the retina. Thus converging, they traverse the anterior chamber of the eye and arrive at the crystalline lens, where they are made still more convergent and may, therefore, be brought to a focus on the retina itself.

The main refraction occurs at the cornea. Next in importance are the refractions at the anterior and posterior surfaces of the crystalline lens. However, there are refractions also in the interior of the lens at the boundaries of its separate layers, since these latter have different densities. All these refracting surfaces may be regarded approximately as surfaces of revolution around a common axis. Although in most human eyes the axis of the various surfaces are apparently not strictly coincident, the variations are so slight as to be negligible in effect so far as the position and size of the image are concerned; so that the eye may be regarded as a *centered* optical system.

The axis of this system, or the *axis of the eye*, as we call it, coincides approximately with the straight line joining the vertex of the cornea with a point on the retina between the yellow spot and the place where the optic nerve enters the eyeball.

The positions of the *focal points, principal points* and *nodal points* of the eye are subject to rather considerable individual variations. Measurements of different eyes and of each of their refracting surfaces are often found to differ to a rather surprising extent, when we consider the refinement of construction and adjustment required in such an organ. Moreover, it will be seen later on that even in the same eye, the cardinal points change their positions during the act of accommodation. About all that can be said definitely as to the positions of the cardinal points in the normal eye, adjusted for distant vision, is that the *two principal points are very close together, as are also the two nodal points*. Both principal points lie near together about midway in the anterior chamber of the eye, while the nodal points are very near the posterior surface of the crystalline lens, and the second focal point is on the retina.

Since it is necessary for many purposes to have at least approximate values of the optical constants of the eye, the values of LISTING's schematic eye are tabulated herewith. They were obtained by compiling the measurements available at the time as well as possible and expressing the results in simple integers.

The data assumed by LISTING are as follows:

1.	Index of refraction of air	1
2.	Index of refraction of aqueous humor	103/77
3.	Index of refraction of crystalline lens	16/11
4.	Index of refraction of vitreous humor	103/77
5.	Radius of curvature of cornea	8 mm
6.	Radius of curvature of anterior surface of lens	10 "
7.	Radius of curvature of posterior surface of lens	6 "
8.	Distance between anterior surfaces of cornea and lens	4 "
9.	Thickness of lens	4 "

From these data he calculated the following:

1. The first focal point is located 12.8326 mm in front of the cornea, and the second focal point 14.6470 mm beyond the posterior surface of the lens.

2. The first principal point is 2.1746 mm, and the second principal point 2.5724 mm, beyond the anterior surface of the cornea, so that their distance apart is 0.3978 mm.

3. The first nodal point is 0.7580 mm, and the second nodal point 0.3602 mm, in front of the posterior surface of the lens.

4. Accordingly, the first focal length of the eye is 15.0072 mm, and the second focal length is 20.0746 mm.

The positions of the principal points h', h'', the nodal points k', k'', and the focal points f', f'', as found by LISTING, are shown in Fig. 44. Among the data employed in these calculations, the only ones about which there might be some question are the index and radii of the lens. However, the focal length of the lens as calculated from these data agrees so well with direct measurements of this magnitude as made by the writer, that the optical effect of the lens in LISTING's schematic eye cannot, at any rate, differ essentially from that of the natural eye. The values that are important for the refraction of the cornea are based on sufficiently reliable measurements. There is no reason to doubt, therefore, that LISTING's model agrees about as well with the actual facts as can be expected with the wide range of variation that exists in individual eyes.

With the cardinal points of the eye as given above, the construction described in §9 can be used to trace the path of a given incident ray after its last refraction, and likewise the position of the image of a luminous point anywhere near the axis of the eye. Since, as just seen, the two nodal points of the eye as well as the two principal points are very close together, each pair may be regarded as a pair of coincident points, without seriously impairing the accuracy of the results. Thus is obtained a still simpler optical model, which LISTING calls the

reduced eye.[1] The double principal point of this eye is 2.3448 mm beyond the anterior surface of the cornea, and the nodal point (κ in Fig. 44) is 0.4764 mm in front of the posterior surface of the lens, the focal points being unchanged. The optical behaviour of this reduced eye is equivalent to that of a single spherical refracting surface whose centre and vertex are at the nodal point and principal point, respectively, the first medium being air and the second medium the aqueous or vitreous humor. The radius of curvature of such a surface would be 5.1248 mm. Many problems, in which only the size and position of the image are required, are greatly simplified by using this equivalent spherical surface. In Fig. 44 the surface is shown as the dotted arc *ll*, with its centre at κ.

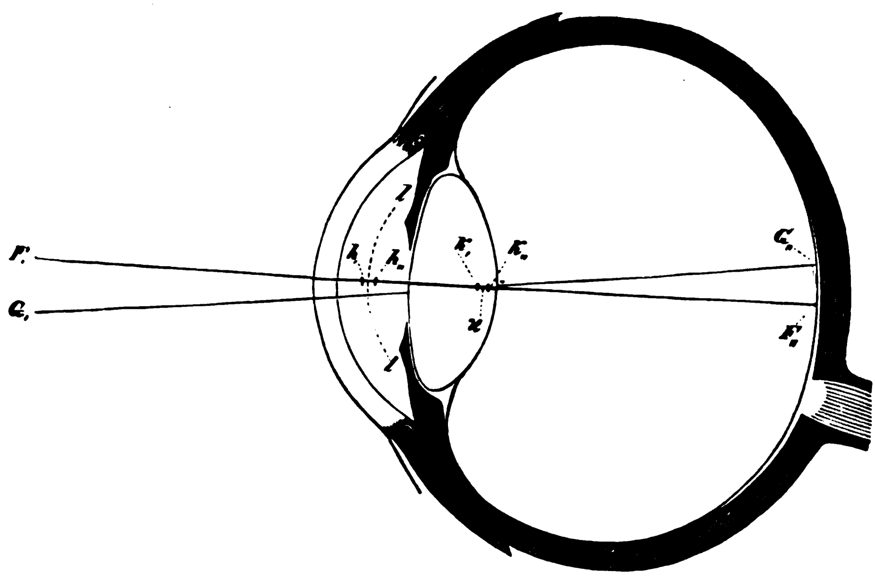

Fig. 44.

If, as often happens, we know in advance that the image is focused on the retina, and all we wish to do is to find the position of the image of a given point, the nodal points are sufficient for the purpose; and if it is permissible to regard the nodal points as coincident, as in the reduced eye, the position of the image may be located by drawing a straight line from the luminous point to the nodal point and prolonging it to meet the retina. A straight line drawn in this way may be called a *line of visual direction.* The nodal point considered as single is therefore the point of intersection of all the lines of visual direction. The two parts of such a line that are in the air in front of the cornea and

[1] ¶Listing's reduced eye is very similar to the "simplified eye" imagined by Huygens in the first part of his *Dioptrica* (1652). (J. P. C. S.)

beyond the lens in the vitreous humor are parts of the actual path of a certain ray, which may be called a *direction ray*. The path of a direction ray coincides with a line of visual direction except along the part of its path that is comprised between the anterior surface of the cornea and the posterior surface of the lens.

If, however, it is desired to make the exact construction, using the two separate nodal points, it is necessary to distinguish between *two lines of direction* which are parallel to each other, one being drawn from the luminous point to the first nodal point and the other through the second nodal point to the image point on the retina. The part of the first line which is outside the eye and the part of the second line in the vitreous humor are portions of the path of a single ray, namely, the so-called *direction ray* above mentioned.

That one of the direction rays which comes from the point of fixation and therefore proceeds to the fovea is called the *visual axis;* or, rather, the visual axis is the line drawn from the point of fixation to the first nodal point of the eye. Formerly, the yellow spot was generally supposed to be located at the terminus of the optical axis of the eye, and the visual axis and optical axis were regarded as identical. However, the writer's investigations have shown that these lines are quite distinct from each other. As a matter of fact, the part of the visual axis lying in front of the eye is on the nasal side and usually somewhat above the optical axis, so that the *fovea centralis* is on the temporal side and usually a little *below* the optical axis. In Fig. 44, which represents a horizontal section of the eye, the visual axis is shown by the lines $G_{\prime}G_{\prime\prime}$, and the optical axis by the line $F_{\prime}F_{\prime\prime}$, in their relative positions as found by the writer in a normal eye. The upper part of the diagram represents the outer or temporal side of the eye, and the lower part the nasal side.

In studying the refraction of the several media of the eye, it is convenient to consider the eye as a compound optical system composed of two partial systems, the cornea and the crystalline lens. The corneal system is bounded by air on one side and the aqueous humor on the other side. The aqueous humor is also the first medium of the lenticular system, and the vitreous humor is the last medium.

Beginning with the cornea, we find the problem of its optical performance is materially simplified by the fact that this system is merely a thin shell whose two surfaces have nearly the same curvature and whose index of refraction is only slightly greater than that of the aqueous humor. In §9 in connection with equations (12), (12a) and (12b) it was shown that at any interface between two media an infinitely thin layer of arbitrary index of refraction, bounded by surfaces of equal curvature, can be inserted in the system without having any effect

on the procedure of the rays. Here let us suppose, therefore, that such a layer of liquid material is interposed in front of the cornea, as in fact is actually found there in the layer of tears that moisten the outer surface of the cornea. Thus the whole corneal system, apart from the air in front of the eye, may be considered as a lens like a watch-crystal surrounded by aqueous humor on both sides. A lens of this description has a very long focus and does not, therefore, appreciably deflect the light traversing it. Consequently, we may just as well consider the aqueous humor as extending clear out to the anterior surface of the cornea; and indeed this assumption is usually made in discussing the geometrical optics of the cornea, and is almost necessary for the reason that, while the measurements of the outer surface of the cornea are accurate enough, the data with respect to the inner surface are not sufficiently reliable.

If the focal length is supposed to be infinite, it follows from equation (13), §9 that

$$n_2(r_2-r_1)+(n_2-n_1)d=0,$$

where n_1 denotes the index of refraction of the aqueous humor, n_2 that of the corneal substance, d the thickness of the cornea, and r_1, r_2 the radii of the anterior and posterior surfaces, respectively. But as a matter of fact this equation is not true in case of the cornea. If it is written

$$(r_2+d)-r_1=\frac{n_1 d}{n_2},$$

where (r_2+d) is equal to the distance of the centre of curvature of the posterior surface from the vertex of the anterior surface, we can see that it implies that the centre of curvature of the posterior surface lies beyond that of the anterior surface, in which case the corneal substance would be thicker in the middle than out towards the edge, which as a rule is contrary to the fact. Regarding the cornea as a lens immersed in aqueous humor, we find that it actually has a very long, negative focal length, and is a meniscus lens of the type described towards the end of §9.

Taking $r_1=8$ mm, $r_2=7$ mm, $d=1$ mm and (according to W. Krause) $n_2=1.3507$, $n_1=1.3420$, the focal length of the cornea immersed in aqueous humor is found by equation (13) to be equal to -8.7 metres; a value which, as compared with the dimensions of the eye, may be regarded as practically infinite. This result has been confirmed by the writer's measurements with the ophthalmometer. The size of an object as seen through a glass vessel with vertical parallel walls which was filled with water was measured with this instrument. When a freshly dissected human cornea was immersed in the water

and the object observed through it, the reduction in the size of the image could not be detected with the ophthalmometer. Whatever change may have occurred in the appearance of the image, it was so slight as to escape notice.

In order to get an idea as to what extent the actual refraction of the eye differs from what it would be if the index of refraction of the cornea were really equal to that of the aqueous humor, the optical constants of the cornea may be calculated by equation (12), §9, by putting $n_1=1$, $n_3=n$, $n_2=n+\Delta n$, $r_1=r$, $r_2=r-\Delta r$, where the magnitudes denoted by Δn, Δr and the thickness of the cornea (d) must all be regarded as very small in comparison with n and r. Substituting these values, at the same time neglecting the higher powers of the small quantities, we find for the focal lengths:

$$F_1=\frac{1}{n}F_2=\frac{r}{n-1}\left\{1-\Delta n\cdot\frac{(n-1)d-n\Delta r}{n(n-1)r}\right\} \quad . \quad . \quad . \quad (1)$$

The difference between this and the value $\frac{r}{n-1}$, obtained by putting $\Delta n=0$ is a small magnitude of the second order. Likewise, the distance (x) of the first principal point from the anterior surface of the cornea calculated as above turns out to be

$$x=\frac{d\cdot\Delta n}{n(n-1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

The interval (a) between the two principal points is indeed of the third order of smallness, namely:

$$a=\frac{d^2\Delta n}{nr} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1b)$$

For calculation of images, therefore, it is accurate enough to assume that refraction occurs simply at the anterior surface of the cornea and to put the index of refraction of the cornea equal to that of the aqueous humor.

The second part of the optical system of the eye is composed of the crystalline lens, bounded by the aqueous humor in front and the vitreous humor behind, The indices of refraction of these two humors are so nearly the same that the difference may be ignored. In an optical system surrounded by the same medium on both sides, the principal points coincide with the nodal points. Thus, just as in an ordinary glass lens surrounded by air, these two pairs of points in the optical system of the crystalline lens are identical. But the crystalline lens differs essentially from a glass lens because the density of its substance is not uniform but increases from the outside towards the central part. Being ignorant of the exact law of this increase, we are not in position

to trace in detail the passage of the rays through the lens, or to determine the exact positions of its focal points and principal points; and we have therefore to be contented with finding their approximate positions. In this connection the following propositions may be stated.

1. *The focal length of the crystalline lens is less than it would be if the index of refraction of the entire lens were uniform and equal to the index of the lens-core.*

In order to demonstrate this important fact, imagine the crystalline lens resolved, according to its natural lamellar structure, into the core, which is an almost spherical double convex lens of positive focal length, and the separate layers surrounding it, which near the axis of the eye correspond to lenses of meniscus type, which get thicker, or at least not thinner, towards the edge; and for which, therefore, $r_1 \geqq r_2 + d$ (see end of §9), where r_1 denotes the radius of the convex surface, r_2 that of the concave surface, and d the thickness of the lens. In such cases, according to equation (13), §9, the focal length is negative. The positions of the principal points h_1, h_2 and of the focal points p_1, p_2 in a lens of this type are shown in Fig. 42.

Referring to Fig. 45, let $a_{\prime}$, $a_{\prime\prime}$, designate the vertices of the lens, $c_{\prime}$, $c_{\prime\prime}$ the centres of the two surfaces, and $h_{\prime}$, $h_{\prime\prime}$ the positions of the principal points. This lens will produce a virtual, erect and reduced image (β) of an object situated at a point b in front of the first (convex) surface. As shown in §9, this image lies not only in front of the second principal point, but invariably also in front of the second surface of the lens; for if $bh_{\prime} > a_{\prime}h_{\prime}$, then $\beta h_{\prime\prime} > \alpha h_{\prime\prime}$, where α is the point on the axis of the lens conjugate to the vertex $a_{\prime}$. But the image

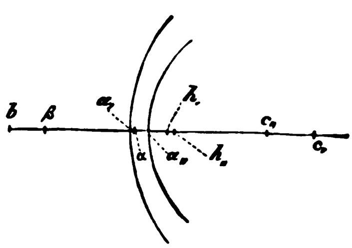

Fig. 45.

of the point $a_{\prime}$ is due to a single refraction at the second surface of the lens, and since the focal length of this surface is negative, the image (α) of $a_{\prime}$ will be nearer to the surface and in front of it. Therefore, β, which is farther back than α, must necessarily be in front of the posterior surface of the lens.

Now it may also be shown that for an object (b) in front of the first vertex ($a_{\prime}$) of the lens, the image (β) will be nearer the second surface when the index of refraction is greater. It is easy to see that this is so when the object is at $a_{\prime}$ and its image at α. Thus, by equations (3), §9, putting $\alpha a_{\prime\prime} = q$, we have:

$$\frac{n_2}{d} - \frac{n_1}{q} = \frac{n_1 - n_2}{r_2}$$

or

$$q=\frac{n_1 r_2 d}{n_2 r_2+(n_2-n_1)d}.$$

Evidently, the value of q decreases as n_2 increases. Now if it can be shown that, when n_2 becomes greater, the image of b approaches nearer to $\mathfrak{a}$, this means that under these circumstances this image also approaches the second surface of the lens. Put $bh_{\prime}=f_{\prime}$, $a_{\prime}h_{\prime}=p$ (which corresponds to the length $-h_{\prime}$ in equations (13a) of §9); then the distance of the second principal point from the image (β) is

$$\beta h_{\prime\prime}=\frac{f_{\prime}F}{F-f_{\prime}}$$

where F denotes the focal length of the lens; and the distance of the same principal point from the point $\mathfrak{a}$ which is the image of $a_{\prime}$ is

$$\mathfrak{a}h_{\prime\prime}=\frac{pF}{F-p}.$$

Subtracting, we find for the interval between these two images:

$$\beta\mathfrak{a}=\frac{(f_{\prime}-p)F^2}{(F-f_{\prime})(F-p)}$$

$$=\frac{f_{\prime}-p}{\left[\frac{F-p}{F}-\frac{f_{\prime}-p}{F}\right]\frac{F-p}{F}}.$$

Putting

$$C=\frac{F-p}{F}$$

and substituting the values of F and $p=-h_{\prime}$ as given by equations (13) and (13a), §9, we have:

$$C=1+\left(1-\frac{n_1}{n_2}\right)\frac{d}{r_2}.$$

If the absolute value of the focal length is denoted by P, that is, if

$$P=-F=\frac{n_1 r_1 r_2}{\left(1-\frac{n_1}{n_2}\right)[n_2(r_1-r_2-d)+n_1 d]},$$

then

$$\beta\mathfrak{a}=\frac{(f_{\prime}-p)}{\left[C+\frac{f_{\prime}-p}{P}\right]C}.$$

From the expressions above, it is evident that C increases and P decreases when n_2 is increased, whereas $(f' - p)$, which does not involve n_2, remains unchanged. Either increase of C or decrease of P results in diminishing $\beta\alpha$, and, hence, increase of the index of refraction causes reduction of the interval $\beta\alpha$.

Thus far we have studied the effect of a single one of the lenses obtained by dividing the crystalline body in its layers. Suppose now that all these meniscus lenses on one side of the core are mounted together in their natural position and surrounded by aqueous humor introduced between each pair of adjacent layers of different densities in the crystalline lens; and isolate that part of it on one side of the nucleus; thereby obtaining a system like that represented in Fig. 46, where ab is the axis, and g and h are the two opposite vertices of the combination. Let a designate the position of a luminous point on the axis in front of the convex side. From what was proved above with respect to a single lens of this type, evidently the image of a in the first lens will lie in front of the second surface of this lens and therefore also in front of the first surface of the second lens. Similarly, the image of this image in the second lens will lie in front of the second surface of that lens, and so on for each lens in succession; and, consequently, the final image of a in the entire system will lie somewhere in front of the last refracting surface, at α, say.

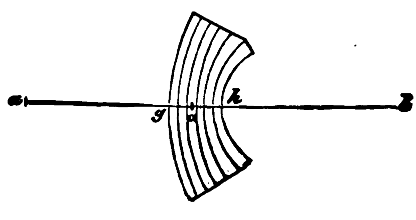

Fig. 46.

Evidently, too, as the point a approaches the vertex g, the point α will approach the vertex (h) of the farther surface. For the image of a real object in a simple negative lens is nearer the lens, when the object is nearer; and since the image produced by each lens of the system acts as object for the next lens, therefore when a approaches the first surface, its image moves along the axis in the same direction, and so on for each image in succession.

The conclusion is that if the index of refraction of one of the layers were increased, the image α would thereby fall nearer h. Until the layer which is supposed to be altered is reached, there would be, of course, no change in the path of the rays or in the successive images; but the image in that layer will be nearer h than it would have been, and, consequently, the last image (α) will be nearer. If, therefore, this final image is to stay where it was originally before the index of one layer is increased, the object a must be moved farther back so as to increase the distance ag.

Consider now the whole crystalline lens as composed of two such systems of meniscus lenses B and C (Fig. 47), with its double convex

core (A) comprised between them. If the crystalline lens as a whole forms a real inverted image at b of a luminous point a in front of the lens, the system of layers B must produce an image at a point α in front of the anterior surface of the core. Similarly, the resultant image at b is the image in system C of a point β lying on

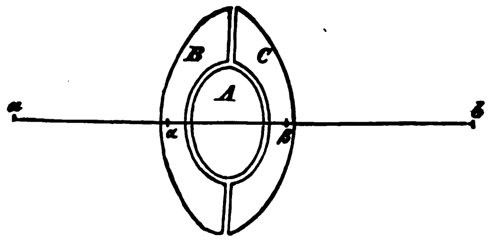

Fig. 47.

the axis beyond the posterior surface of the core at the place where the rays intersect after having traversed the core and before being refracted by system C. Like any double convex lens, therefore, the core itself must produce a real inverted image of α at the place marked β; which it will do, provided α lies in front of its first focal point. If a were removed to an infinite distance, b would be at the second focal point of the optical system of the crystalline lens.

If now the index of refraction of one of the layers of B is supposed to be increased, α will thereby be brought closer to the anterior surface of A, and, consequently, the image of α formed by B at β and the image of β formed by C at b will be displaced forwards in the direction of the incident light. Similarly, an increase of the index of refraction of one of the layers of C will produce a displacement of b away from the crystalline lens, without having any effect on the position of β.

If, therefore, the index of refraction of a single layer of the crystalline lens is increased, the second focal point of the lens will thereby be made to recede farther from its posterior surface.

Thus even if the index of refraction of each layer of the crystalline lens were increased until it was equal to that of the core, the focal point would not be infinitely far away, because ultimately the entire lens in this case would be formed of the same material as the core and would be a simple homogeneous double convex lens of finite focus.

The same argument applies, of course, as to the position of the first focal point; and so it has been shown that the focal points of the crystalline lens are both nearer the lens than they would be if all its layers were of the same density and index as the core.

2. *The interval between the principal points of the crystalline lens is less than it would be for a lens of the same external form with constant index of refraction equal to that of the core.*

The *optical centre* of the crystalline lens is a point on the axis which is conjugate in turn to each of the principal points.[1] Wherever it is,

[1] ¶The optical centre of the crystalline lens is the point where the curved line crosses the axis which represents the path of the ray inside the lens that was directed originally towards the first principal point. (J. P. C. S.)

the same method as was used above for finding the focal point may be employed to show that the effect of increasing the index of refraction of the single layers of the lens will be to draw the two images of the optical centre closer to the external surfaces of the lens; and hence the greater, algebraically speaking, will be the value of the distance between them. Suppose now that the layers have the same index of refraction as that of the core; the optical centre of this new homogeneous lens will not, in general, be at the same place with that of the original crystalline lens. Since the interval between the principal points of a convergent or positive lens is the maximum of all intervals between pairs of conjugate points (see end of §9), the interval between the principal points of this new homogeneous lens will be in any case greater than that between the images of the optical centre of the original lens that are formed by the new lens; and hence it is also greater than the distance of the principal points of the unaltered lens from each other.

Moreover, it may be shown that the interval between the principal points of the crystalline lens is positive; that is, that the second principal point lies beyond the first; provided we assume, as is evident from the form of the layers, that the radii of curvature of their surfaces, taken near the axis, exceed the distances of these surfaces from the core. The image in a spherical refracting surface of a point between the vertex and the centre is nearer the surface than the point itself. Therefore the image of the centre of the core formed by the anterior half of the lens falls in front of, and that formed by the posterior half falls beyond, that point. The two corresponding images of the centre of the core are thus separated by a positive distance. And since the interval between the principal points is algebraically greater than that between any other pair of conjugate points, it likewise must be positive.

The principal points of a lens with the same external form as the crystalline lens of the human eye and with an index of refraction the same as that of the core would be about $\frac{1}{4}$ mm apart; so that the separation of the principal points of the crystalline lens itself must be very small indeed.

Measurements of the indices of refraction of the transparent media of the human eye were made many years ago by CHOSSAT[1] and BREWSTER,[2] who seem, however, to have examined only a small number of eyes; and quite recently W. KRAUSE[3] has carried out an extensive series of such measurements. BREWSTER introduced the substance under examination between the curved surface of a convex lens, mounted as the objective of a microscope, and a plane glass plate set perpendicular to the instrumental axis. This

[1] *Bulletin des sc. par la Société philom. de Paris.* A. 1818. June. p. 294.

[2] *Edinburgh Philos. Journal.* 1819. No. 1. p. 47.

[3] *Die Brechungsindices der durchsichtigen Medien des menschl. Auges von* Dr. W. KRAUSE, Hannover 1855.

altered the focal length of the microscope. BREWSTER measured the object-distance of the microscope before and after the insertion of the refracting substance, and also after inserting water of known index of refraction. CAHOURS and BECQUEREL[1] suggested measuring the size of the microscope image, which is also the method used by W. KRAUSE, whose process is the one here described.

An ordinary KELLNER microscope, the lower part of which is represented by Fig. 48, was arranged for the purpose by substituting a double convex crown glass lens of about 30 mm focus in place of the objective, its fastening b being screwed into the tube a of the microscope. The lens was set in a black hollow depression in which it was screwed tight by means of the casing d in the middle of which there was an opening 2.6 mm in diameter. The lens was pressed air-tight against the rim of this aperture, and below it was mounted a plane crown glass plate e, held in place by a conical ring f, fitted upon the conical surface of d but not so tightly as entirely to exclude the air in between. The specimen of ocular medium to be tested was placed in the ring f on the middle of the flat plate, and the ring pressed against the mounting d with sufficient force to insure that the glass plate was perpendicular to the instrumental axis. The lens could thus be easily removed and cleaned after each measurement.

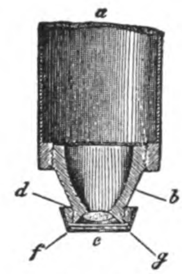

Fig. 48.

The eyepiece was provided with a glass micrometer divided in thirtieths of a Vienna line[2]; another scale divided in tenths of a line was mounted upon the microscope stage. The instrument being adjusted so that both scales were clearly in focus at the same time, the number of divisions of one micrometer that corresponded to one scale division of the other micrometer was determined. Similar measurements were made with air and with distilled water substituted for the transparent humor.

Equations (12) of §9 may be employed for the reduction of the results. It is true that these equations apply to two refracting surfaces only, whereas KRAUSE's apparatus involves four, namely, the two surfaces of the plate and the two surfaces of the lens. But the system may be divided into two parts, the first being the plate and the second being the lens, the focal lengths of the former being infinite. Denoting the first (lower) and second (upper) focal lengths of the plate by $f_{\prime}, f_{\prime\prime}$, respectively, corresponding to the notation in equations (11a) to (11f), §9, those of the lens by $\varphi_{\prime}, \varphi_{\prime\prime}$, and the distance of the second principal point of the plate from the first principal point of the lens by d; we find, from the last of equations (11f), when $f_{\prime\prime}$ is made infinite, that the second (upper) focal length of the whole system is

$$F_{\prime\prime} = \varphi_{\prime\prime}.$$

The first focal length has the same value, since the first and last media are the same (air). Putting $f_{\prime\prime} = \infty$ in equation (11e), we find that the second principal point of the whole system coincides with that of the lens. Accordingly, in this case the second principal point and the second focal point are the same as if the medium contained between the flat plate and the lens extended upwards indefinitely. According to the notation used in equation (12), §9, let us denote the index of refraction of the substance that is being investigated by n_1, that of the glass lens by n_2, and that of the air by $n_3 = 1$. The expression for F_2 as given by that equation will then correspond to the focal length F of our system:

[1] *L'Institut. Scienc. math., phys. et natur.* 1840. p. 399.

[2] ¶Vienna line = 2.195 mm. (J. P. C. S.)

$$F=\frac{n_2r_1r_2}{n_2(1-n_2)r_1+[n_2r_2-(1-n_2)d](n_2-n_1)}.$$

If F_0 denotes the focal length of the objective, when distilled water of index n_0 is inserted between the plate and the lens, and if Φ denotes the focal length when this space contains air, we have two other similar formulae, which, together with the one above, may be put in the following form:

$$\left.\begin{aligned} FA-n_2r_1r_2&=n_1FB \\ F_0A-n_2r_1r_2&=n_0F_0B \\ \Phi A\ -n_2r_1r_2&=\Phi B \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which A and B are abbreviations for the following expressions:

$$A=n_2[(1-n_2)r_1+n_2r_2-(1-n_2)d],$$
$$B=n_2r_2-(1-n_2)d.$$

Subtracting the second of equations (2) from the first, and the third from the second, we obtain:

$$(F-F_0)A=(n_1F-n_0F_0)B$$
$$(F_0-\Phi)A=(n_0F_0-\Phi)B.$$

Eliminating A and B by division, we have:

$$\frac{F-F_0}{F_0-\Phi}=\frac{n_1F-n_0F_0}{n_0F_0-\Phi}.$$

Accordingly,

$$n_1=1+(n_0-1)\,\frac{F_0(F-\Phi)}{F(F_0-\Phi)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

Thus knowing the three focal lengths F, F_0 and Φ and the index of refraction (n_0) of distilled water, we are in a position to determine the index of refraction (n_1) of the substance to be measured. The focal lengths, however, may be calculated from measurements of the images under the different conditions. Thus, if b denotes the size of a scale-division of the lower micrometer, and if β denotes the absolute size of its image in the focal plane of the ocular (that is, disregarding the change of sign due to inversion), then by equation (8b), §9:

$$\frac{\beta}{b}=\frac{f_2-F}{F}$$

or

$$F=\frac{f_2b}{b+\beta}\ , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2b)$$

where F denotes the focal length of the optical system of the objective and f_2 denotes the distance of β from the second principal point of this system. Having measured b and β, we would, therefore, still have to know f_2 in order to find F. But on the assumption that f_2 is constant, which is practically the case in KRAUSE's apparatus, this quantity may be eliminated from the expression for n_1, and consequently does not have to be known. If β, β_0 and $\mathfrak{b}$ denote the values corresponding to the three focal lengths F, F_0 and Φ, we have three equations of the form of equation (2b) which combined with equation (2a) enable us to eliminate the focal lengths as well as f_2, and thus to obtain finally;

$$n_1=1+(n_0-1)\,\frac{\mathfrak{b}-\beta}{\mathfrak{b}-\beta_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2c)$$

By this method, therefore, it is not even necessary to know the size of the object b under the microscope; and all we need to have is some object of constant dimensions.

In these measurements the value of f_2 is constant, provided the positions of the eyepiece micrometer and the second principal point of the objective system are not altered. The latter will not be absolutely stationary, when different liquids are introduced between plate and lens, unless the upper surface of the lens is plane. The distance of the second principal point from the second surface of the lens is denoted by h_2 in equation (12a), §9. Except when r_2 is infinite, this distance depends on the index of refraction (n_1) of the substance that is inserted on the lower side of the lens. But if r_2 is infinite, then

$$h_2 = -\frac{n_3 d}{n_2},$$

in which therefore h_2 is independent of n_1. Accordingly it might be better to use a plano-convex lens for this purpose with its plane side up, instead of a double convex lens. However, the error introduced by employing a double convex lens is extremely small, provided the thickness of the lens is negligible in comparison with the length of the instrument.

In BREWSTER's measurements the index of refraction of distilled water was taken as 1.3358, which according to FRAUNHOFER's data about corresponds to the E line in the green, that is, to rays of medium refrangibility. KRAUSE, taking LISTING's advice, based his work on the rays of greatest intensity, for which, according to FRAUNHOFER, the index of refraction of water is 1.33424. The results obtained by CHOSSAT, BREWSTER and KRAUSE for the human eye are here tabulated. W. KRAUSE measured twenty eyes from ten individuals, and found very considerable variations in them.

Table of indices of refraction of media of human eye

Observer		Cornea	Aqueous humor	Vitreous humor	Crystalline lens: Outer layer	Middle layer	Core
CHOSSAT		1.33	1.338	1.339	1.338	1.395	1.420
BREWSTER $n_0 = 1.3358$			1.3366	1.3394	1.3767	1.3786	1.3839
W. KRAUSE $n_0 = 1.3342$	Max.	1.3569	1.3557	1.3569	1.4743	1.4775	1.4807
	Min.	1.3431	1.3349	1.3361	1.3431	1.3523	1.4252
	Mean	1.3507	1.3420	1.3485	1.4053	1 4294	1.4541
HELMHOLTZ $n_0 = 1.3354$			1.3365	1.3382	1.4189		

The writer's own measurements, the results of which are also included in the table above, were made in the following way. The fluid specimen to be measured was inserted between a plane glass plate and the concave surface of a small plano-concave lens, and the image produced by this system was measured with the ophthalmometer; whence the focal lengths were calculated. Incidentally, the radius of the curved surface of the lens could be found directly with the ophthalmometer, by the same method as was described in §2 for measuring the radius of curvature of the cornea. Under these circumstances, it was unnecessary to assume a value for the index of refraction of distilled water, because it was measured also and found in this way to be 1.3351, which is intermediate between that assumed by BREWSTER and that assumed by KRAUSE.

Krause made another series of measurements of the indices of refraction of calves' eyes, especially with a view to finding out whether the indices suffered any noticeable change of value in the first twenty-four hours after death. He made measurements on twenty specimens immediately after the animals had been killed, and on twenty others after they had been kept twenty-four hours at a temperature of 19° C. The average results are as follows:

	Immediately after death	24 hours after death
Cornea	1.3467	1.3480
Aqueous humor	1.3421	1.3415
Vitreous humor	1.3529	1.3528
Outer layer of lens	1.3983	1.4013
Middle layer of lens	1.4194	1.4211
Core of lens	1.4520	1.4512

Accordingly, there is not any noticeable change in the indices of refraction of the media of a calf's eye in the first twenty-four hours after death, and it is a reasonable inference that the same is true in case of the human eye.

Since it is not possible to calculate the focal lengths of the several layers of the crystalline lens directly from their forms and indices of refraction, the author has inserted below the results of direct measurements of the optical constants of the crystalline lenses of two human eyes, which he was able to make about twelve hours after death.

When the lens is extracted from the eye and exposed to the air its outer surface soon dries and shrivels up, and if kept in water it swells and loses its transparency. The author therefore adopted the plan of surrounding the lens with vitreous humor. The lens is, moreover, remarkably sensitive to every little tension or pressure, but as long as it is surrounded by its elastic and tight-fitting capsule, these effects are only temporary. During the measurements, therefore, the lens has to be kept in such a position that it will not be subjected to external stresses of any sort. This was accomplished by using the apparatus shown in section in Fig. 49, which is actual size. In the centre there is a hollow brass cell divided on the inside at *bb* by a horizontal partition, concave above, with a circular opening in the middle. The mounting of the objective from an old microscope was convenient for the purpose. Upon the lower rim of this mounting was cemented a plane-parallel glass plate, *cc*, care being taken, however, not to use any appreciable thickness of cement. The lower part of the brass cylinder having been first filled with vitreous humor, the crystalline lens was very carefully transferred from the eye, without being injured or bruised, to the position shown, and laid, flat side downwards, on the opening in the diaphragm *bb*. The upper part of the brass tube was then filled level to the brim with vitreous humor and covered over with a second glass plate *dd*, thus giving the humor a plane upper surface. As it was not convenient to set the ophthalmometer in a vertical position, a right-angle isosceles glass prism *f* was mounted on the glass plate *dd*, which reflected the light from below in a horizontal direction. The whole apparatus was conveniently mounted on the body of a microscope, from which all the lenses and the small diaphragm below had been removed. A brass plate provided with a Gravesand slit, the opening of which was to serve as the object to be viewed through the crystalline lens, was mounted first on the stage of the microscope, and afterwards close

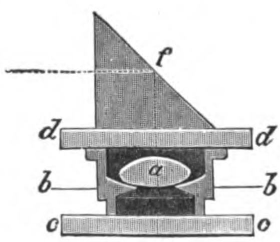

Fig. 49.

to the under side of the glass plate cc, between it and the upper end of the microscope tube. The illumination was produced by adjusting the microscope mirror so as to reflect light from below up through the slit in the brass plate. The width of the image of the slit in the crystalline lens was measured with the ophthalmometer.

In making the calculation it is necessary to know the distance between the knife-edges of the slit and the under surface of the plate cc. Let this be denoted by a_1 when the brass plate lies on the microscope stage, and by a_2 when it is close to the under surface of the glass plate. The greater a_1 and the smaller a_2 can be made, the more reliable are the results. We must also know the thickness c of the plate cc, its index of refraction n_c (approximately at least), the distance b between its upper surface and the upper edge of the opening bb, and the index of refraction n_2 of the vitreous humor with respect to air. Let b_1 denote the interval between the edges of the slit as it lies on the microscope stage at a distance a_1 from the plate cc; β_1 the width of the image of this interval as seen through the crystalline lens (which here is negative, on account of the inversion); and b_2, β_2 the corresponding magnitudes for the other position of the slit; and, finally, let f denote the required focal length of the lens in vitreous humor, and x the distance of its first nodal point from the plane of the upper edge of the opening bb. From equations (3e) and (6c), §9, concerning refraction in plane plates, it follows that the rays proceed in the vitreous humor before passing through the lens as if they had come from a source of size b_1 or b_2, located at the distance

$$\left(na_1+\frac{n}{n_c}c+b+x\right) \text{ or } \left(na_2+\frac{n}{n_c}c+b+x\right),$$

respectively. The size of the image β_1 or β_2 will evidently not be altered further by the subsequent refractions at the surfaces of the upper glass plate. Accordingly, we may write

$$\frac{\beta_1-b_1}{\beta_1}=\frac{na_1+\frac{n}{n_c}c+b+x}{f}$$

$$\frac{\beta_2-b_2}{\beta_2}=\frac{na_2+\frac{n}{n_c}c+b+x}{f};$$

which give by subtraction:

$$\frac{\beta_1-b_1}{\beta_1}-\frac{\beta_2-b_2}{\beta_2}=\frac{n(a_1-a_2)}{f},$$

whence the focal length of the crystalline lens surrounded by vitreous humor is found to be:

$$f=\frac{n\beta_1\beta_2(a_1-a_2)}{b_2\beta_1-b_1\beta_2},$$

and the value of x can be found by substituting this value in either of the two original equations. In making the calculation it is necessary to keep in mind that when a_1 is greater than the focal length, the image will be inverted, and hence β_1 will be negative. A slight correction has to be made in the value of x thus obtained, due to the fact that the curved surface of the lens extends a little below the plane of the opening bb on which it rests, so that x is not exactly equal to the distance of the nodal point from the anterior surface of the lens.

This correction is easily calculated from the diameter of the opening and the radius of curvature of the lower or front lens surface.

By simply turning the lens over, the distance of the second nodal point from the other surface of the lens may be found in the same manner.

The reduced thickness of the glass plate c/n_c may be determined from observations with the ophthalmometer, by inserting it between the slit and a small glass lens, for which the focal length and positions of the nodal points are known, just as it was originally inserted between the slit and the crystalline lens. The value of b also is similarly obtained, and the same equations as were used in calculating x and f may be employed to find b and c/n_c when the two former magnitudes have been ascertained.

The curvatures of the surfaces of the lens in the vicinity of the axis may be found either by reflection, as already explained, or by refraction. For this purpose, the lens is left in the brass cell, and the part of the vitreous humor over its upper surface is removed. The slit is then placed in front of the prism f, a little to one side of the axis of the ophthalmometer, and its reflected image measured; or it is left lying on the microscope stage and the measurement made on the dioptric image thus formed. It has already been explained how the measurement of the reflected image is utilized. In the dioptric method, suppose the symbols b_1, β_1, and f have the same meanings as before, but let β_3 denote the size of the image formed with the upper layer of vitreous humor removed, and let y denote the distance of the second nodal point from the upper surface of the lens (supposed to be surrounded by the vitreous humor). Then, if the radius of curvature at the vertex of the upper surface is denoted by R, it may be found from the equation

$$R.\frac{n(\beta_1-\beta_3)}{(n-1)\beta_3}=f\,\frac{b_1-\beta_1}{b_1}-y.$$

The focal length of the crystalline lens with its peculiar structure has been found to be shorter than that of a lens of the same external form made of homogeneous substance of the same density and index of refraction as those of the core of the lens. Consequently, a homogeneous lens exactly like the crystalline lens in shape and size and of the same focal length would have to be made of some material of even greater index of refraction than that of the core itself. The index of refraction of this imaginary lens fulfilling these specifications has been called by SENFF the *total index* of the crystalline lens. It is quite different from the average index of refraction obtained by taking the arithmetical mean of the values for all the layers, exceeding, as it does, the highest of all the values of the index of refraction in the crystalline lens. Herewith is appended a summary of the data derived from measurements of human lenses made by the author, the dimensions being in millimetres. The focal lengths and principal points are given on the supposition that the lens is surrounded by vitreous humor. The radii of curvature were obtained by the reflection method.

1. Focal length	45.144	to	47.435
2. Distance of first principal point from anterior surface	2.258	"	2.810
3. Distance of second principal point from posterior surface	1.546	"	1.499
4. Thickness of lens	4.2	"	4.314
5. Radius of curvature of anterior surface at vertex	10.162	"	8.865
6. Radius of curvature of posterior surface at vertex	5.860	"	5.889
7. Total index of refraction	1.4519	"	1.4414

However, whether the form and focal length are the same for lenses measured after death as in the unaccommodated living eye is rendered doubtful by certain of the writer's measurements. The smallest values of the thick-

ness of the lens as found by measurements of dead eyes are occasionally more than half a millimetre greater than the writer found for the same distance in the eyes of three live individuals.[1] The method of measuring the distance between the pupil and the front of the cornea was explained in §3. The anterior surface of the lens is close against the edge of the pupil, and so the thickness of the lens can be found as soon as the distance of the posterior surface of the lens from the cornea is obtained.

The cornea and lens are represented in Fig. 50 by *AA* and *B*, respectively. Light comes to the eye along some direction such as *Cc*, and, after refraction first at the cornea and then at the anterior surface of the lens, is partially reflected at *i* at the posterior surface of the lens. The reflected ray emerges by the path *idD*, so as finally to enter the eye of the observer. If now the source of light *C* and the observer's eye *D* are interchanged, the light will again proceed along the same path, only in the reverse direction, *DdicC*, being reflected, as before, at the same point *i* on the posterior surface of the lens. The patient's eye is directed steadily towards a point of fixation and the straight line *Gg* represents the visual axis of his eye. These things having all been previously determined by suitable measurements, we can find the angles between the lines *Cc*, *Dd* and *Gg*. In order to locate the points *c* and *d* on the cornea, suppose the observer's eye is at *D*, and let a small source of light be so adjusted at a place *E* in front of the eye that the observer at *D* will see the reflex of this light in the anterior surface of the cornea and at the same time the reflex from *C* in the posterior surface of the crystalline lens. This coincidence occurs when the ray *Ed* is reflected to *D*, that is, when the bisector *de* of the angle *EdD* is normal to the cornea. Now if the angle *EdD* or the angle between *Ed* and *Gg* has been found by suitable measurement, it is easy to calculate the angle between *ed* and *Gg*; and hence, from the form and curvature of the cornea, as obtained by previous measurements, the length of the arc *dg* can be found, that is, the position of *d* with respect to *g*. The position of the point *c* is ascertained in the same way. Thus, the positions of the points *c*, *d* and the directions of the lines *Cc*, *Dd* are known; and the point *h* where these lines meet is the apparent place of the reflex image in the posterior surface of the crystalline lens, that is, the place where it appears to be as seen through the intervening ocular media.

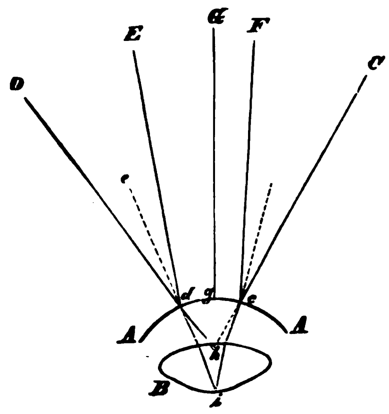

Fig. 50.

In making the measurement the sources of light *C* and *E* are arranged on a horizontal graduated bar several feet from the eye under examination. The source *C* should be as large and bright as possible, but *E* should be small and coloured by a blue glass to facilitate observing its reflection. The observer looks through a small telescope, which is also mounted on the grauated bar to enable him to locate its position. The telescope and lamp *C* can then be interchanged, as desired.[2]

The mean apparent position of the posterior surface of the lens, as found by observations of this kind with three different eyes, was not far in front of the centre of curvature of the cornea. The displacement produced by the

[1] v. Graefes *Archiv. für Ophthalmologie.* Bd. I. Abt. 2. S. 56.

[2] The details of this method are described in Graefes *Archiv.* I, 2, p. 51.

refraction at the cornea may be calculated. Since a spherical refracting surface has very little effect on the apparent position of an object located near its centre, individual variations in the value of the index of the refraction of the aqueous humor are unimportant so far as the final result is concerned. The same is true with respect to the refraction of the lens for a point on its posterior surface, since this surface too is very near the second principal point of the lens. The results of the author's measurements of the interval between the principal points of the dead crystalline lens were not reliable because in the case of such very small magnitudes all the other errors are cumulative; and so he has borrowed the correction that has to be made for the lenticular refraction from the data of LISTING's schematic eye. The apparent forward displacement of the posterior surface of the lens, due to refraction through the lens, is somewhat less than the interval between the principal points. It has been shown that the distance between the principal points in the natural lens is less than in a homogeneous lens of the same form and of index of refraction equal to that of the core; consequently, the correction deduced from LISTING's lens must be rather too large, and tends therefore to give slightly too large a value of the calculated thickness.

The mean results for the three eyes examined, as deduced from two series of concordant observations, were as follows:

	O.H.	B.P.	J.H.
Radius of curvature of cornea	7.338	7.646	8.154
Apparent distance of posterior surface of lens from vertex of cornea	6.775	7.003	6.658
Actual distance	7.172	7.232	7.141
Distance of pupillary plane from vertex of cornea	4.024	3.597	3.739

Assuming the anterior surface of the lens to lie in the pupillary plane, these results give for the thickness of the lens in the unaccommodated living eye: 3.148 3.635 3.402

Adding to these the correction necessary because of the convexity of the anterior lens surface, the edge of the pupil itself being assumed to have no appreciable thickness, we obtain the values: 3.414 3.801 3.555

The values of the pupillary diameter and the curvature of the anterior surface of the lens used in calculating this correction were obtained by actual measurements in each of the eyes concerned. These final results also are still less than the smallest values of the thickness heretofore obtained from dead lenses, which vary, according to the elder KRAUSE, from 4.0 to 5.4 mm.

The fact, observed by the younger KRAUSE, that the index of refraction of the lens from a calf's eye remains practically unchanged 24 hours after death, makes it improbable that the thickening of the lens is due to absorption of water; for in that case we should expect a decrease of the index of refraction. It seems likely, therefore, that the observed change is of the same nature as what takes place in the act of accommodation. This will be referred to again in §12.

There still remains to be stated what has been learned to date about the cardinal points of the eye. The conclusions here given are based on LISTING's schematic eye, which certainly departs very little from the actual average, as is verified in some measure by the writer's investigations. At any rate, whereever it is necessary or permissible to use mean values in the computations of physiological optics, the numerical data for the individual eye in question being unknown, it is probable, in view of the very large individual variations that exist, that the data of LISTING's schematic eye will be found just as reliable and satisfactory as the actual mean values, supposing that the latter were known. Accordingly, LISTING's constants will be used in this treatise, with occasional comments, when necessary, explaining how they are sometimes different from what seem to be the actual mean values.

LISTING gives 8 mm as the radius of the cornea; although, according to the measurements of SENFF as well as those of the author, it appears to be somewhat less than this. The average of the values of the index of refraction of the cornea obtained by W. KRAUSE is rather higher than BREWSTER's value $103/77 = 1.3379$ which is adopted by LISTING. On both accounts LISTING's data for the focal lengths of the cornea are somewhat greater than the observed average. According to equations (3a) and (3b), §9, the first focal length of the cornea is

$$F_1 = \frac{r}{n-1},$$

where n denotes the index of refraction of the aqueous humor and r denotes the radius of the cornea; and the second focal length is

$$F_2 = \frac{nr}{n-1}.$$

In LISTING's schematic eye,

$$F_1 = 23\tfrac{9}{13}, \quad F_2 = 31\tfrac{9}{13}.$$

If we put $r = 7.8$, which is the result of SENFF's measurements and which agrees also approximately with the mean of the writer's determinations, and if, following W. KRAUSE, we take $n = 1.342$, we get:

$$F_1 = 22.81, \qquad F_2 = 30.61.$$

The index of refraction of the lens in LISTING's schematic eye is 16/11; the thickness of the lens is 4 mm, and the radii are 10 and 6 mm. Accordingly, for a lens immersed in aqueous humor, we find by equations (13), (13a), and (13b), §9, that the focal length is 43.796 mm, the interval between the principal points 0.2461 mm, the distance of the first principal point from the anterior surface 2.3462 mm, and the distance of the second principal point from the posterior surface 1.4077 mm. These values agree very closely with the results of the writer's earlier direct measurements of two dissected human lenses. He is not aware of any other direct measurements of the focal lengths of eyes of human beings. The reason why it is not practicable to calculate the focal lengths from the form and indices of the component layers of the lens has already been explained. It follows from the theorem proved in the earlier part of this section, that the assumption, made by most earlier opticians, of an equivalent homogeneous crystalline lens having the same figure and the average index of refraction of the actual lens, is essentially incorrect; and that, on the contrary, such an artificial lens would have to have a higher index than that of the densest part of the actual crystalline lens. For example, SENFF[1] found this *total index of refraction* of the crystalline lens of an ox to be 1.539, whereas the actual indices for the outer layer and for the core were 1.374 and 1.453, respectively. The two values of the total index found by the author from his measurements, namely, 1.4519 and 1.4414, are both less than the above and correspond, say, with the average of the values given by W. KRAUSE for the core. (His results were: maximum, 1.4807; minimum, 1.4252; mean, 1.4541.) LISTING had previously chosen the value $16/11 = 1.4545$, which agrees well enough with both KRAUSE's work and the writer's investigations.

If we assume that there is always the same difference between the optical properties of the crystalline lens before and after death as was shown by the writer's experiments, LISTING's schematic eye would probably correspond to an eye that was accommodated for near vision, the focal length of the lens when relaxed being somewhat longer, and its thickness somewhat less.

The value assumed by LISTING for the distance between the anterior surfaces of cornea and lens is 4 mm. This corresponds to the near-sighted eye designated as O. H. in the author's measurements. In near-sighted eyes the

[1] VOLKMANN, Art. "Sehen" in R. WAGNERS *Handwörterbuch d. Physiologie.* Bd. III. S. 290.

anterior chamber is usually deeper and the iris flatter. The other two eyes in the author's experiments were normal, and in both of them the depth was less; but in all three the posterior surface of the lens was in front of the centre of curvature of the cornea. This leads the writer to suspect that the lens is somewhat nearer the cornea in normal eyes than LISTING has assumed. The difference is too small, however, to be of much importance.

Being given the focal lengths of the cornea and lens and the positions of the principal points of the latter, the cardinal points of the eye as a whole may be found by means of equations (11a) to (11f), §9, The values calculated by LISTING from his data have already been recorded.

For finding the position of the image on the retina, the nodal points of the eye are the most convenient of all the cardinal points, and, luckily, the locations of these points is known now with considerable certainty.

By the method given in §9 for finding the nodal points, the point on the axis of the eye which is conjugate to each of the nodal points in succession is found to lie between the nodal point of the cornea (which is its centre of curvature) and the first principal point of the lens; its distances from these points being to each other as the lesser focal length of the cornea, is to that of the lens, namely, about 1 to 2. In LISTING's schematic eye, the distance of the first principal point of the lens from the centre of curvature of the cornea, which he found to lie on the posterior surface of the lens, is 1.627 mm. But according to measurements of living eyes as made by the author, the posterior surface of the lens may be as much as 1 mm in front of the centre of curvature of the cornea, which might make the above distance about 2.6 mm. Thus, the point, which is conjugate to each nodal point in succession, would be from 0.54 to 0.87 mm in front of the centre of curvature of the cornea, the range of variation being, therefore, very small. The first nodal point is its image as formed by the cornea. The image of an object just a little in front of the centre of curvature of a spherical refracting surface is only a very short distance in front of the object itself. It we take LISTING's values of the focal lengths of cornea and lens, the first nodal point proves to be 0.758 mm in front of the centre of curvature of the cornea. On the other hand, if the point, which is conjugate to this nodal point with respect to the optical system of the cornea is assumed to be 0.87 mm in front of the centre of curvature of the cornea, the first nodal point will be found to be 1.16 mm in front of the centre. We shall, therefore not go far wrong by assuming that in normal eyes the first nodal point is from 3/4 to 5/4 mm in front of the centre of curvature of the cornea.

VOLKMANN[1] endeavoured to determine experimentally the position of the nodal point in the human eye. The fact was mentioned above that when the light from a candle enters the eye sidewise, the image of the flame, especially in the case of a blond individual, can be seen at the inner corner of the eye. VOLKMANN measured the distance of this image from the cornea, and at the same time the angle between the incident rays and the visual axis. He then constructed a horizontal section of the eye to a suitable scale, marked on it the point where the retinal image was visible through the sclerotica, and drew through this point a line intersecting the optical axis at the same angle as that between the incident rays and the visual axis. The point of intersection was taken as the nodal point. The mean of his measurements with five persons gave the position of the nodal point as 8.93 mm beyond the cornea. Undoubtedly, this value is somewhat too large, as it makes the nodal point lie beyond the centre of curvature of the cornea, whereas it must necessarily lie in front of that point. The error in VOLKMANN's result is explained by the fact that he was as yet unaware of the distinction between the optical axis

[1] R. WAGNERS *Handwörterbuch d. Physiologie*. Art. "Sehen." S. 286.*

and visual axis, and also because the definition of nodal points and principal points are valid only for much smaller angles of incidence than he used in these experiments. BUROW[1] found, moreover, in repeating VOLKMANN's experiments with the eyes of white rabbits, that for very wide angles of incidence the retinal image falls nearer the optical axis than it should if all the direction lines intersected in a point. Both of these influences would have the effect of making the distance from cornea to nodal point, as determined by VOLKMANN's method, somewhat too large.

We shall now explain how the centering of the eye and the positions of the optical axis and the visual axis may be determined. The method utilizes the reflex images formed by the cornea and lens surfaces of a bright source of light placed in front of the eye.

Concerning the appearance of these reflex images and the best methods of observing them, the reader is referred to §12. In Fig. 51, *cd* represents the axis of an accurately centered eye; the eye of the observer is at *a* and the source of light at *b*. Suppose *ab* is perpendicular to *cd*, and $ac = cb$. With this arrangement, everything being symmetrical, it is clear that the light from *b*, falling upon the three reflecting surfaces at their vertices, where they intersect the axis *cd*, will in each case be reflected to *a*. If the observer and the source of light are interchanged, the same thing will occur again, and the three reflected points will appear in the same perspective, on account of the bilateral symmetry of the whole apparatus. Since the anterior surface of the lens is about half-way between the cornea and the posterior surface of the lens, the image in the anterior surface of the lens should, for either position of the light and observer, appear about halfway between the other two images.

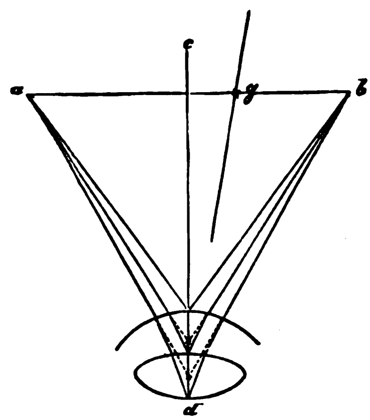

Fig. 51.

The following method may now be employed to ascertain the adjustment of any given eye. Let a horizontal graduated scale be placed along the line *ab*, suitable openings being provided at *a* and *b* for the observer and the source of light. The eye under examination is brought into some convenient position *d*, on the perpendicular bisector *cd*, and its owner is directed to look fixedly at some adjustable object, *g*, which is then moved up or down and to right or left until the observer can see the reflex from the anterior surface of the lens lying between those from the cornea and the posterior surface of the lens. The observer then changes place with the light and notes whether, with the fixation mark unchanged, the three images remain in the same relative positions as before. If the eye under examination were correctly centered, it would be possible to find a position for the object *g* such that this would be the case.

The writer has never examined a human eye that quite fulfilled this condition. If the three reflexes are in the right positions when viewed from one side, they are not so when viewed from the other, and in order to adjust them in the right positions again, the point of fixation (*g*) has to be shifted more or less. For each of the three eyes on which this method was tried, it was found necessary to place the fixation point *g* somewhat above the plane *abd*. The visual axis was invariably found to lie on the nasal side of the line *cd*, its

[1] *Beiträge zur Physiologie d. menschl. Auges.* S. 56-60.

horizontal projection making the following angles with *cd* under the given conditions:

Eye	Light coming from the nasal side	from the temporal side
O. H.	3° 47′	4° 57′
B. P.	5° 6′	8° 12′
J. H.	5° 43′	7° 44′

These results show that *the human eye is not exactly centered.* But the differences between the corresponding angles for different eyes is comparatively small; and we may, therefore, assume the line *cd*, found from the experiments, as the approximate position of the so-called optical axis, and take the arithmetical mean of the above results as the angle between this assumed optical axis and the horizontal projection of the visual axis. This optical axis also coincides well enough with the axis of the cornea as found by the author and passes through the centre of its circumference.

The pioneer in physiological optics who was the first to have a clear conception of the refraction of light in the eye and of the formation and position of the image on the retina was KEPLER. It is true that MAUROLYCUS had previously compared the crystalline lens of the eye with a glass lens, and asserted that it converges the rays towards the axis, but he could not admit that it forms an inverted image on the retina, because we should then have to see everything upside down. PORTA also, who invented the *camera obscura*, compared the eye with that instrument, but he supposed that the image was formed on the crystalline lens. KEPLER, who had already investigated the general theory of optical instruments, was the first to realize the existence of the inverted retinal image and the condition for distinct vision, namely, that the rays from each point of the object shall be brought to a focus at some point of the retina. KEPLER's theory was extended by the work of the celebrated Jesuit philosopher SCHEINER,[1] who made further investigations on the structure of the eye and the refraction in its transparent humors. He verified the fact that optical images are projected on the retina in the case of the eyes of certain beasts, by exposing the back part of the retina so that the image could be seen, and in 1625 in Rome he performed the same experiment on a human eye. He assumed that the refractivities of the aqueous humor and the crystalline lens were the same as those of water and glass, respectively, while the vitreous humor was intermediate between the other two in this respect. Finally HUYGENS[2] constructed an artificial model of the eye, by means of which he demonstrated the principal phenomena of vision, the application of spectacles, etc.

With the exception of a few amateurish and wholly impossible propositions that have been put forth in opposition to it, KEPLER's theory has received practically universal acceptance from the first. For example, N. TH. MÜHLBACH[3] and CAMPBELL[4] denied the existence of the retinal image, and LEHOT[5] advanced the idea that a three-dimensional image of the object is formed within the vitreous humor. PLAGGE[6] worked on the theory that the eye is a mirror

[1] *Oculus.* Inspruck 1619.
[2] *Dioptrica* in *Opera posthuma.* Lugduni 1704. p. 112.
[3] *Inquisitio de visus sensu.* Vindob. 1816.
[4] *Annals of philosophy.* X. 17.—*Deutsches Archiv.* IV. 110.
[5] *Nouvelle Théorie de la Vision.* Paris 1825.
[6] HECKERS *Annalen.* 1830. S. 404.

and that the image used in vision is the reflection in the cornea. J. Reade[1] concurred in this opinion and attributed vision to the pr sence of nerves in the cornea. Mayer[2] opposed Plagge's view, but advanced an equally remarkable one of his own, namely, that the retina acts as a concave mirror. Likewise, Andrew Horn[3] imagined the vitreous humor to be the reflector and the resulting image to act upon the optic nerve.

As to the positions of the cardinal points of the eye, there was some difficulty at first concerning the second focal point. According to calculations based on the measured dimensions and indices of refraction, this point seemed to be beyond the retina. This error was due to using in the calculation the mean index of refraction of the layers of the crystalline lens, which we now know to be incorrect.[4] Vallée[5] concluded that it was necessary to assume a gradual increase in the index of refraction of the vitreous humor towards the back of the eye. Pappenheim[6] was willing to admit this explanation, provided even a slight change of the sort can be demonstrated to exist. Until the development of Gauss's theory, considerable confusion prevailed among physicists and physiologists as to the location of the nodal points of the eye. This was because the theory of optical instruments up to that time had dealt exclusively with systems of refracting surfaces at comparatively negligible distances apart, as, for example, the lenses of the objective of a telescope; whereas in the eye the distances between the refracting surfaces are quite considerable as compared with the focal length of the whole system. Owing to the imperfect development of the theory, there appears to have been some uncertainty as to the proper standpoint from which to attack the problem. Many attempts were made to locate a point on the optical axis of the eye that would correspond to the optical centre of a glass lens; such that a ray directed towards it would not be ultimately deviated by refraction in traversing the ocular media. If it is permissible to consider the two nodal points as coincident, their common position would correspond to the required optical centre. One trouble, especially, was that this point was also confused with the point of intersection of the so-called *lines of sight* drawn from points at different distances in the field of view that appeared to be all in the same line of vision. This point, however, is essentially different from the nodal point of the eye, and is, in fact, the centre of the entrance-pupil of the eye, as will be shown in the next section. Muncke[7] supposed that the two points were identical and located them in the middle of the lens; whereas Bartels[8] placed them at the centre of the cornea. The place where the straight lines meet that connect the various points of the object with the conjugate points of the retinal image was called by Volkmann[9] the *point of intersection of the direction rays*; and, subsequently, after Mile's objections, the *point of intersection of the direction lines*. By means of experiments upon the eyes of white rabbits, he showed that these lines all intersect at a point, and found its location, which necessarily lies between the two nodal points. His result showed that, in the

[1] *Annals of philos.* XV. 260.

[2] Muncke, Art. "Gesicht" in Gehlers *Wörterbuch*. The reference given there is wrong.

[3] *The seat of vision determined.* London 1813.

[4] Moser in Doves *Repertorium*. V. 337-349.*—Forbes, *Proc. Edinb. Roy. Soc.* 1849. Dec. p. 251.

[5] *Comptes rendus.* 1845. XIV. 481.

[6] Ibid. XXV. 901.

[7] Gehlers *physik. Wörterbuch* (*neu bearb.*) Leipzig 1828. Art. "Gesicht." Bd. IV. 2. S. 1434.*

[8] *Beiträge zur Physiol. d. Gesichtssinns.* Berlin 1834. S. 61.

[9] *Neue Beiträge zur Physiol. d. Gesichtssinns.* Leipzig 1836. Kap. IV.—Poggendorffs *Ann.* XXXVII. 342.

rabbit, at least, it was beyond the lens. By a different method he tried to find the same point in the case of the living human eye. Two hair-sights about 6 inches from the eye were viewed through two small peep-holes placed nearer the eye, the latter being so adjusted that both hairs were seen at the same time in the centres of the openings. Each hair and its corresponding peep-hole thus constituted a line of sight. VOLKMANN might therefore have been able to find the point of intersection of the direction lines in this way if it had been possible for the subject to see both hairs through their respective openings at the same time without turning the eye. This is, however, a very difficult thing to do, because the subject can look directly at only one of the hairs at a time, the other being seen indirectly on a less sensitive portion of the retina. The probable result was that the subject looked first at one hair and then at the other, the lines of fixation therefore intersecting at the centre of rotation of the eye; and accordingly this was the point that VOLKMANN took to be the place of intersection of the direction lines.

MILE,[1] KNOCHENHAUER[2] and STAMM[3] took exception to VOLKMANN's conclusions. MILE pointed out that the direction lines and lines of sight are not necessarily identical, and defined the point of intersection of the direction lines as being the centre of the cornea, assuming the effect of refraction by the lens to be negligible. Thence he concluded that the direction lines do not necessarily have to pass through the centre of the blur circle projected on the retina by a luminous point out of focus. KNOCHENHAUER tried to simplify MILE's proof that the coincidence of images in the field is independent of the direction lines, and thereby to avoid the assumption, at that time generally accepted but in fact only approximately true, that the point of intersection of the direction lines is the same for objects at different distances. BUROW[4] also rejected VOLKMANN's conclusions, but used his method to determine the centre of rotation of the eye, and worked out an independent method of finding the point of intersection of the direction lines. This method, however, was not successful, for reasons subsequently given by LISTING.

The first to apply the theoretical work of GAUSS[5] and BESSEL[6] to the optical system of the eye was MOSER,[7] who, from the available data at that time as to the form of the refracting surfaces and the indices of refraction, computed the positions of the nodal points (which, by the way, he called principal points). The values which he found for the distances of these points from the cornea were 3.19 and 3.276 Paris lines (7.18 and 7.37 mm). But since he had assumed BREWSTER's average value of the index of refraction of the crystalline lens (1.3839), which means that the rays from a distant source come to a focus beyond the retina, he concluded that the radius of the cornea should be diminished from 3.39''' to 2.88''', and on this supposition he deduced the new values 2.835''' and 2.890''' (6.38 and 6.50 mm) for the distances from cornea to nodal points.

LISTING[8] investigated the properties of the principal points and nodal points of the eye (giving the latter their name), found their approximate positions, and particularly called attention to the fact that if the lens is to be treated as made of some homogeneous material, the index of refraction as-

[1] POGGENDORFFS *Ann.* XLII. 37-71. 235-263.* Reply by VOLKMANN, ibid. XLV. 207 to 226.*

[2] Ibid. XLVI. 248-258.*

[3] Ibid. LVII. 346-382.*

[4] *Beiträge zur Physiologie u. Physik d. menschl. Auges.* Berlin 1841. S. 26-93.

[5] *Dioptrische Untersuchungen.* Göttingen 1841.

[6] *Astronomische Nachrichten.* XVIII. Nr. 415.

[7] DOVE, *Repertorium d. Physik.* V. 337, 373.

[8] *Beitrag zur physiologischen Optik.* Göttingen 1845.

signed to it must exceed the actual index of its densest part. It was then that VOLKMANN[1] made his later attempt, as above mentioned, to find experimentally the positions of the nodal points in living human eyes. Finally, LISTING[2] published a complete mathematical theory of the subject, including a calculation of the numerical data based upon the best measurements available at the time.

1575. FR. MAUROLYCI *Photismi de lumine et umbra ad Perspectivam et radiorum incidentiam facientes.* Venetiis 1575. Messinae 1613.—A later complete edition of his optical writings has the title: FR. MAUROLYCI, *Abbatis Messanensis, theoremata de lumine et umbra, ad Perspectivam et radiorum incidentiam facientia; Diaphanorum partes seu libri tres, in quorum primo de perspicuis corporibus, in secundo de Iride, in tertio de organi visualis structura et conspicillorum formis agitur: Problemata ad Perspectivam et Iridem pertinentia. His accesserunt* CHRISTOPH. CLAVII *e. S. J. notae.* Lugduni 1613.

1583. JO. BAPT. PORTAE *Neap. de refractione Optices parte libri novem.* Neapoli 1583. Liber III-VIII.

1602. *JO. KEPLER *ad Vitellionem paralipomena, quibus astronomiae pars optica traditur.* Francofurti 1604. Cap. V.

1611. KEPLER, *Dioptrice, seu demonstratio eorum, quae visui et visibilibus, propter conspicilla non ita pridem inventa, accidunt.* Augustae Vindelicorum 1611.

1619. C. SCHEINER, *Oculus, sive fundamentum opticum.* Innspruck 1619. London 1652.

1695. HUYGENS (†1695), *Opera posthuna. Dioptrica.* Lugduni 1704. p. 112.

1759. W. PORTERFIELD, *A treatise on the eye.* Edinb. 1759. Vol. I. Book 3. Chap. 2*.

1776. J. PRIESTLEYS *Geschichte der Optik*; translated in German by G. S. KLUEGEL. Leipzig 1776 (Early history; calculation of focal lengths, S. 465)*.
RUMBALL, *Annals of Philos.* II. 376.

1813. ANDREW HORN, *The seat of vision determined.* London 1813.

1816. N. TH. MÜHLBACH, *Inquisitio de visus sensu.* Vindobonae 1816.
MAGENDIE, *Précis élémentaire de Physiologie.* Paris. Vol. I. p. 59.

1817. CAMPBELL, *Annals of Philos.* X. 17. *Deutsches Archiv.*, IV. 110.
J. READ, *Annals of Philos.* XV. 260.

1825. C. J. LEHOT, *Nouvelle Théorie de Vision.* 1825.

1828. G. R. TREVIRANUS, *Beiträge zur Anatomie und Physiologie der Sinneswerkzeuge.* Bremen 1828. Kap. I*.
MUNCKE in GEHLERS *Physikalischem Wörterbuch; neu bearbeitet.* Leipzig 1828. Art. Gesicht. IV. 2. S. 1364*.

1830. PLAGGE, HECKERS *Annalen* 1830. S. 404.

1834. BARTELS, *Beiträge zur Physiolgie des Gesichtssinns.* Berlin 1834. S. 61.

1836. A. W. VOLKMANN, *Untersuchung über den Stand des Netzhautbildchens.* POGGENDORFFS *Ann.* XXXVII. 342*.—*Neue Beiträge zur Physiologie des Gesichtssinns.* Leipzig 1836. Kap. IV.

1837. JOH. MILE, Über die Richtungslinien des Sehens. POGGENDORFFS *Ann.* XLII. 37 and 235*.

1838. VOLKMANN, POGGENDORFFS *Ann.* XLV. 207*. (Reply to above.)

1839. GERLING, Über die Beobachtung von Netzhautbildern. POGGENDORFFS *Ann.* XLVI. 243*.
KNOCHENHAUER, Über die Richtungsstrahlen oder Richtungslinien beim Sehen. POGGENDORFFS *Ann.* XLVI. 248*.

1841. A. BUROW, *Beiträge zur Physiologie und Physik des menschlichen Auges.* Berlin 1841. S. 16-93*.

1842. VALLÉE, *Comptes rendus.* XIV. 481.
W. STAMM, Über VOLKMANNS Richtungslinien des Sehens. POGGENSDORFFS *Ann.* LVII. 346*.

[1] R. WAGNERS *Handwörterbuch d. Physiologie.* Art. "Sehen." S. 286.*

[2] Ibid., Art. Dioptrik des Auges.

1843. A. W. VOLKMANN, J. MÜLLERS *Archiv. f. Anat. u. Physiol.* 1843. S. 9 (reply to BUROW).
1844. *L. MOSER, Über das Auge, in DOVES *Repertorium d. Physik.* S. 337-349*.
1845. J. B. LISTING, *Beitrag zur physiologischen Optik.* Göttingen 1845 (abstract of *Göttinger Studien*). S. 7-21*.
L. L. VALLÉE, *Comptes rendus.* XX. 1338.—*Institut.* No. 393. p. 166.
1846. *A. W. VOLKMANN, Art. "Sehen" in R. WAGNERS *Handwörterbuch der Physiologie.* III. 1. S. 281-290*.
1847. F. C. DONDERS, *Holländische Beiträge zu den anat. u. physiol. Wissensch.* I. S. 107-112.
1849. J. D. FORBES, Note respecting the dimensions and refracting power of the eye. *Proceedings Edinb. Roy. Soc.* 1849. p. 251.—SILLIMAN'S *Journal.* (2) XIII. 413.
1851. *J. B. LISTING, Art. "Dioptrik des Auges" in R. WAGNERS *Handwörterbuch d. Physiol.* IV. 451-504*.
1854. H. HELMHOLTZ, GRAEFES *Archiv. für Ophthalmologie.* I. 2. S. 1-74*.

Measurements of the indices of refraction:

1710. HAWKSBEE, *Phil. Transact.* 1710. p. 204.
1785. A. MONRO. II. *On the structure and physiology of fishes.* p. 60.
1801. TH. YOUNG, *Phil. Transact.* 1801. I. 40*.
1818. CHOSSAT, *Bulletin des sc. par la Société philomat. de Paris.* A. 1818. June. p. 294.—*Ann. de ch. et de ph.* VII. p. 217.
1819. D. BREWSTER, *Edinb. Philos. Journ.* 1819. No. 1. p. 47.
1840. CAHOURS et BECQUEREL, *Institut.* 1840. p. 399.
1847. S. PAPPENHEIM, *Comptes rendus.* XXV. 901. — *Arch. d. sc. ph. et natur.* VII. 78.
QUESNEL, *Revue scient.* XXXII. 144.
1849. BERTIN, *Comptes rendus.* XXVIII. 447. — *Institut.* 1849. No. 796. p. 105. — *Ann. d. ch. et de ph.* XXVI. 288. — *Arch. d. sc. ph. et nat.* XII. 45. — POGGENDORFFS *Ann.* LXXVI. 611.
1850. ENGEL, *Prager Vierteljahrsschrift für prakt. Heilk.* 1850. I. 152.
H. MAYER, ibid. 1850. IV. Beilage and 1851. IV. 92.
1852. RYBA, ibid. 1852. II. 95.
1855. W. KRAUSE, *Die Brechungsindizes der durchsichtigen Medien d. menschl. Auges.* Hannover 1855*.

Supplement.—DONDERS gives the following summary of the results of a large number of measurements of the curvature of the cornea at its intersection with the visual axis. The values are given in millimetres.

A. Males

1.	20 under 20 years	7.932
2.	51 under 40 "	7.882
3.	28 over 40 "	7.819
4.	11 over 60 "	7.809
	Mean	7.858
	Maximum	8.396
	Minimum	7.28

B. Women

1.	6 under 20 years	7.720
2.	22 under 40 "	7.799
3.	16 over 40 "	7.799
4.	2 over 60 "	7.607
	Mean	7.799
	Maximum	8.487
	Minimum	7.115

C. According to Static Refraction

1. 27 Emmetropes.................................7.785
2. 25 Myopes......................................7.874
3. 26 Hypermetropes.............................7.96

1852—61. L. L. Vallée, Théorie de l'œil. *C. R.* XXXIV, 321-323; 718-720; 720-722; 789-792; 872-876. XXXV, 679-681. LI, 678-680. LII, 702-703; 1020 to 1021. *Mém. des savants étrangers.* XII, 204-264. XV, 98-118; 119-140.

1857. W. Zehender, Über die Brewster'sche Methode zur Bestimmung der Brechungsexponenten flüssiger und festweicher Substanzen. *Archiv für Ophthalmol.* III, 2 S. 99.

1858. N. Lubimoff, Recherches sur la grandeur apparente des objets. *C. R.* XLVII, 24-27. *Ann. de chimie* (3), LIV. 13-27.

1860. Breton, Note sur une propriété du cristallin de l'œil humain. *C. R.* L, 498-499.

1864. Giraud Teulon, Nouvelle étude de la marche des rayons lumineux dans l'œil. *Ann. d'oculistique, 1864.*

— F. C. Donders, *On the anomalies of accommodation and refraction of the eye.* London. p. 38-71.

§11. Blur Circles on the Retina[1]

When light from a luminous point reaches the eye, the part of the bundle of rays that are admitted through the pupil forms a cone of rays beyond it, with its circular base in front and with its apex, representing the image of the luminous point, turned away from the incident light.

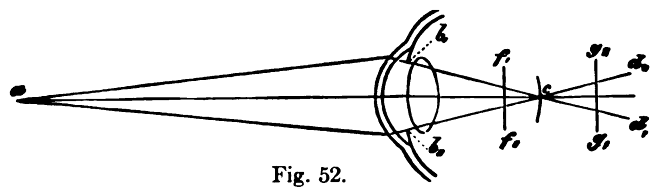

Fig. 52.

Beyond this focus, the rays diverge again. In Fig. 52, the luminous point is supposed to be at *a*, and the pupil is represented by *b,b,,*; the rays converge to the point *c*, *cd,* and *cd,,* being the prolongations of the rays *b,c* and *b,,c*, respectively. When the point of convergence is exactly on the surface of the retina, the luminous point (*a*) affects just this single point (*c*) of the retina, and thus the image of *a* is distinct. But when the retina lies a little in front of or beyond the focus, so as to cut the cone of rays at *f,f,,* or at *g,g,,*, it will be illuminated, not at one point only, but over a small circular area corresponding to the cross section of the cone. An area of the retina thus illuminated by an external luminous point-source is called a *circle of diffusion* or *blur circle*. The circular form corresponds, of course, to the roundness of the pupil. If its form or the base of the cone of incident rays is altered, as may be virtually accomplished by placing just in front of the cornea an opaque screen having a small hole of any desired outline whose diameter is less than that of the pupil, the area of diffusion will assume a correspondingly different shape. If the spot falls on the central part of the retina, it will be

[1] Consult Appendix III at the end of Part I. G.

geometrically similar to the opening. The small diffusion images formed when the focus is close to the retina present, however, striking exceptions to these rules, and will be discussed later in §14.

The production of blurred images may be easily reproduced experimentally by adjusting a small source of light, or, better still, an illuminated pinhole in a screen, at a suitable distance in front of a convex lens, and catching the light on a white card beyond the lens which can be moved to and fro parallel to its axis. It will be found that a distinct, punctual image is formed only when the card is in a certain definite position; elsewhere it expands into a round spot of light or blur circle. When the luminous object is a bright line, as, for example, a narrow illuminated slit, the circles of diffusion from the various points of this line will overlap, as shown in Fig. 53*b*, and there will appear on the screen, in place of the sharp line *a*, a bright figure similar to that of *c*. When the source of light is a clear-cut, uniformly bright area, the centre of the diffusion image will be uniformly bright, but the outer portions will fade and seem to blend gradually into the brightness of the surroundings.

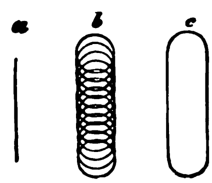

Fig. 53.

Diffusion images of this same sort may be projected on the retina of the eye. Naturally, we cannot move the retina arbitrarily to and fro with respect to the lens like the paper card in the above experiment, but we can move the luminous point nearer the eye or farther from it, so that its image moves back and forth in the vitreous humor. As in the case of any optical system of spherical refracting surfaces, the images formed by the eye of objects at different distances lie at different points along the axis. The image of an infinitely distant point lies in the second focal plane, and that of a nearer point lies beyond it. Therefore, if one of these images lies on the retina, it is sharply defined, while the other necessarily produces a blur circle on it. Evidently, therefore, *objects at different distances from the eye cannot be seen distinctly at the same time.* To verify this, one has simply to hold a veil or some other transparent texture about six inches in front of one eye, and about two feet beyond it an open book. He can see at pleasure either the threads of the mesh or the letters of the printed page with perfect distinctness, but he cannot see both distinctly at once. When the threads stand out sharply, the letters are blurred, and *vice versa*. Moreover, if the eye looks steadily in the same direction, first, at the farther object and then at the nearer, it will be noticed that every such change requires a conscious effort.

The experiment may be varied in many ways. Go to a window, for example, and hold a needle vertically about six inches in front of the eye, so that it appears to cross the horizontal window-bar at right angles. If you look at the needle, you will also see the window-bar as an indistinct dark band, or if you look at the window-bar or at the view outside, you will see the needle simply as a blurred vertical band in the field of view. Again, look through a small hole at the more distant objects beyond, and you may see either those objects or the edge of the hole distinctly, but never both at the same time. The first form of the experiment, however, is the most striking and also the best calculated to make it clear that the phenomenon is not due to any change in the direction of vision.

It is to be noticed, in all these cases, that while one may not see clearly two unequally distant objects at once, still they may be seen distinctly one at a time, and that the transition from one to the other is entirely under the control of the observer.

This peculiar process whereby the eye is enabled to see distinctly objects at different distances is called *accommodation* ~~*or adaptation*~~[1] of the eye to distance.

When the object is far away, a very considerable change of distance will make but slight alteration in the position of its image. Thus, if an eye is accommodated for an infinite distance, the blur circles of object-points even as close as twelve metres say, are so tiny that the distinctness is not seriously impaired. But when the eye is accommodated for near vision, a slight change of distance one way or the other will cause the object to be entirely out of focus. That segment of the visual axis where, for a given state of accommodation, an object can be seen without being indistinct is what J. Czermak has called the "line of accommodation." The length of this segment increases with its remoteness from the eye, and becomes infinite when its distance is very great.

These effects may be easily verified by keeping the eyes fixed on a needle or similar object erected an inch or more in front of a printed page. If the observer moves his eye as near the needle as he can without being unable to see it distinctly, the page appears blurred; but if the observer now moves his eye away from the needle, still fixing it steadily, the printed page becomes clearer and clearer.

The reason we are able to "sight," and to tell whether two points at different distances are exactly in line with the eye, is just because the blur circle of a distant object is very small when the eye is accommodated for another distant object. Strictly speaking, only one of the points sighted can be seen distinctly at one time, while the others

[1] The latter term is no longer used in this sense. G.

appear more or less blurred; and the exact alignment of two points may be obtained by simply getting the sharp image of one point into apparent coincidence with the centre of the blur circle of the other. A line passing through two such apparently coincident points is called a *line of sight*. All lines of sight intersect at one point within the eye, namely, at the centre of the image of the pupil formed by the cornea, known as the *point of intersection of the lines of sight.*[1]

That the change taking place during the process of accommodation is an actual alteration of the optical image itself, and not simply a mode of sensation of the retina, as some physiologists have supposed, may be proved in the most convincing manner by the use of the ophthalmoscope. With this instrument, which will be described in §16, the fundus of the eye can be seen distinctly, including the retina with its blood vessels and the images projected on it. If the patient's eye is fixed on a given object, what we find is that the image of a light at the same distance away will be sharply focused on the retina, and at the same time the veins and other anatomical details of the retina will be clearly visible in the vicinity of this bright image. Now suppose the light is moved much nearer; its image will become indistinct, but the details of the retinal membrane remain as sharply defined as before. Attempts at seeing the changes of the retinal images in dead eyes from which the rear portions of the sclerotica and choroid have been removed or in the living eyes of white rabbits, whose sclerotica is nearly transparent, are both unsatisfactory and likely to fail, because the images thus observed are usually not exact enough to enable the investigator to recognize minute variations in them. Even in a living eye noticeable alterations of the image, due to accommodation, do not occur unless the object itself is comparatively small and precise like a thread or a printed word. Large objects may be still recognized by form even when the accommodation is not correct.

But on the retina of a dead eye all the finer details are effaced, as will be seen by artificially magnifying the image until it looks to the observer as large as it would have seemed to the observed eye, when its retina was still sensitive.

These accommodation phenomena and the varying positions of the ray-focus with respect to the retina are still better demonstrated by an experiment due to SCHEINER. Two pinholes are made in a card at a distance apart less than the diameter of the pupil of the eye. With one eye closed, the observer looks through both holes at a small

[1] ¶This point coincides, therefore, with the centre of the entrance-pupil of the eye. (J. P. C. S.)

[2] Since the act of sighting requires central visual acuity, we are justified in speaking only of one line of sight; which is the one along the ray that after refraction in the eye proceeds to the *fovea centralis*. G.

object sharply delineated against a contrasting background, for example, at a needle held in front of a bright window. The needle should be adjusted at right angles to the line joining the two holes. If the eye is focused on the needle itself, it appears single; but if it is focused on something else, nearer or farther away, the needle appears double. In the former case, one of the openings may be covered with the finger without producing any effect except a darkening of the whole field. But when the needle is seen double, one of the images will disappear when one of the holes is closed. Thus, if the eye is focused on an object beyond the needle, the image that vanishes is the one on the opposite side from the hole that is closed; whereas if the eye is focused on an object nearer than the needle, the image that vanishes will be on the same side as the hole that is closed. If any difficulty is experienced in accommodating the eye without having definite objects to look at, it is a good plan to use two pins, a vertical one about six inches away and a horizontal one about two feet away, and to look at one in order to see the other double, adjusting the card so that the line joining the holes is always placed at right angles to the needle that is to be seen double.

Now suppose three holes are made in the card, all within an area whose diameter is less than that of the pupil; then there will be three images of the needle. If the holes are arranged as in Fig. 54*a*, and if the eye is accommodated for a point nearer than the needle, its three images will appear as shown in *b*, with their knobs in the same relative positions as the holes. But if the accommodation is for a point beyond the needle, the images will be reversed, as in *c*, so that their knobs form an inverted pattern of the holes. It makes no difference whether a needle is used against a bright background, or some bright object against a dark background, or an illuminated hole or slit in an opaque screen; the results are always the same.

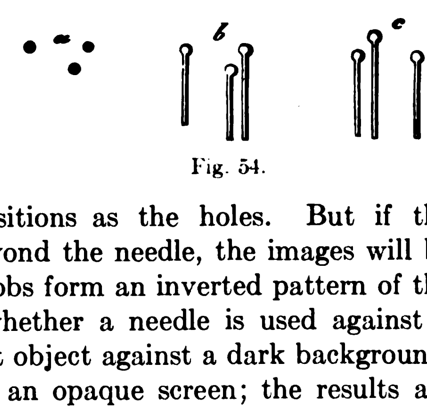

Fig. 54.

The explanation of these effects is easily found by making corresponding experiments with a glass lens. In Fig. 55, *b* represents a convex lens in front of which there is an opaque screen with two holes in it at *e* and *f*. Rays diverging from a luminous point *a* are focused by the lens on the other side of it at the point *c*. Consequently, all the rays of the two bundles that go through the openings *e* and *f* will meet at *c;* and hence if a white card be placed perpendicular to the axis of the lens at *c*, there will appear on it merely a single bright spot, which is the image of *a*. But if the card is moved either way,

so as to come into the position *mm* or *ll*, the two bundles of rays will fall on it separately, and two bright spots will appear.[1] We have only to imagine the glass lens replaced by the optical system of the eye and the white card by the retina. If the two pencils converge on the retina, only one image is seen; if before or beyond it, there are two. The position of the card at *mm* corresponds to the accommodation of the eye for an object farther than *a*, and the position at *ll* for a nearer object. There is one apparent contradiction. In the experiment with the glass lens, when the upper opening *e* is covered, it is the upper spot on *m*, or the lower spot on *l*, that disappears; and this seems at first sight to be exactly the reverse of what occurs in the eye. The discrepancy is easily explained. For it must be remembered that the retinal image is always inverted, and that the image of what appears

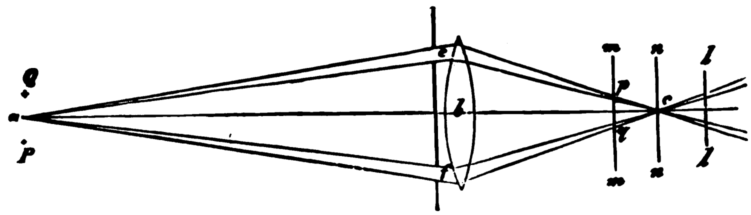

Fig. 55.

to us as the higher of two objects really lies lower on the retina, and *vice versa*. When, therefore, the retina, supposed to be at *m*, intercepts light coming from *a* at two places *p* and *q* on opposite sides of the axis, the impression produced on the observer is that *p*, which is above the axis, is due to an object *P* lying below the axis in the field of view, and that *q*, which is below the axis, is due to an object *Q* lying above the axis. Consequently, if the opening *e* is closed, the effect on the retina at *p* ceases and causes the observer to suppose that the object at *P* has been extinguished. It is just the reverse when the eye is focused for an object nearer than *a*, in which case the retina corresponds to the card placed at *ll*.

Similar results are obtained with a screen perforated with three openings, as in Fig. 54*a*. There will be three bright spots on the white card if it be placed either at *m* or at *l*; if at *m*, the spots will be arranged in the same manner as the pin-holes, if at *l*, in the reverse order. The same apparent contradiction occurs in comparing this with the eye, and is explained in the same manner.

If a screen with one pin-hole in it is moved laterally up or down in front of the glass lens (Fig. 55), the bright spot at *c* will remain stationary when the card is placed at *nn;* but when the

[1] ¶L. D. Weld, Some Precise Methods of Focusing Lenses. *School Science and Math.*, XVIII, p. 547 (1918); in which a modification of this principle is given as a simple method of finding the focus of a lens. (J. P. C. S.)

card is at mm in front of c, the spot moves on the card up or down just in the same way as the opening; and when the card is at ll beyond c, the spot moves up or down just opposite to the way the hole moves. Likewise, if the eye be directed, through a small hole in a card, towards a needle and accommodated for a more distant point in the same line, any slight lateral movement of the card will result in the apparent displacement of the pin in the opposite direction; but if the accommodation be for a point nearer than the needle, the displacement will be in the same direction. These phenomena are easily explained, with reference to Fig. 55, imagining that the perforated screen has only one opening which is situated first at e and then at f.

A screen with a narrow opening in it may be placed in front of the eye to enable it to get a distinct view of an object for which the eye is not accommodated. In such case the cone of rays entering the eye has a very small aperture and a correspondingly small cross section anywhere along its route, so that the blur circles on the retina will be small also.

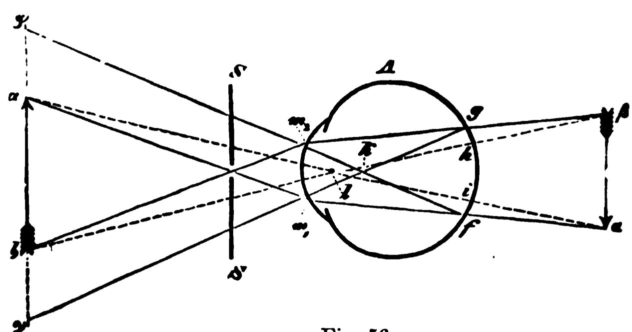

Fig. 56.

An object held close in front of the eye gives a blurred image; but if it is viewed through a small opening, the image gets more distinct, as above stated, and also appears larger. The apparent magnification increases as the opening is moved away from the eye. These phenomena may be explained by reference to the diagram in Fig. 56; where a and b represent two points of the luminous object, SS the perforated screen, and A the eye. The only rays that reach the eye from a and b are am_1 and bm_2, respectively. If $\alpha\beta$ represents the image of ab formed by the optical system of the eye, these rays am_1, bm_2 proceed after their refraction towards α and β, and meet the retina at f and g, respectively. Connect each of these latter points by straight lines with the nodal point of the eye, designated by k; these lines correspond to rays which seem to enter the eye from the points γ and φ where they intersect the plane of the object; and the resultant sensation on the retina is naturally interpreted as corresponding to an object $\gamma\varphi$, larger therefore than the real object ba.

As the perforated screen S is moved from the eye towards the object, evidently the points m_1, m_2, and likewise the lines $m_1\alpha$, $m_2\beta$ that determine the positions of the points f and g on the retina will recede from the axis; and accordingly the retinal image fg becomes larger than before.

If the screen is removed, each point of the object will be reproduced by a blur circle on the retina. The centres of the blur circles corresponding to a and b will then be nearer each other on the retina than f and g, where the images of a and b appeared when the screen was in place. The centre of the blur circle is determined by the chief ray of the bundle of rays, that is, by the ray which goes through the centre (l) of the pupil. The straight lines $a\alpha$, $b\beta$, intersecting at l, meet the retina at i and h, which are, therefore, the centres of the blur circles corresponding to the points a and b when the screen is removed. The distance ih is evidently less than fg.

On the other hand, if one looks through a small opening towards a distant object with the eye accommodated for near vision, the object will appear diminished in size in proportion as the opening is farther from the eye.

The range of distances for which the human eye can accommodate varies greatly with the individual. The limits of this range are called the *near point* and *far point of accommodation*. In normal eyes the near point is usually about four or five inches away, while the far point is very much farther, perhaps sometimes even infinitely far away. However, it seems to be very unusual for the far point to be at an infinite distance, even in the case of persons who live in the open and are accustomed to look only at distant objects. People almost always describe a ray-shaped figure as a star, and the fact that it appears so to most persons indicates that that is the way a star looks to them; which shows that their eyes are not accommodated for infinity, as will be explained in §14.[1]

A *near-sighted* or *myopic* eye is one for which the far point is a short distance away, sometimes only a few inches from the eye; the near point being, of course, even closer. A *far-sighted* or *presbyopic* eye, on the other hand, is one for which the near point is quite a little distance away, perhaps several feet from the eye; but the far point does not generally seem to have receded to the same relative extent, but rather to have remained stationary. Thus, the amplitude of

[1] [The argument here proves nothing. Even if geometrical optics alone were competent to decide the intricate questions that are involved in the appearance of a star, a sufficient answer to the above reasoning would be to remind the reader that the image on the retina is never absolutely punctual in the sense of point-to-point correspondence; and that a luminous point for which the eye was able to accommodate, might easily look like a star. (J. P. C. S.)]

accommodation of the eye is greatly reduced, that is, the capacity of varying its refracting power is perhaps almost entirely gone. In certain exceptional cases where either the eye has become malformed by disease, or the crystalline lens has been extracted in the operation for cataract, the eye is so far-sighted that it is only able to focus on the retina a bundle of converging incident rays and requires, therefore, a weak convex lens to see distinctly an object at infinity. Near-sightedness is usually the result of occupation or habits requiring the close and minute examination of small objects. Far-sightedness is more commonly met with in old age, hence the Greek name presbyopia (from πρεσβυς meaning an old man). Moreover, among sailors, shepherds, hunters and other persons whose attention is concentrated chiefly on remote objects, there are cases where the eye is unable to accommodate for near vision and seems to have become incapacitated by lack of practice. This defect is commonly remedied by the use of spectacles. Concave glasses are used to correct near-sightedness by bringing the image of a distant object so near the eye that it is not farther than the far point. On the other hand, a far-sighted individual needs a convex lens which will produce an image of a near object farther away where the eye can be accommodated to see it.

When the eye is immersed under water, the refraction at the cornea becomes almost negligible, and the crystalline lens by itself is unable to focus the image on the retina; and now the eye is, to so speak, extremely far-sighted and requires a powerful convex spectacle glass to discern clearly anything at all.

In order to calculate the size of a blur circle on the retina, it should be remarked in the first place that all rays which outside the eye are aimed at the apparent pupil (which is the image of the pupil as seen through the cornea)[1] pass through the real pupil after having been refracted at the cornea, and that they proceed in the vitreous humor as if they had come from the virtual image[2] of the pupil in the lens. This follows immediately from the theory of optical images. A given point of the actual pupil and the corresponding point of its image in the cornea are conjugate points so far as the refraction at the cornea is concerned. Rays originating at any point of the actual pupil and emerging from the eye appear to be coming from the image of that point; and, conversely, rays proceeding in the air and converging towards a point of the apparent pupil must, after refraction at the cornea, come to a focus at the corresponding point of the actual pupil.

In LISTING's schematic eye the iris is supposed to be half a millimetre in front of the anterior surface of the lens, and accordingly its

[1] The *entrance-pupil* as it is now called. G.

[2] The *exit-pupil*. G.

image in the crystalline lens is found to be 0.055 mm beyond the iris itself and magnified in the ratio of 16 to 15. If, however, the pupil is supposed to be in contact with the anterior surface of the lens, as is more natural, the enlargement is found to be only about 1/18 (3/53, to be exact), and the image is 0.113 mm beyond the pupil. Retaining the other data of LISTING's schematic eye, we must, therefore, put the distance of the lenticular image of the pupil from the cornea equal to 18.534 mm. On the other hand, this same pupil is magnified by the cornea by 1/7 (13/90, to be exact) and appears to be 0.578 mm in front of its actual position.

The magnitudes of blur circles on the central part of the retina may be found now as follows. In Fig. 57 let the straight line gf represent the optical axis, and let the straight line gq perpendicular to gf

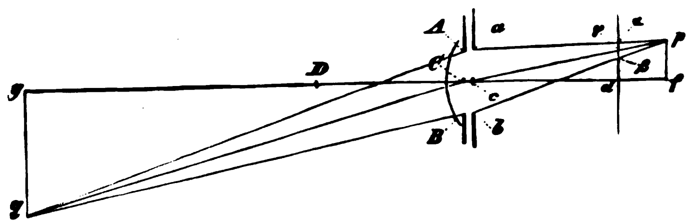

Fig. 57.

represent an object in front of the eye, whose image in the eye is fp. Since only a small portion of the retina in the vicinity of the axis is to be considered, it may be represented by a straight line $d\alpha$ perpendicular to the axis. Moreover, let the image of the pupil in the lens and its image in the cornea be represented by ba and BA, perpendicular to the axis at c and C, respectively. The rays ap and bp, coming from the edge of the (exit) pupil, intersect the retina at α and β, and $\alpha\beta$, therefore, is the diameter of the blur circle on the retina corresponding to the object-point q. By geometry, since ab and αd are parallel,

$$ap : \alpha p = ab : \alpha\beta$$

$$ap : \alpha p = cf : df,$$

and therefore

$$\alpha\beta = \frac{ab \cdot df}{cf} \qquad (1a)$$

If the retinal plane coincides with the second focal plane of the eye, and if D designates the position of the first focal point, as in equation (8), §9, let us put $CD = H_1$, $cd = H_2$, $Cg = h_1$ and $cf = h_2$ (instead of h_{m+1}); so that we may write immediately:

$$\frac{H_1}{h_1} + \frac{H_2}{h_2} = 1,$$

or

$$\frac{H}{h_1} = \frac{h_2 - H_2}{h_2} = \frac{df}{cf},$$

and hence
$$a\beta = ab\cdot\frac{H_1}{h_1}. \qquad (1b)$$

If c designates the centre of the image of the pupil in the lens, that is, if $bc = ca$, the point γ where the straight line cp meets the retina is the centre of the blur circle. For since

$$ac:bc = a\gamma:\beta\gamma,$$
$$ac = bc,$$

consequently,
$$a\gamma = \beta\gamma.$$

Accordingly, the ray that proceeds to the centre of the blur circle is the ray in the vitreous humor that comes from the centre of the so-called exit-pupil of the eye. In the anterior chamber of the eye this same ray goes through the centre of the real pupil, and its prolongation in the air goes through the centre of the so-called entrance-pupil.[1]

One result of this is, that if the centres of the blur circles on the retina corresponding to two points at unequal distances from the eye coincide, the ray proceeding from the centre of the exit-pupil to this common centre on the retina must belong to both bundles of rays. The part of this common ray which lies outside the eye must necessarily pass through both luminous points and its prolongation must go through the centre of the entrance-pupil of the eye. Even if one of the blur circles on the retina contracts to a single point at the centre of the other, the above statements will still be correct.

When two points at different distances are aligned with each other as when we "sight" from one to the other, the image of one falls on the retina at the centre of the blur circle of the other; or, supposing both points are indistinct, the centres of their blur circles coincide. The straight line connecting such a pair of points is what we have already called a *line of sight.* As just explained, it must coincide with the ray that is aimed at the centre of the entrance-pupil, so that this centre is therefore the point of intersection of all lines of sight.

This subject is closely related to the *visual angle.* When it is said that objects subtending equal visual angles have the same apparent size, the common vertices of the angles must be at the point of intersection of the lines of sight. However, it is usual to assume that the vertex of the visual angles is at the point of intersection of the direction lines, that is, at the first nodal point. On the other hand, if the extremities of the object are viewed in succession one after the other, the vertex of the visual angle in this case should be taken at the centre

[1] ¶Although the terms "entrance-pupil" and "exit-pupil" were suggested by Abbe long after the first edition of Helmholtz's *Optik* was published, and, of course, do not occur in the original text, they are employed in the translation. Incidentally, it may be noted here that Huygens in his *Dioptrica* constantly makes use of the same ideas as conveyed by these terms. (J. P. C. S.)

of rotation of the eye. Of course, when the object is very far away, these distinctions do not affect the visual angle, but they are not unimportant when the object is near by.

Appended below is a brief table, computed by LISTING for his schematic eye on the assumption that the retina coincides with the second focal plane of the eye and that the pupil is 4 mm in diameter. In this table l_1 denotes the distance of the luminous point in front of the first focal plane, l_2 the distance of the image beyond the retina, and z the diameter of the blur circle. The calculation is based on equation (8c), §9, namely,

$$l_1l_2 = F_1F_2$$

and on equation (1a) §11. The product F_1F_2 in LISTING's schematic eye is 301.26, or 300 in round numbers, the distances being expressed in mm.

l_1 in metres	l_2 in mm	z in mm
∞	0	0
65	0.005	0.0011
25	0,012	0.0027
12	0.025	0.0056
6	0.050	0.0112
3	0.100	0.0222
1.5	0.200	0.0443
0.75	0.40	0.0825
0.375	0.80	0.1616
0.188	1.60	0.3122
0.094	3.20	0.5768
0.088	3.42	0.6484

This table shows how little the position of the image varies as long as the object is far away, and how rapidly it recedes beyond the retina when the object is near at hand and comes nearer and nearer.

Various instruments called *optometers* have been devised for measuring the range of accommodation of the eye.

Here may be mentioned first the simple, every-day method of distinguishing near-sight and far-sight by finding how far off the letters on a printed page or details of some other suitable object can be seen most conveniently. Of course, no great accuracy can be attained in this way. Printed letters are never so small as not to be legible over a considerable range of accommodation. For example, the author can read print like that on this page at a distance of at least 13 inches, with his eyes accommodated for his far point 3 feet away. Again, not being able to accommodate nearer than 3.6 inches, he can read the

same print when it is only 2.7 inches from his eye. Another thing to be noted here is that the apparent size of an object is bigger when it is brought near the eye, so that, other things being equal, its details can be more clearly made out than when it is farther away. Thus, tiny objects which are hard to see are sometimes examined even nearer than the near point of the eye, because the larger apparent size may more than compensate for the slight indistinctness of the details, and it may be better seen than it could be with exact accommodation under a smaller visual angle. It is evident that in order to use this method for measuring the range of accommodation, we must take different objects for different distances, each of them being just large enough to be recognized distinctly when the eye is accommodated for that particular distance.

PORTERFIELD[1] was the first to propose using SCHEINER's experiment for measuring the range of accommodation distances, and designed an optometer on this principle, which was afterwards improved by THOMAS YOUNG.[2] The latter uses a fine white thread stretched on the black surface of a graduated bar. One end of the thread is close to the eye, which looks along the thread through two small peepholes or slits in an opaque screen, and sees the thread as a pair of straight lines which intersect each other at the point for which the eye is accommodated. This point may be located without difficulty, and its distance from the eye is the distance of accommodation of the eye at that particular instant. The method may be modified by using any other small objects whose distances from the eye can be varied. The objects chosen for the experiment must be small enough to be just distinctly visible through the holes in the screen, as, for example, slender needles against a bright background, or fine holes and slits in dark screens. The eye must be sure to look through both openings at once, otherwise errors are liable to be made. The field of view in these experiments is contracted to the comparatively large blurred images of the two apertures, which partly overlap each other as shown in Fig. 58, *a* and *b*. Double images of a needle, as indicated at *g*, can be seen only in the bright middle region *c* that is common to both openings, but not on either side in any part of the field that corresponds to one aperture alone. In this latter region the image is always single, as at *h*; which is the reason why persons sometimes have difficulty in succeeding with the experiments unless they have practised with the apparatus.

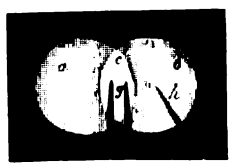

Fig. 58.

Another similar method for finding the range of accommodation and especially for ascertaining the far point is more accurate, in the writer's opinion, than that of looking through two openings. Daylight or light from a candle flame is admitted through a small opening in a screen. To an eye not accommodated for it a small point of light like this looks like a star with five or six rays (see §14 below), whereas, when the eye is properly accommodated, it appears as a fairly well defined bright spot even if it is not uniformly round. Now move a screen slowly from one side in front of the pupil; and what generally happens in this case is that the pattern of light that is seen begins to fade away on one side, and this occurs on the same side as that from which the screen is interposed, provided the eye is accommodated for a point nearer than the luminous point, but on the opposite side if the eye is focused for a point beyond the source of light. But if the eye is accommodated for the luminous point, the image gets uniformly darker all over, or else it is eclipsed

[1] An Essay concerning the Motions of our Eyes. *Edinb. medical Essays.* Vol. I. p. 423. IV. 185 (1747).

[2] *Phil. Transactions.* 1801 P. I. p. 34.

in some irregular way, as for example, by closing in from both top and bottom when the shutter is introduced from one side.[1]

An easier way than SCHEINER's, especially for an inexperienced observer, which depends on the chromatic aberration of the eye, will be described in §13.

RUETE's optometer is designed to insure against intentional deception on the part of the person examined. It is a box-shaped hood through which passes a tube. The patient looks with one eye through the tube at a book, only a few words of which can be seen, and whose distance he has no means of estimating except by the accommodation of the eye itself. Various sizes of type are used and at various distances; and any attempt at deception is practically certain to involve the subject in self-contradictions.

HASNER's optometer consists of a board mounted horizontally on a tripod, provided at one end with a mask fitting over the upper part of the face, in order to hold the eyes steadily in one position. The board has a graduated scale, to measure the distance from the eyes, and another scale also to register the angle of convergence of the two eyes when they are directed towards any point along its length. The instrument is intended to facilitate measurements of the range of accommodation and various experiments concerning the phenomena of single and double binocular vision.

Artificial eye-models for the demonstration of KEPLER's theory of vision and the use of spectacles have been described by HALLER,[2] HUYGENS,[3] WOLF,[4] ADAMS[5] and KRIES.[6]

KEPLER[7] was not only the first to give a correct interpretation of the refraction of light in the eye, but he was also aware of the necessity of an accommodation of the eye for different distances and supposed that blurred vision was due to the lack of proper accommodation. SCHEINER[8] described the phenomena that occur in imperfect accommodation when the observer looks through two holes in a screen. Explanations of these experiments were given by DE LA HIRE,[9] who, however, denied the possibility of accommodation for different distances; and later by J. DE LA MOTTE[10] and PORTERFIELD,[11] the latter of whom also corrected DE LA HIRE's erroneous conclusions. MILE[12] first noted the singular apparent motions of an object lying outside the range of accommodation of the eye, as seen through a narrow aperture which is itself in motion; and these phenomena were afterwards described more fully by H. MAYER,[13] as connected with the theory of accommodation. A complete discussion of the cause of blur circles on the retina, their overlapping, etc., was given by JURIN.[14]

As to the use of spectacles, there is a place in PLINY's *Naturalis historia*[15] which seems to be a reference to them. He mentions certain concave emeralds

[1] ¶The description of these phenomena appears to have been first given by CZERMAK. (J. P. C. S.)

[2] *Elem. Physiolog.* V. 469.

[3] *Dioptrica.* Lugduni 1704. p. 112.

[4] *Nützliche Versuche.* III. 481.

[5] *Essay on vision.* London 1792.

[6] German translation of the preceding essay. Gotha 1794.

[7] *Paralipomena.* p. 200.

[8] *Oculus.* p. 37 and 41. Similar experiments, p. 32 and 49.

[9] *Journal des Sçavans,* 1685; and in *Accidens de la vue.* 1693.

[10] *Versuche und Abhandl. der Gesellschaft in Danzig.* Bd. II. S. 290.

[11] *On the eye.* Vol. I. Book 3. Chap. 3.

[12] POGGENDORFFS *Ann.* XLII. 40.

[13] *Prager Vierteljahrsschrift.* 1851. Bd. IV. S. 92.

[14] *Essay on distinct and indistinct vision.* SMITH's *Optics.* Cambridge 1738.

[15] L. XXXVII. c. 5.

that had a curious property of "converging the vision" (*visum colligere*), and for that reason should never be cut. The emperor NERO, who was near-sighted (PLINY, I. II, c.34), is said to have viewed gladiatorial combats through an emerald of this kind. Subsequently, from the beginning of the 14th century we come across accounts in which spectacles are mentioned as a new discovery. A Florentine nobleman, SALVINUS ARMATUS, who died in 1317, is referred to on his epitaph as the inventor of spectacles.[1] ALEXANDER DE SPINA, a monk of Pisa, who died in 1313, is said to have seen a pair of spectacles worn by somebody who made a secret of the invention; but they were imitated and used by many persons.[2] MAUROLYCUS (1494 to 1575) afterwards tried to explain the theory of such lenses, but the explanation, like his theory of vision, was incorrect. He supposed that the visual rays, each emanating from a separate part of the object, are made more convergent or more divergent by the lens, as in fact is the case only for rays proceeding from a single point of the object; and it was not until KEPLER began to study the question that a correct and complete theory of the use of spectacles was published.[3]

Literature:
1575. FR. MAUROLYCUS, *De lumine et umbra.* Lib. III.
1583. J. B. PORTA, *De refractione.* Lib. VIII.
1604. J. KEPLER, *Paralipomena ad Vitellionem.* p. 200.
1619. SCHEINER, *Oculus.* p. 32–49.
1685. DE LA HIRE, *Journal des Sçavans.* Ann. 1685. — *Accidens de la vue* 1693. §II. (Consequences of SCHEINER's experiment.)
1709. DE LA HIRE, *Mém. de l'Acad. de Paris.* An.1709. p. 95 (Vision in water).
.... DE LA MOTTE, *Versuche und Abhandlungen der Gesellschaft in Danzig.* Bd. II. S.290. (Theory of SCHEINER's experiment.)
1738. JURIN, *Essay on distinct and indistinct vision in* SMITH, *System of optics.* Cambridge 1738.
1759. PORTERFIELD, *On the eye.* p. 389–423.* (Theory of SCHEINER's experiment.)
1792. G. ADAMS, *An essay on vision.* London. 2d edition, translated in German by F. KRIES. Gotha 1794. (Particularly about spectacles.)
1800. J. BISCHOFF, *Praktische Abhandlung der Dioptrik.* Stuttgart. 2. Aufl. (About spectacles).
1801. TH. YOUNG, *Philos. Transact.* P. I. p. 34. (Optometer.)
1810. GILBERT in his *Annalen d. Physik.* XXXIV. 34 and XXXVI. 375. (Vision in water.)
.... WOLLASTON, Improved periscopic spectacles. *Phil. Mag.* XVII. NICHOLSON's *Journal,* VII. 143. 241.
.... JONES *on* WOLLASTON's *spectacles.* NICHOLSON's *Journal.* VII. 1902 and VIII. 38.
1821. G. TAUBER, Anweisung für auswärtige Personen, wie dieselben aus dem optisch-okulistischen Institute zu Leipzig Augengläser bekommen können. Leipzig. 3 Aufl.
1824. MUNCKE, Über Sehen unter Wasser. POGGENDORFFS *Ann.* II. 257. GEHLERS *physik. Wörterbuch,* neu bearb. Leipzig 1828. Art. Gesicht. S. 1383–1386.* About spectacles, see also 1403–1410.*
1825. PURKINJE, *Zur Physiologie der Sinne.* II. S. 128.*
1830. HOLKE, *Disquisitio de acie oculi dextri et sinistri in mille ducentis hominibus.* Lipsiae.
1837. J. MILE in POGGENDORFFS *Ann.* XLII. S. 51.*
1840. HENLE in J. MÜLLERS *Lehrbuch der Physiologie.* Bd. II. S. 339–341.*

[1] VOLKMANNS *Nachrichten von Italien.* Bd. I. S. 542. The tablet in the Church of Mary Magdalene at Florence has since been removed, but it read:

Qui giace Salvino degli Armati
Inventore degli Occhiali.
Dio gli perdoni le peccata.

[2] SMITH's *Optics.* *Remarks* p. 12.

[3] *Paralipomena.* p. 200.

1845. YOUNG'S *optometer*. *Phil. Mag.* XXVI. 436.
1850. J. CZERMAK, *Verhandl. d. Würzburger physik. Gesellschaft*. Bd. I. S. 184.
1851. PEYTAL, Nouvel instrument à l'usage de la vue myope. *Institut*. No. 841. p. 53. No. 857. p. 180.
H. MAYER, *Prager Vierteljahrsschrift für prakt. Heilkunde*. XXXII. S. 92.*
v. HASNER, ibid. S. 166. (Optometer.)
1852. TH. RUETE, *Der Augenspiegel und das Optometer*. Göttingen. S. 28.*
1854. JO. CZERMAK, *Wiener Sitzungsberichte*. Bd. XII. S. 322.*

Supplement

The theory of the individual variations of the refraction of the eye has been very thoroughly worked out by DONDERS in his valuable treatises on this subject, which has resulted already in the most useful applications in ophthalmic practice, not simply directly for the correction of defective vision by means of glasses, but also indirectly for the relief of a whole series of obscure ailments that are due to imperfect refraction and accommodation.

The great contribution made by DONDERS consists particularly in the distinction which he has made between the phenomena pertaining to *abnormal degrees of refraction* when the eye is passive and relaxed and those which depend on the amplitude of *accommodation* and hence on refractive conditions brought about by muscular effort.

The idea that the passive eye is accommodated for far vision, which is very decidedly supported by the evidence of the subjective sensation, and which is also at the basis of the argument as presented above, is carried still further by DONDERS, who has pointed out that, when the sphincter of the pupil and the accommodation have been paralyzed by certain mydriatics, particularly by atropin (the alkaloid of belladonna), the eye is automatically adjusted for its far point, without being able to alter its state of refraction. If there were some muscular mechanism whose contraction could reinforce the accommodation for far vision, we should be forced to admit the very unlikely assumption that this muscle had not been paralyzed by the atropin but was subjected to a sort of permanent spasm.

Moreover, pathological observations indicate that whenever the mechanism of the eye becomes affected by paralysis of the *nervus oculomotorius*, the eye invariably is focused all the time for its normal far point. And there is no case on record to show that when the movements of the eye are paralyzed the far point had come nearer the eye.

Accordingly, the maximum distance of accommodation occurs when the eye is relaxed. The normal position of the far point may be considered as infinitely distant.[1] DONDERS uses the term *emmetropic*

[1] ¶The author here records a change in his views on this subject, as previously expressed. (J. P. C. S.)

(from ἔμμετρος, *modum tenens*, and ὤψ, *oculus*) to describe an eye of this sort, so as to avoid the vagueness of the word "normal" and the expression "normal vision." An emmetropic eye may be subject to various defects, and is not necessarily "normal" by any means.

If the far point of the eye is at a finite distance in front of it, it is said to be *brachymetropic* or *myopic*, the latter being the more ancient name for this state of refraction. A myopic eye cannot focus rays on the retina unless they were originally divergent. On the other hand, an eye which can focus on the retina not simply parallel but even convergent incident rays is said to be *hypermetropic*.[1]

Without the aid of spectacles, a myope cannot focus for distant objects; and so he is at considerable disadvantage as compared with an emmetrope. On the other hand, an hypermetrope has always to exert some accommodation in order to see any actual object distinctly, which frequently results in numerous, sometimes distressing, symptoms of fatigue. Both defects therefore impair the efficiency of vision, and hence DONDERS classified them together under the name *ametropia*.

More often than not these defects are due to abnormal length of the axis of the eye,[2] which is shorter in an hypermetropic than in an emmetropic eye. The matter appears to be closely connected also with the position of the centre of rotation of the eye, which is too far back in a myopic eye and too far forward in an hypermetropic eye. As a general thing there are no abnormal curvatures of the cornea and crystalline lens that can account for ametropia.[3]

In order to ascertain the exact condition of an ametropic eye, it is necessary to determine the range of adjustment that can be produced in its refraction by active muscular effort. For example, if we compare an emmetropic eye which can be accommodated for objects anywhere from an infinite distance to six inches with a very myopic eye whose range is only between six inches and three inches, naturally, it would be inferred at once that the latter has a much narrower margin of accommodation than the former. But suppose now that the myopic eye uses a concave spectacle lens six inches in focus, so as to enable it to see clearly to infinity; this eye has then the same actual range of accommodation as the other, namely from six inches to infinity. The action of the spectacle lens in this illustration is to form a virtual image nearer than the actual object, so that an object six inches away, for instance, appears to be only three inches, and thus

[1] ¶The far point of an hypermetropic eye is at a finite distance behind or beyond the eye, that is, its far point is "virtual." (J. P. C. S.)

[2] ¶"Axial ametropia." (J. P. C. S.)

[3] ¶"Curvature ametropia," whereas abnormalities of the indices of refraction of the ocular media are included under what is called "indicial ametropia." (J. P. C. S.)

is brought to the near point where it can be seen distinctly by the myopic eye here supposed.

Accordingly, the interval between the far point and the near point does not afford a method of comparison of the amplitudes of accommodation of two far-sighted eyes; and in order to compare them with each other we must first equalize their refractions by the use of suitable lenses.

If a spectacle glass is not to magnify or reduce the objects viewed through it, its second principal point should coincide with the first nodal point of the eye.[1] This could be accomplished, if it were worth while, by using a thick meniscus lens (see latter part of §9). Let F and N denote the distances of the far point and near point of the eye, and A the distance of the nearest point for which the eye can accommodate when provided with a lens of negative focal length F, all these distances being measured from the first nodal point of the eye; then

$$\frac{1}{A}=\frac{1}{N}-\frac{1}{F}$$

and the amplitude of accommodation, according to DONDERS, is measured by the reciprocal of A.[2]

The amplitude of accommodation, therefore, is measured in units of curvature, that is, in reciprocals of the units of length. The unit of length corresponding to the spectacle numbers that are in vogue at present is the inch, either Paris inch or Prussian inch; and we might call the reciprocal unit a "Zolltel."[3] For instance, an emmetropic eye

[1] ¶The statement in the text is not entirely clear: no actual spectacle glasses are used in this way. The condition that the so-called "spectacle magnification" shall be unity requires that the second principal point of the correction glass and the first focal point of the eye shall be coincident; which until quite recently was supposed to be the proper adjustment of the glass in front of the eye; but in modern "Punktal" lenses and similar types of spectacle glasses, this condition is not fundamental and is only approximately satisfied. (J. P. C. S.)

[2] ¶The amplitude of accommodation, as defined by DONDERS, is usually given by the formula

$$A=\frac{1}{p}-\frac{1}{r},$$

where p and r denote the far and near point distances, respectively, both measured from the first principal point of the eye, and A (not the reciprocal of A) denotes the amplitude of accommodation. According to this expression, the amplitude of accommodation of the eye is equal to the refracting power of a thin lens which when placed at the first principal point of the eye would produce by itself an image of the far point at the near point. (J. P. C. S.)

[3] ¶Professor L. D. WELD suggests the term "reciprocal inch." Opticians have long since adopted the "dioptry" for all such purposes, corresponding to the metre as unit of length; and the use of "Kilodioptry," "Hektodioptry," etc., corresponding to millimetre, centimetre, etc., respectively, has also been advocated. It is indicative of the author's singular perspicacity that he recognized immediately the need of units of this kind, as soon as he became familiar with DONDERS' method of measuring the amplitude of accommodation and similar magnitudes. (J. P. C. S.)

with a range of accommodation from infinity to 6 inches, a myopic eye with a range from 6 inches to 3 inches, and an hypermetropic eye with a range from 12 inches behind the eye to 12 inches in front of the eye, all have equal amplitude of accommodation, namely 6 units, because

$$\frac{1}{6}-\frac{1}{\infty}=\frac{1}{3}-\frac{1}{6}=\frac{1}{12}-\left(-\frac{1}{12}\right)=\frac{1}{6}.$$

The amplitude of accommodation continually diminishes with advancing years, at a rate which for emmetropic or nearly emmetropic eyes is approximately proportional to the age, being about 14 dioptries at 10 years,[1] and practically nothing at the age of 65. Loss of the power of accommodation, therefore, occurs naturally in old age, and Donders confines the term *presbyopia* to this state. But it should be remarked that from about 50 years of age the far point also recedes from the eye, and an eye that was emmetropic in youth will eventually become hypermetropic, and an eye that was slightly myopic may in the same way get to be emmetropic.

The gradual decrease of the amplitude of accommodation is probably due to an increase of rigidity of the external layers of the crystalline lens, the lens itself becoming less flexible. Moreover, according to the discussion in §10, an increase of the indices of refraction of the outer layers of the lens must produce a decrease of the refracting power of the lens and consequently cause the second focal point of the eye to be moved farther back.

Another thing to be mentioned is that generally the efforts of convergence and accommodation are made simultaneously, and hence quite involuntarily there is some definite connection between the two innervations. Anyone who has not practised the voluntary control of his accommodation finds it much easier to accommodate for distant binocular vision when the axes of the two eyes are parallel and to make the greatest effort of accommodation when the eyes are converged. Donders, therefore, differentiates between:

(1) The *absolute amplitude of accommodation*, in which the far point is located with the axes of the two eyes parallel or even divergent, and the near point with the axes as convergent as possible. The near point of accommodation is farther off than the near point of convergence. This is the utmost possible amplitude of accommodation, and in the case of a certain young emmetrope 15 years old was found to be as much as 1/3.69 "Zolltel."

[1] The text reads here "3/8 Zolltel" instead of "14 dioptries." (J. P. C. S.)

(2) The *binocular amplitude of accommodation.* In this case the convergence is not made any stronger than is necessary to fixate the point for which the eyes are accommodated. The power of accommodation thus attained is not quite as great as in the first case, the amplitude of binocular accommodation for the same individual being 1/3.9.

(3) The *relative amplitude of accommodation* for a given degree of convergence. For the same individual, with the axes of the eyes parallel, this quantity was only 1/11, reaching its maximum value 1/5.76 for a convergence of 11°, and changing more and more slowly as the convergence increased. At 23° it was 1/6.4; with 38° convergence at the binocular near point, it was 1/9; and at the absolute near point, with 73° convergence, it became zero.

In ophthalmic practice, therefore, it is necessary to choose some definite degree of convergence in order to make comparisons of powers of accommodation, and suitable lenses must be used to try to enable the patient to accommodate with this degree of convergence.

For the determination of the far point, the most appropriate convergence is zero, the eyes being directed to a very distant object. The distance of the far point is directly equal to the focal length of the weakest concave lens in the case of a myopic eye, or of the strongest convex lens in the case of an hypermetropic eye, with which it is possible to see a very distant object with perfect distinctness. For the determination of the near point, DONDERS recommends using a convex lens of such power as will bring this point invariably to about 8 inches from the eye (supposing it to be naturally farther away than that), in order to insure a sufficient effort of accommodation. The effect produced by the lens must, of course, be taken into account in the subsequent calculation.

Printed letters and figures of various sizes serve very well as test objects in measuring the refraction of the eye of an inexperienced patient.[1]

On the whole it is advisable to employ suitable glasses in cases where the eyes have not sufficient accommodation for the occupation of the patient. Presbyopic eyes require a convex lens for reading or writing, and for near objects generally, in order to reduce the size of the blur circles. A stronger glass is needed in the evening or in weak illumination, when the pupil is wide open and the blur circles

[1] The following, for example, have been published: *Schriftskalen*, by JAEGER, Jr., of Vienna, 1857; and SNELLEN'S *Test types for the determination of the acuteness of vision:* London, WILLIAMS & NORGATE; GERMER BAILLIÈRE of Paris, PETER'S of Berlin and GREVEN of Utrecht. In the latter the type are arranged in a graduated scale of sizes and for each size the number of Paris feet is indicated, at which a normal eye should be able to read it. Similar charts have been made also by GIRAUD TEULON (Paris, NACHET).

consequently larger, than is necessary in daylight or with bright illumination. The general rule is to use a lens which will bring the near point within 10 or 12 inches of the eye; except that for very old people, between 70 and 80 years of age, whose visual acuity is considerably reduced it is desirable to bring the image to 8 or 7 inches so as to increase the apparent size of the object.

It is particularly important to prevent myopes from holding the head down to look at near objects and from converging their eyes too much, because the stretching, pulling and distension of the membranes at the back of the eye, which thus result from the increased blood-pressure and muscular strain, soon get worse, and the increased myopia may seriously impair the vision and be dangerous. In the milder degrees of near-sightedness for which the distance of the far point exceeds 5 inches, it is generally permissible to use concave spectacles, and wear them constantly, that have the effect of removing the far point to infinity. This virtually transforms the myopic eye into an emmetropic one. The patient must, however, never hold books, writing, sewing, etc. nearer than 12 inches from the eye. If the eyes are otherwise in good condition, it is possible to read and write without difficulty at this distance; but if it becomes necessary to engage in work requiring closer examination, the patient should employ weaker concave, or perhaps even achromatized prismatic, lenses (the latter being thicker on the side towards the nose), so that very near objects may be seen with less effort of accommodation and convergence.

Near-sighted people who have never worn spectacles must sometimes become accustomed to them gradually, in order to allow the connection between accommodation and convergence to adjust itself to the new conditions. Weaker lenses are provided at first, and after the patient has had some experience with them, stronger ones may be substituted which will completely neutralize the myopia. In the case of shorter ranges of accommodation or of considerably reduced visual acuity, it will usually be found better to use weak lenses for viewing the ordinary near-by objects and to have a lorgnette for seeing at a distance.[1]

In the higher degrees of myopia the eye is usually painful and dangerously affected. There are numerous other considerations that cannot be discussed here, and a competent oculist should by all means be consulted. The general indifference of near-sighted people to the condition of their eyes is responsible for many subsequent disorders and much blindness, and the patient cannot be too earnestly warned against such neglect.

[1] ¶ Bifocal lenses are now generally employed instead. (L. D. W.)

Hypermetropic eyes require convex glasses. It is a good plan to give a patient of this kind at first a glass that is a little too strong for him, so that he cannot see a distant object distinctly with it until he has learned how to relax the accommodation that he has always been in the habit of using. In proportion as he ceases to exert accommodation, the strength of the lens should be increased. As the amplitude of accommodation diminishes, an hypermetrope requires stronger convex glasses for near vision and weaker lenses for far vision. Thus, by proper choice of spectacles, the very severe disability due to continual strain of accommodation can be entirely relieved, and one of the greatest triumphs, in a practical way, that have been achieved by the new ophthalmology is this simple remedy that can now be used in the most obstinate cases of asthenopia resulting from far-sightedness, which used to be the despair of both patient and oculist.

1855. Stellwag v. Carion, Die Akkommodationsfehler des Auges. *Wiener Sitzungsber.* XVI. 187.

— Czermak. Akkommodationslinien. Ibid. XV. 425, 457.

1856. A. v. Graefe, Über *Myopia in distans* nebst Betrachtungen über das Sehen jenseits der Grenzen unserer Akkommodation. *Archiv für Ophthalmol.* II, 1, p. 158-186.

1857. J. J. Oppel, Über das Sehen durch kleine Öffnungen und das Geramsche Diaskop. *Jahresber. d. Frankfurter Vereins.* 1856-1857. p. 37-42.

1858. F. C. Donders, Winke betreffend den Gebrauch und die Wahl der Brillen. *Archiv für Ophthalmol.* IV, 1, 286-300.

1859. M. Mac-Gillavry, *Onderzoekingen over de hoegrootheid der accommodatie.* Dissertat. Utrecht 1858. Henle u. Pfeufer. *Zeitschrift für ration. Medicin.* (3) VI, 612-613.

1860. F. C. Donders, Beiträge zur Kenntnis der Refraktions- und Akkomodationsanomalien. *Archiv für Ophthalmol.* VI, 1. S. 62-105. VI, 2. S. 210-283. VII, 1. p. 155-204. *Verslagen en Mededeelingen der K. Acad.* Amsterdam 1861. p. 159-201. *Jaarlijksch Verslag betrekkelijk het Nederlandsch Gasthuis voor Ooglijders.* I, 63-205. II, 25-68. IV, 1-118.

— C. Landsberg, Beschreibung eines neuen Optometers und Ophthalmodiastometers. Poggendorffs *Ann.* CX, 435-452. *Polytechn. Zentralbl.* 1860. p. 405-406.

— A. Burow, Über den Einfluss peripherischer Netzhautpartien auf die Regelung der akkommodativen Bewegungen des Auges. *Archiv für Ophthalm.* VI, 1, 106-110.

1861. Ch. Aeby, Die Akkommodationsgeschwindigkeit des menschlichen Auges. Henle u. Pfeufer. *Zeitschrift.* (3) XI, 300-304.

— Giraud Teulon, Des mouvements de décentration latérale de l'appareil cristallin. *C. R.* LII, 383-385. *Inst.* 1861. p. 82. *Cosmos.* XVIII, 284-286.

— H. Dor, Des différences individuelles de la réfraction de l'œil. *J. d. la Physiologie.* XI, XII. *Arch. d. sciences phys.* (2) X, 82-85.

— H. de Brieder, *De stoornissen der accommodatie van het oog. Dissertat.* Utrecht. — *Jaarlijksch Verslag betr. het Niederl. Gasthuis.* II, 69-142.

— v. Jaeger Jr., *Über die Einstellungen des dioptrischen Apparats im menschlichen Auge.* Wien 1861.

— Stellwag v. Carion, Zur Literatur der Refraktions- und Akkommodationsanomalien. *Zft. d. K. K. Ges. d. Ärzte.* 1861.

1862. de Haas, *Geschiedkundig onderzoek omtrent de Hypermetropia en hare gevolgen. Dissert.* Utrecht. — *Jaarlijksch Verslag betr. het Nederl. Gasthuis* III, 157-208.

1863. A. Burow, Vorläufige Notiz über die Konstruktion eines neuen Optometers. *Archiv für Ophthalmol.* IX. 2, 228-231.

— A. Burow, *Ein neues Optometer.* Berlin 1863.
— Idem, *Über die Reihenfolge der Brillenbrennweiten.* Berlin 1864.
— A. v. Graefe, Ein Optometer. *Deutsche Klinik.* 1863. S. 10.
1864. F. C. Donders, *On the anomalies of accommodation and refraction of the eye.* London. pp. 1–635.
1865. E. Javal, Une nouvelle règle à calcul. *Ann. d'ocul.* Bruxelles. LIII, 181.
1866. J. W. Verschoor, Optometers en Optometrie. *Zesde Jaarlijksch Verslag van het Nederl. Gasthuis voor Ooglijders.* p. 97–160.

§ 12. Mechanism of Accommodation[1]

The known changes that occur in the different parts of the eye during the accommodation are as follows:

1. The pupil contracts in accommodation for near vision and dilates for far vision. This change is readily observed and has long been known. It is to be observed in every eye which alternates between near and far vision in the same direction. A lasting contraction of the pupil due to too long exposure to an excessive degree of illumination must, however, be avoided.

2. The pupillary margin of the iris and the centre of the anterior surface of the lens move forward slightly in incipient accommodation for near vision. In order to observe this, a well-defined distant fixation point is chosen, and a needle point interposed as the object for near vision. With one eye occluded, the point of the needle is brought into exact alignment with the distant fixation point. The eye must be kept steadily focused in this position and not allowed to wander to one side, because the success of the experiment depends entirely on keeping the direction of the eye unchanged. The observer is placed so as to look at the cornea of the subject's eye from one side and slightly from behind, where he can just see the black pupil of the eye protruding about halfway in front of the corneal margin of the sclerotica when the eye is focused on the distant object. If the subject then fixes his vision on the nearer object, that is, on the needle point, the observer will notice immediately that the black oval of the pupil, and perhaps also a portion of the margin of the iris nearest him, becomes visible in front of the sclerotica. Fig. 59*a* shows the appearance of the eye for the case of far vision, and Fig. 59*b* its appearance for near vision. The alteration in the position of the black spot becomes

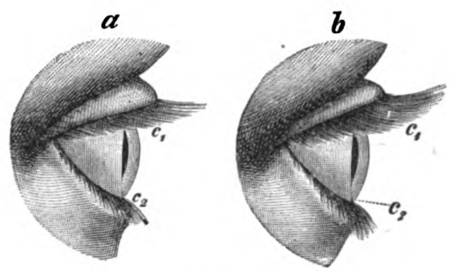

Fig. 59.

[1] See also Appendix IV to Part I. G.

most noticeable by watching the width of the bright interval between it and a darker band c_1c_2 that appears on the anterior edge of the cornea. This band is the distorted image of the margin of the sclerotica projecting beyond the iris on that side of the eye as made by the refraction through the cornea. As its inner surface is usually in shadow, it looks darker than the iris, which is illuminated from in front. When accommodation for near vision begins, the interval between this strip c_1c_2 and the dark pupil becomes perceptibly smaller. If the pupillary margin did not move forward, this interval would, on the contrary, become wider in near vision, because the pupil contracts uniformly along all diameters. It would become wider likewise if the forward movement of the pupil occurred as a result of an accidental turning of the subject's eye towards the observer. Therefore, by watching the dark band above mentioned, there is no possibility of any deception. It was pointed out in §3 that the anterior surface of the lens is always in close apposition to the pupil.

3. The anterior surface of the crystalline lens becomes more convex in near vision, less convex for far vision. This fact can be demonstrated by the reflex in the anterior surface of the lens. As in the previous experiment, two well-defined objects serving as fixation points have to be aligned by the eye. The room must be completely dark and, except for a large bright flame, which is adjusted on a level with the eye to one side of the line of sight, no larger or brighter object should be in front of the eye, so as to prevent all disturbing corneal reflections. In

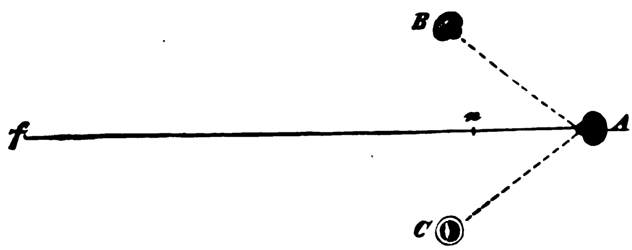

Fig. 60.

Fig. 60, the eye of the subject is supposed to be at A, a cross section of the flame is shown at C, and the nearer and farther points of fixation are designated by n and f, respectively The observer must now adjust his own eye (B) on a level with the eye of the subject and place the lamp so that the angle BAf is about equal to the angle CAf. He moves his eye back and forth in the neighbourhood of B until he catches the reflexes from both surfaces of the lens. These images (Fig. 61, b and c) are not as bright as the reflex from the cornea (a). The image (b) in the anterior surface of the lens is erect and somewhat larger than that in the cornea, but it is usually so faint that it is almost impossible to recognize the exact form of the flame. Its apparent position is far behind the pupil (8 or 12 mm). Consequently,

it vanishes behind the edge of the iris on the slightest movement of either the observer's eye or of the light. This image will be called the *first* lens reflex, to distinguish it from the *second* lens reflex in the posterior surface. The latter (Fig. 61, *c*) is inverted and much smaller than the corneal reflex and the first lens reflex, and therefore it appears as a bright, fairly well defined little spot. Its apparent position is just beyond the surface of the pupil, about a millimetre away. Hence, its displacement with respect to the pupil and the corneal reflex is comparatively slight when the observer moves his head. When the patient's eye is accommodated for near vision, the first lens reflex becomes considerably smaller and usually approaches the centre of the pupil. The reduction in the size of the image may be seen best by using a shield, instead of a flame, with two openings in it in a vertical line which are both illuminated from behind. The same result can be obtained by using a plane horizontal mirror adjusted under a flame so that the flame and its image in the mirror are like two similar sources of light. Each of the reflex images in the eye will appear then as a pair of luminous spots, and the way in which the pair belonging to the anterior surface of the lens approach each other or separate, according as the eye looks at the near object or the far object, may be easily and clearly seen. In Fig. 62, the reflexes for far vision are illustrated in *A*, and for near vision in *B*; the corneal reflex is marked *a*, the first lens reflex *b*, and the second lens reflex *c*. The source of light in these diagrams is supposed to be a pair of rectangular slits illuminated from behind.

Fig. 61.

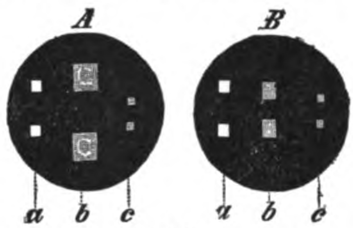

Fig. 62.

The size of the image of a given object in a convex mirror is proportional to its radius, and consequently the experiment above proves that the anterior surface of the crystalline lens becomes more convex in accommodation for near vision. It is true that a very slight reduction in the size of the reflex would occur as a result of the refraction of the rays in the cornea if the anterior surface of the lens merely approached the cornea without altering its convexity, but a simple calculation is sufficient to show that the reduction in the size of the mirror image from this cause would be extremely insignificant as compared with what actually occurs.

4. The reflex in the posterior surface of the lens likewise becomes somewhat smaller in accommodation for near vision. In order to verify this, more exact methods of observation have to be employed,

which will be described in a supplement to this section. These methods show that the apparent position of the posterior surface of the lens (as seen through the lens and cornea) is not appreciably altered. As the apparent position of the posterior surface of the lens is but slightly different from its actual position, and as the shifting of the cardinal points of the eye during accommodation, as will be pointed out presently, is of a kind that is calculated to have at least a partly counteracting influence on this apparent position, it may be assumed that the actual position of the posterior surface of the lens is not appreciably altered in accommodation. The shifting of the cardinal points has also a partially counteracting influence on the size of the second lens reflex. However, it may be demonstrated that no assumption as to possible variations of the optical constants is sufficient to account for the amount of reduction in the size of the image in near vision that is actually observed. It may, therefore, be inferred that the posterior surface of the lens also becomes more convex in near vision, but only to a slight extent.

As these observations indicate that, in addition to the changes of curvature, the anterior surface of the lens must become thicker in the middle in near vision; and since its volume cannot alter, it must be also inferred that the equatorial diameter of the lens is shortened.

In the cross section of the anterior portion of the human eye shown in Fig. 63, the cornea and lens are drawn on a 5 to 1 scale from actual measurements of a living eye as made by the author. On the side marked *F*, it shows these structures when accommodated for

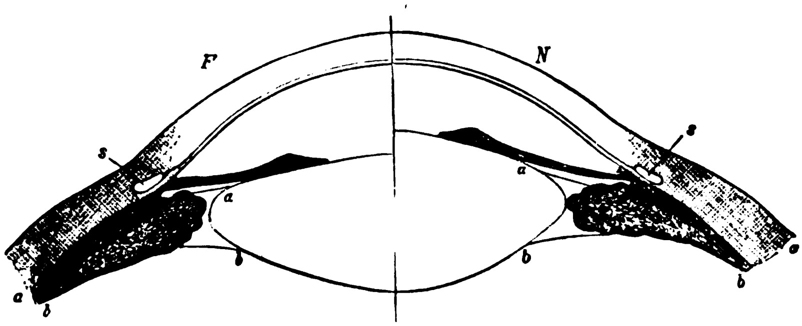

Fig. 63.

distant vision, and on the side *N* for near vision. The ciliary processes in this figure are drawn as if the section passed through the folds of the zonule situated between the processes. This is done so that the relations of the zonule may be seen. The anterior and posterior edges of its folds are marked *aa* and *bb*, respectively.

The effect of the increased convexity of the surface of the lens is to shorten its focal length, its two principal points being shifted

forwards at the same time. This is partly due to the forward displacement of the anterior surface of the lens and partly because the increase of curvature of the anterior surface is greater than that of the posterior. Both circumstances conspire to bring rays, coming from the external source of illumination and converging from the cornea on the lens, to a nearer focus than they would have in the far seeing eye. The magnitude of the changes observed in the lens of a living eye seems also to be sufficient to account for the amplitude of accommodation.

No other changes in the ocular media, which could have an effect on accommodation, have been observed. In particular, the curvature of the cornea remains entirely unaffected throughout the process. On the other hand, it is conceivable that accommodation for near vision might be aided by an elongation of the eyeball as a whole produced by the simultaneous contraction of all six eye-muscles. However, there is no indication of any such process, nor does it seem to be necessary. Furthermore, it seems to be contrary to the results of the author's experiments as described in § 2, which show that a change of pressure in the eye is accompanied by a change in the convexity of the cornea; whereas no such change in its convexity during accommodation has been observed. As a further objection to this conjecture, it might be pointed out that even a slight continuous pressure on the eye reduces the amount of blood in the vessels of the retina and makes it insensitive to light.

As to the mode by which the deformation of the lens is produced, no one has yet been able to answer this question with certainty. Earlier investigators, like THOMAS YOUNG, supposed that the lens is composed of muscle fibres which was therefore called the *musculus crystallinus*. Even if the fibres of the lens could possibly be considered homologous to muscle fibres, in spite of their entirely different form, no nerve fibres run to the lens. Their presence in such transparent structures as are here under consideration could hardly have escaped observation. Besides, all experiments in which induced electric current has been applied to fresh animal lenses have failed to produce any change in form, though this produces contraction in all known muscular structures. Such experiments were carried out, for example, by CRAMER[1] on the eyes of freshly killed seals and birds. Changes in the form of the lens were obtained as long as the iris and ciliary apparatus were uninjured, but the extirpated lens failed to give them. In collaboration with v. WITTICH, the author has conducted similar experiments on the lenses of freshly killed rabbits and frogs, with the same negative results.

[1] *Het Accommodatievermogen.* p. 58 and 86.

However, CRAMER found that the changes of accommodation could be reproduced on extirpated eyes if an induced current were passed through the anterior portion of the eye. His experiments may be described as follows: A wooden ring of a proper size was placed upon the stage of a microscope having a flat illuminating mirror, and on this ring the eye of a seal (*phoca littorea*), five weeks old, was placed with the cornea downwards. The animal had just been killed by strangulation. The eyeball was freed from muscles, fat and other adjacent parts, and a portion of the sclerotica, choroid and retina carefully removed from the dorsal surface without injury to the vitreous humor. By ingenious adjustment of the microscope and its mirror, CRAMER was able to observe the image of a candle flame, about 35 cm away, clearly depicted on the posterior surface of the vitreous humor when the latter was viewed under a magnification of 80 diameters. As soon as the current of a little electro-magnetic machine was transmitted between the two sides of the cornea, the image became larger and more indistinct. CRAMER then passed a cataract needle through the edge of the cornea, inserting its point through the pupil behind the iris and cut the iris so that it had a radial cleft which passed from its base to the pupil. After this procedure the electric current produced no change in the image. These experiments were not successful when carried out on the eyes of dogs and rabbits, because right after death the pupil becomes very small, and rather strong electric currents cause the lens to become opaque (probably through electrolysis). CRAMER found that the action of electric currents on the eyes of pigeons changed the reflex on the anterior surface of the lens, but not on the cornea. The change of the image on the anterior surface of the lens could be observed better in extirpated eyes from which the cornea had been removed. The increased convexity of the lens continued as long as the current of the induction apparatus was passed through it and disappeared when it stopped. It could not be produced after the iris had been removed.

CRAMER's conclusions were, primarily, that the form of the lens is altered by contractile structures contained in the eye itself; and, moreover, he regarded the iris as the special organ that was chiefly responsible for these changes. He attributed a marked convexity to the iris because he regarded its origin as being on the inner surface of the *musculus ciliaris* farther back than previous anatomists had supposed. According to his assumption, both the circular and radial fibres of the iris contracted simultaneously to produce accommodation of the eye for near vision. Thus the circular fibres would give the radial fibres a fixed point of attachment at their central end, which, being tense, would exert pressure on the parts behind them (edge of

lens and vitreous body). The result would be a tendency on the part of the very yielding elastic lens to bulge out through the pupil at the only place where there was no pressure against it, and so to become more convex. The contraction of the pupil in near vision would also be explained as due to the compression of the annular muscle of the pupil necessary to give the radial fibres of the iris a hold at the inner end.

DONDERS called attention to the fact that the elastic tissue on the inner wall of SCHLEMM's canal, to which the periphery of the iris is attached, might have some significance in accommodation. The iris and the ciliary muscle are attached in common to this wall of the canal and the muscle fibres pass backwards to be attached to the choroid. With the choroid as its fixed point, any contraction of this muscle might stretch the elastic tissue in the wall of SCHLEMM's canal and thus draw the base of the iris backwards. As a result, it would be in a more favourable position to exert pressure on the structures lying behind it.

Indeed, it is easy to see that the peripheral parts of the iris must go backwards if the centre of the lens and the pupillary margin of the iris move forwards. For the volume of the aqueous humor contained in the anterior chamber of the eye is constant. If it has to give way in the middle on account of the bulging out of the lens, it will be forced to the sides and consequently push back the peripheral parts of the iris.

CRAMER has remarked that the mode of extension of the anterior chamber in near vision in the case of children may be seen with the naked eye. The author has found that this can be observed also in

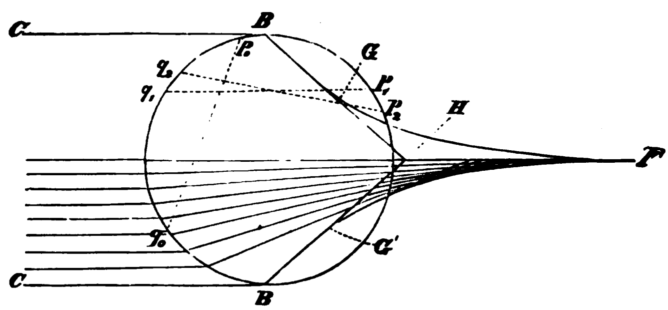

Fig. 64.

adults by means of a special way of illuminating the eye. If the light falls on the eye entirely from the side so that the iris is mostly in shadow, a curved bright band or caustic line can be seen by an observer who adjusts his eye at the proper place on the side opposite the light. In the lower half of Fig. 64 the paths are shown of a pencil of parallel rays refracted at a spherical surface separating air from a medium of the

same index of refraction as the aqueous humor. The focus of the central rays is designated by F. The outer rays do not go through this focus, but intersect the adjacent rays and form thus a caustic surface, a meridian section of which is represented by the arc GF. The outermost ray CB is refracted along BH. The caustic curve GF ends at the point G at the middle of the chord in which the straight line BH cuts the circle. Suppose now that planes are passed through the refracting sphere situated like the iris in the aqueous humor. For example, if the straight line q_0P_0 is the trace of a plane perpendicular to the plane of the diagram, its entire anterior surface would be illuminated with light. But if the plane were passed through q_1P_1, part of it would lie in front of the outside refracted ray BG, and be illuminated. The part beyond it would remain dark. If the plane were passed through q_2P_2, it would cut the caustic surface. Here again a part would be light and a part dark, but the boundary between the illuminated and non-illuminated portions would stand out as a bright line corresponding to the line in which the plane q_2P_2 cuts the caustic surface. It may be seen from the figure that, if the portion of the plane, q_2P_2, which cuts the caustic surface, were to move backwards, that is, away from the refracting surface, the bright line would approach the border.

Now this can be observed on the iris when the eye is accommodated for near vision. If the patient's eye is illuminated from the side so that the caustic curve appears at the ciliary margin of the iris, and if he looks alternately at a near and a far object in the same line, the caustic curve will be seen to approach the margin of the iris on accommodation for near vision and move away from it in far vision. This illumination of the iris is shown in Fig. 65. The light falls on the eye from the side in the direction of the arrow. The corneal reflection of the light is visible at b on the side towards the light. The caustic curve with its light shining partly through the projecting edge of the sclerotica is seen on the other side at a.

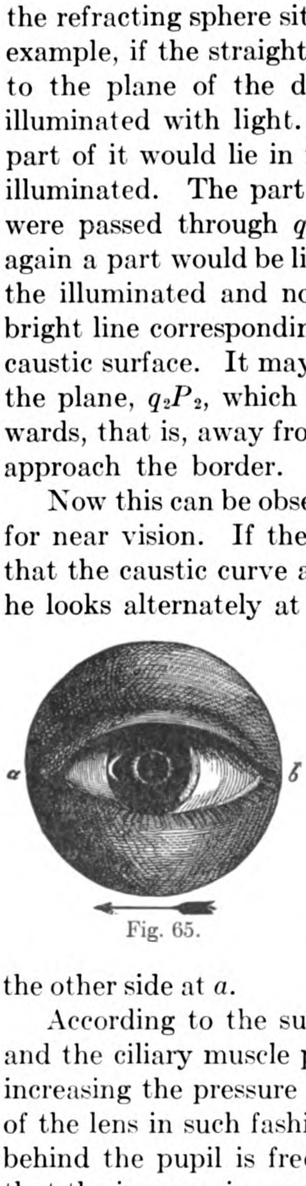

Fig. 65.

According to the suppositions of CRAMER and DONDERS, the iris and the ciliary muscle produce the change of the form of the lens by increasing the pressure in the vitreous body and on the outer margin of the lens in such fashion that only the centre of its anterior surface behind the pupil is free from the increased pressure. It is possible that the increase in convexity of the anterior surface of the lens which CRAMER noticed first might be explained on this basis.

On the other hand, the change of the shape of the lens, as shown by the writer's measurements, cannot be explained without the aid of some other force. The lens could not become thicker in the middle as a result of the hydrostatic pressure acting on its posterior surface and edge. Such a pressure would tend to cause the equatorial plane of the lens to bulge forwards and, consequently, to make its posterior surface flatter.

An assumption which would appear to avoid this difficulty is that in the passive, far seeing state of the eye the lens is stretched by the zonule attached to its edge. The folds of the zonule pass outwards and backwards from their insertion on the capsule of the lens, thereby forming sheaths for the ciliary processes, and are attached to the posterior extremities of these processes and to the ciliary muscle, where they finally coalesce with the hyaloid membrane, retina and choroid. On contraction, the ciliary muscle could pull the posterior end of the zonule forwards nearer the lens and reduce the tension of the zonule. But the effect of the tense zonule is to exert a pull on the lens along its equatorial diameter, and thus to shorten its axis and make its surfaces flatter. If the pull of the zonule is relaxed in accommodating for near vision, the equatorial diameter of the lens will diminish, and the lens will get thicker in the middle, both surfaces becoming more curved. If the pressure of the iris is superadded, the equatorial plane of the lens will bulge forwards, increasing the curvature of the anterior surface and decreasing that of the posterior surface, so that the latter may be approximately equal to its original amount when the lens is accommodated for far vision.

It would seem that the changes in form of the lens could be explained on this basis. Besides, it is relatively easy to change the form of the lens in dead eyes by cutting the zonule. This would also agree with the fact that the writer has found the thickness of the lens of a live eye adjusted for distant vision less than it ever is in the lens of a dead eye. This difference can hardly be attributed to a swelling of the lenses of cadavers by the absorption of water, since, according to W. KRAUSE's observations, the indices of refraction of the outer, middle and inner layers of the lenses from calves 24 hours after death are exactly the same as directly after death. One would expect a decrease of the index of refraction in consequence of the absorption of water.

In order to give a summary of the probable variations of the optical constants and cardinal points of the eye that occur in accommodation for near vision and, at the same time, to show that the observed changes in the form of the lens are sufficient to account for accommodation, the author has calculated the optical constants for two accommodations

of a schematic eye that correspond closely to certain accommodations which he measured. The only difference between the eye as adjusted for far vision and LISTING's schematic eye is that the surfaces of the lens are slightly more to the front, and the lens itself is supposed to be thinner. The indices of refraction of the aqueous and vitreous humors and of the crystalline lens are assumed to have the same values as given by LISTING, namely, 103/77 and 16/11, respectively. Distances are expressed in mm, and the positions of the various points are given by their distances from the vertex of the anterior surface of the cornea. On the assumption that when accommodated for distance this schematic eye can see distinctly at infinity, the axial point of the retina will be 22.231 mm beyond the anterior surface of the cornea. The calculation for the other state of accommodation is based on the assumption that the image of an object 118.85 mm in front of the first focal point of the eye, or 130.09 mm from the vertex of the cornea, is sharply focused on the retina. This agrees with the amplitude of accommodation of a normal eye.

	Accommodation for far vision	Accommodation for near vision
Assumed:		
Radius of curvature of cornea	8.0	8.0
Radius of curvature of anterior surface of lens	10.0	6.0
Radius of curvature of posterior surface of lens	6.0	5.5
Position of anterior surface of lens	3.6	5.6
Position of posterior surface of lens	7.2	7.2
Calculated:		
Anterior focal length of cornea	23.692	23.692
Posterior focal length of cornea	31.692	31.692
Focal length of lens	43.707	33.785
Distance of anterior principal point of lens from anterior surface	2.1073	1.9745
Distance of posterior principal point of lens from posterior surface	1.2644	1.8100
Distance between the two principal points of lens	0.2283	0.2155
Posterior focal length of eye	19.875	17.756
Anterior focal length of eye	14.858	13.274
Position of the anterior focal point	−12.918	−11.241
Position of the first principal point	1.9403	2.0330
Position of the second principal point	2.3563	2.4919
Position of the first nodal point	6.957	6.515
Position of the second nodal point	7.373	6.974
Postition of the posterior focal point	22.231	20.248

Some earlier observers[1] employing less precise methods of investigations were led to suppose that they had found variations of the curva-

[1] J. P. LOBÉ, *Diss. de oculo humano.* Ludg. Batav. 1742. p. 119. — HOME, *Philos. Transact.* 1796. p. 1.

ture of the cornea. Subsequent more accurate measurements of this convexity with the aid of reflex images have shown that it remains entirely unchanged. Such measurements have been made by SENFF,[1] CRAMER[2] and the author himself. These experiments can be made very exactly with the ophthalmometer, inasmuch as a change of 1/200 of the length of the radius may be observed. It would require a change of the radius of the cornea from 6.8 mm to 8 mm to produce by itself a range of accommodation of the eye from 5 inches to infinity. The author's own results have invariably been negative. A very ingenious experiment of THOMAS YOUNG which gives the same result should be mentioned here. He describes it as follows: "From a small botanical microscope I take a double convex lens having a radius and focal length of 0.8 inch, which is fastened in a socket one-fifth of an inch deep; securing its edges with wax, I drop into it a little moderately cold water till it is three-fourths full, and then apply it to my eye, so that the cornea projects into the socket, and is everywhere in contact with the water. Forthwith my eye becomes far-sighted, and the refracting power of the glass lens, which is reduced by water to a focal length of about 1.6 inches, is not sufficient to supply the place of the cornea, which has been disqualified by the superposition of the water; but the addition of another lens, of five inches and a half focus, restores my eye to its natural state, and somewhat more. I then use the optometer, and find now the same inequality in the horizontal and vertical refractions as without the water; and in both azimuths my power of accommodation enables me to focus an object four inches away, as formerly. At first sight, indeed, my accommodation appeared to be a little less than it was, and sufficient only for a range extending from infinity to five inches from the eye; which led me to believe that possibly the cornea did have some slight effect in the natural state; but, reflecting that the artificial cornea was about a tenth of an inch before the place of the natural cornea, I calculated the effect of this difference, and found it exactly sufficient to account for the diminution of the range of vision."

An approximate estimate, at any rate, may be obtained of the amount of the forward displacement of the pupillary margin of the iris in near vision, after having calculated the dimensions and convexity of the cornea and the distance of the pupillary plane from it. In Fig. 66 the cornea is represented by C and its outer edges are marked c and d; and ab is the pupil for far vision. Suppose the observer is so placed with respect to this eye that its entire pupil is just hidden; then cb must be the path of the observer's line of vision in the aqueous humor of the patient's eye. If, when the latter is focused for near vision, the entire pupil becomes just visible in front of the rim of the sclero-

[1] WAGNER, *Handwörterbuch der Physiologie*. Art. Sehen.

[2] *Het Accommodatievermogen der Oogen*. Haarlem 1853. p. 45.

tica, and if its width $\alpha\beta$ is perceived, it must lie in front of the line cb. Yet it must touch this line, as was shown by Fig. 60. Thus the amount of the displacement may be at least approximately found. In the eyes which the writer measured this displacement amounted to 0.36 mm for O. H.'s eye and to 0.44 mm for B. P.'s eye. In case the pupil does not move forwards the whole way in near vision, but, say, only half-way or two-thirds, the amount of forward movement must be estimated, and a calculation made accordingly.

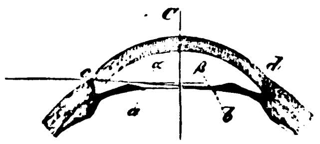

Fig. 66.

The radius of curvature of the anterior surface of the lens may be measured by the aid of the reflex image in it. However, the reflex in this case is too faint and vague for its distance to be measured exactly with the ophthalmometer. If, on the other hand, there is produced a corneal reflex of variable size along with the first lens reflex, the sizes of the two images may be readily compared by the naked eye, and they may be made equal. It is easy to find the size of the corneal image either by measurement or by calculation. For example, the author adjusted two bright flames in a vertical line with each other so as to be reflected in the anterior surface of the lens; and at the same time two other flames, smaller and fainter, were reflected in the cornea, so that their images appeared to be close to the reflexes in the lens, and at the same distance apart. Instead of using a pair of flames, a single flame and its reflected image in a horizontal mirror is more convenient.[1] The reflex images in the anterior surface of the lens for both near and far vision were measured in this way; and it was found that, in eyes with good accommodation, the image in the anterior surface of the lens in near vision is only about 5/9 as large as that in far vision. The image is formed by an optical system composed of a refracting and a reflecting surface; whose focal length may be calculated by equation (8b), §9, from the size of the image and the size and distance of the object; since the above mentioned formula applies to reflecting systems also. The radius of the reflecting surface may be found when the focal length of the compound system has been ascertained. The two focal lengths of the refracting system which lies in front of the reflecting surface will be denoted by f_1, f_2, and the radius of the reflecting surface will be denoted by r, to be reckoned positive or negative according as the mirror is concave or convex, respectively. Hence, if, finally, the distance of the vertex of the reflecting surface from the second principal point of the refracting system is denoted by d, the focal length of the compound reflecting system will be:

$$q=\frac{f_1 f_2 r}{2(f_2-d)(f_2-d+r)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now by this formula, q will diminish when d becomes smaller, that is, when the anterior surface of the lens approaches the cornea. But when q becomes smaller, the size of the reflex image of a distant object will also decrease in the same proportion. However, as the change of d amounts to only about 0.4 mm, and that of (f_2-d) to about 28 mm, and that of (f_2-d+r) to about 38 mm, the change in q is found to be extremely small and amounts only to about 1/40 of its total value; whereas the direct observation of the image gives a reduction of about 4/9. It is obvious, therefore, that the reduction in the size of the image is not to be explained as due to the forward movement of the anterior surface of the lens, but must, as a matter of fact, be produced simply by an increase of the curvature of this surface. The following results were obtained in this way by measurements of living eyes:

[1] Graefes *Archiv f. Ophth.* Bd. I. Abt. 2. S. 45.

Eye	Radius of curvature of anterior surface of lens		Displacement of the pupil in accommodation
	for far vision	for near vision	for near vision
O. H.	11.9	8.6	0.36
B. P.	8.8	5.9	0.44
J. H.	10.4		

In order to calculate the radius of curvature of the anterior surface of the lens by the above equation, the radius of curvature of the cornea and the distance of the anterior surface of the lens (pupil) from the cornea must both be known. Each of these magnitudes had already been measured in the eyes mentioned above.

The reflex image of a distant object as seen in the posterior surface of the lens likewise changes in size with changes of accommodation of the eye, but to a very slight extent. The writer watched this variation through the ophthalmometer by observing the reflex images in the posterior surface of the lens due to two illuminated apertures in a screen, in a vertical line on one side of the axis of the eye. The cornea reflexes were formed alongside the second lens reflexes, as shown in Fig. 67; where a_0 and a_1 are the two images of the lower light, and b_0 and b_1 of the upper. The images a_1 and b_0 were not superimposed but appeared close together side by side so that they could be distinguished apart. In accommodation for near vision, b_0 shifted a little towards a_0, and a_1 towards b_1; the amount thereof being estimated as equal to half the width of each spot of light; and since the distance between the centres of the apertures in the screen was six times the length of one of them, the reduction of the size of the image was about 1/12 its total size.

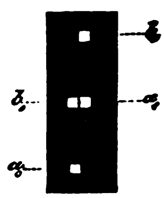

Fig. 67.

Finally, the author tried to determine whether the posterior surface of the lens was also shifted forward in accommodation for near vision. The method employed for this purpose was the same as that used for finding the apparent distance of the posterior surface of the lens from the cornea. With the same arrangement of the apparatus, an examination was made to see whether the reflex in the posterior surface of the lens is altered when the accommodation of the eye is varied without changing the direction of the optic axis. For this purpose, the light and the telescope were placed one on the right and the other on the left, and afterwards interchanged. However, the author was not able to detect any changes of the position of this reflex under these circumstances. The apparent distance of the posterior surface of the lens from the cornea is therefore not appreciably altered by change of accommodation.

From these observations of the reflex image in the posterior surface of the lens and of the apparent place of this surface, what can be inferred as to the real changes of this surface? Its apparent place is certainly very little affected by the refraction through the lens, because this surface is quite close to the posterior nodal point of the lens. Hence, it may be concluded that any possible variations of this apparent position due to changes of the refracting power of the lens in accommodation are negligibly small. For example, in the two schematic eyes whose optical constants were computed in this section by way of illustration, the posterior surface of the lens was apparently shifted forward 0.191 mm in far vision, and 0.113 mm in near vision. Thus, whereas,

as a matter of fact, it stayed fixed, yet when the eye was acommodated for near vision, this surface apparently moved backwards 0.078 mm. But this is too small to be perceived. Besides, this calculation is merely useful to show that the movements and their differences are after all very minute, without indicating at all the real sense of the variation in the actual crystalline lens itself, because in this case the separation of the principal points of the lens is an essential consideration, which is certainly smaller in the actual lens than in the schematic homogeneous lens.

All that can be said, therefore, is that the actual position of the posterior surface of the lens is not appreciably altered by the changes incident to acommodation.

In order to ascertain the effect of changes of the ocular media on the reflex image in the posterior surface of the lens, suppose that the reflecting surface were separated from the last refracting surface of the eye by an extremely thin layer of vitreous humor. Under such circumstances, the cardinal points of the refracting system will be the same as the cardinal points of the eye. Let the index of refraction of the vitreous humor be denoted by n; and, also, let the distance of the posterior focal point of the eye from the posterior surface of the lens be denoted by p, and let the distance of this surface from the second nodal point of the eye (measured therefore towards the front of the eye) be denoted by ϵ. In equation (1), which gives the focal length of a combined refracting and reflecting system, the following substitutions have to be made:

$$f_1 = p + \epsilon$$
$$f_2 = n(p + \epsilon)$$
$$f_2 - d = p.$$

Here, therefore, the focal length of the compound system becomes:

$$q = \frac{nr}{2} \cdot \frac{(p+\epsilon)^2}{p(p+r)} \qquad (2)$$

In accommodation for near vision ϵ invariably gets greater because the nodal point of the eye must move forwards with the change of shape of the lens. For this reason, if r and p did not change at all, the value of q and the size of the reflex image must both increase. But in accommodation for near vision p does diminish, which involves a decrease in the value of q as these magnitudes are related in the eye. Differentiating with respect to p, we find:

$$\frac{dq}{dp} = \frac{nr}{2} \cdot \frac{p+\epsilon}{p^2(p+r)^2} [pr - (2p+r)\epsilon].$$

Only the last factor in this expression, contained in square brackets, can be negative, though in the normal eye it will probably not be so, because ϵ is very small as compared with p and r. Therefore $\frac{dq}{dp}$ will be positive, that is, q and p will both increase or decrease together. In accommodation for near vision, p becomes smaller (if we neglect for the time being the change in ϵ and consider r constant) and therewith both q and the mirror image on the posterior surface of the lens become smaller also. Indeed, it might be supposed that the observed reduction in the size of this image was due to this cause, were it not for the fact that calculation by equation (2) shows that this is not the case. If we take from LISTING's schematic eye the values $p = 14.647$, $\epsilon = 0.3601$, $r = 6$, then p would have to diminish to 10.597 for q to be reduced to 1/12 of its total value. This implies, therefore, that the posterior focal point of the eye has receded to 4 mm in front of the retina, which exceeds the possible variation of the position of this point. But as a part of the reduc-

tion in size of the image produced by this means would be counteracted by the forward displacement of the nodal point due to increase of ϵ, as already explained, undoubtedly, the reduction in size of the reflex in the posterior surface of the lens cannot be what it is seen to be unless there is some increase in the curvature of this surface, slight as it may be.

When the values of q for the focal lengths of both of the schematic eyes of this section are computed, the results are 5.6051 for the case of far vision and 5.3562 for the case of near vision, these magnitudes differing from each other by no more than about 1/21 of their average values, whereas the corresponding radii of curvature (6 and 5.5 mm) differ by 1/12. Here the change of the refracting medium partially conceals that of the radius of curvature and makes it appear to be smaller than it really is. The conclusion is that the posterior surface of the lens becomes more curved in accommodation for near vision.

It is important for the mechanism of accommodation to know exactly the point of origin of the iris. In Fig. 3 SCHLEMM's canal with its surrounding area was represented as it appears in very thin sections of the tunics of the eye. A is the cross section of the canal which probably forms an elongated cleft in the living eye accommodated for far vision; C is the cornea, S the sclerotica, D the conjunctiva, B the choroid, E a ciliary process and J the iris. The inner wall of the canal is made up of different tissues. The most posterior portion of this wall at a very evidently consists of the same type of closely interwoven white fibrous tissue as the sclerotica from which it arises. The anterior portion, on the contrary, consists of another type of tissue which is more opaque than fibrous tissue, and consists of fibres which are more sharply defined and are very resistant to the action of acetic acid and lime. Consequently, it may probably be considered elastic tissue. In front this portion of the wall is inserted between the membrane of DESCEMET and the cartilaginous *substantia propria* of the cornea. Behind it is partly attached to the posterior fibrous portion of the wall and partly unites with the fibre bundles of the tensor muscle of the choroid. The choroid is attached only to the posterior half of the inner wall of SCHLEMM's canal at the point where the fibrous and elastic portions unite. However, from the anterior portion of the wall of the canal there originates also a loose network of fibres which exhibit the characteristics of elastic tissue and which are inserted on the periphery of the iris. The mass of fibres which belong to the tensor muscle and the iris seem to arise partly from the wall of the canal, though some seem to pass directly from the choroid to the iris. In the tissue of the ciliary processes a large number of sectioned blood vessels of wide lumen are visible and on their surface towards the vitreous humor is the layer of black pigment.

In order to be convinced of the correctness of the structure of the iris as here stated, it is necessary, on the one hand, to examine thin sections of the dried tunics of the eye, taking into consideration the fact that the process of drying produces much distortion, and that the elastic fibres are very easily torn or broken away from the attachment of the iris when the latter is sepasated from the cornea. On the other hand, it is necessary to examine fresh specimens, in the preparation of which it is best to insert a bristle in SCHLEMM's canal, at the same time carefully avoiding any traction on the iris or the choroid, for thereby the muscle mass which binds these parts together may be given any form at all. If the iris is lifted lightly and laid back on the ciliary processes, the fine elastic threads which pass over from it to the anterior edge of the canal become visible. If the bristle is then pulled forwards, the elasticity of the anterior portion of the canal wall will be evident. If, however, the iris and the choroid are turned forward and the bristle pulled backward, the posterior portion of the wall is seen to be inelastic.

The author regards the method of attachment described above as very important with respect to the backward movement of the lateral portions of

the iris in near vision. Thus, if the iris is relaxed, it will be held fast to the inner wall of SCHLEMM's canal by the net-work of elastic fibres extending from *b* to the anterior edge of the canal. If, however, the circular and radial fibres of the iris contract simultaneously, the fibrous mass on the posterior border of the canal provides a sufficiently firm opposition; and hence we may say that the relaxed iris is attached to the anterior, the contracted iris to the posterior edge, of SCHLEMM's canal, which on the average are 0.45 mm apart. In Fig. 63 an attempt has been made to represent the different mode of attachment of the iris in far vision (side *F*) and in near vision (side *N*), SCHLEMM's canal being designated by *s* on both sides of the diagram.

The ciliary processes are another part of the eye, whose actions during accommodation must still be considered. L. FICK[1] has shown that they contract under the influence of an electric current and empty themselves of blood which can readily flow out through quite large vascular channels into the *vasa vorticosa* of the choroid. He assumes that, by this transfer of blood into those portions of the eye which lie behind the partition formed by the lens and the zonule, the hydrostatic pressure will be increased there and diminished in front. Consequently, the centre of the lens will be pushed forwards, producing an increase of curvature of its anterior surface. On the other hand, FICK asserts that as a result, the posterior surface will become flatter. This is not in accordance with the author's observations. J. CZERMAK[2] has also attempted to explain the mechanism of accommodation by supposing there was a swelling of the ciliary processes in addition to the contraction of the iris and the ciliary muscle assumed by CRAMER. This might exert pressure on the border of the lens.

The opinion that the eye-muscles changed the form of the eyeball by their pressure on it, especially elongating it in the direction of the axis of the eye and thus separating the retina farther from the lens, had many notable adherents prior to the discovery of the change in form of the lens. The following facts militate against this view. First, any increase of the hydrostatic pressure in the eye tends to flatten the cornea, as the author has found by measurements with the ophthalmometer. Were this the case, it might have been observed in the living eye. Second, the ophthalmoscope reveals how even a slight pressure with the finger on the eyeball causes the blood vessels of the retina to become smaller in diameter, only permitting an intermittent blood stream to pass through with each beat of the pulse, and finally collapsing entirely. As soon as this intermittent movement (visible pulsation of the arteries) begins,[3] the sensitivity of the retina disappears, probably as a result of insufficient blood supply, and the field of vision becomes entirely dark.

Finally, the experiments of THOMAS YOUNG should be mentioned. These can hardly leave any doubt concerning the fact that not the slightest increase in length of the axis of the eye occurs in near vision. It is possible to touch the surface of the conjunctiva between the eye-lids with a smooth well-polished piece of metal without appreciable discomfort. A smooth iron ring (as of a key) is placed in the inner angle of the eye upon the conjunctiva and is pressed against the inner edge of the orbit. The eye is then turned inwards so that the subject looks at a distant object through the ring and over the bridge of the nose. As a result, the inner border of the cornea comes to lie close to the key, and the possibility of the forward movement of the eyeball in accommodation is thus prevented. Now the ring of a very small key is

[1] J. MÜLLERS *Archiv*. 1853. S. 449.
[2] *Prager Vierteljahrsschr*. XLIII. S. 109.
[3] DONDERS in *Nederl. Lancet*. 1854. Nov. S. 275.

inserted between the eyeball and the bone at the outer angle of the eye. As a result of pressure from this key on the eyeball the retina is stimulated, and a dark impression, which looks at first perhaps like a bright spot, appears in the field of view and, as it were, in front of the bridge of the nose. In YOUNG's own case it extended as far as the place of clearest vision, so that he was able to note that straight lines contained within the boundaries of this pressure image appeared slightly bent. He judged that this was caused by the slight indentation of the sclerotica due to the pressure of the key. Inasmuch as the pressure image appeared at the place of clearest vision, the little key must have touched the posterior side of the eyeball in the region of the yellow spot. Under such circumstances, obviously, no axial elongation of the eye could occur without forcing the key from its position. If, therefore, accommodation were accompanied by an axial elongation of the eye, either it could not have occurred at all, or the key would have had to be forcibly displaced, resulting in an extraordinary increase in the area of the retinal impression on account of the greater indentation of the posterior wall of the eyeball. But nothing like this happened. The eye can accommodate as well as ever, and the retinal impression remains entirely the same during altered accommodation.

THOMAS YOUNG must have had somewhat protuberant eyes, as may also be inferred from other experiments which he describes. When the above experiment was made by the writer, only one edge of the retinal impression extended as far as the place of clearest vision. However, there was no doubt as to the possibility of accommodation and the constancy of the retinal impression.

A direct corollary from this experiment is that the distance of the inner contour of the cornea from the yellow spot or from a point on the posterior wall of the eye a little towards the outside from the yellow spot is absolutely fixed. But it would not be possible for the distance from the cornea to the yel'ow spot to change without marked asymmetry of the eye, unless the distance from the edge of the cornea changed likewise.

FORBES supposed that in accommodation for near vision the interior of the inner eye was under increased pressure, and that change of form of the lens was due to its different elasticity in different directions on account of the varied form and density of its layers. On the other hand, DE HALDAT could find no variation of the focal length of the optical system of the eye or of extirpated lenses when compressed in water.[1]

There is no other subject in physiological optics about which so many antagonistic opinions have been entertained as concerning the accommodation of the eye. One reason for this is that many of the decisive facts were not discovered until very recently, so that formerly there was much more room for hypotheses. In order to take a rapid survey of these views, instead of following the chronological order, as is done, however, in the bibliography at the end of this section, the various views will be grouped together here, chiefly to show their bearing on each other.

1. *Theories that deny not only the necessity but the existence of any change of the optical system of the eye.* Many naturalists believed that the eyes of human beings and other animals, as distinguished from artifical lenses, were endowed with the faculty of focusing images of objects at different distances all at the same place or approximately so. MAGENDIE[2] maintained that he had verified this in the eyes of white rabbits in which pigment is absent from

[1] *Comptes rendus.* XX. p. 61, 458 and 1561.
[2] *Précis élémentaire de Physiologie.* I. p. 73.

the choroid so that the retinal image can be seen through the posterior part of the sclerotica. As a matter of fact, however, the image cannot be seen through the sclerotica well enough defined to observe the slight differences that are involved in accommodation. RITTER,[1] HALDAT,[2] and ADDA[3] corroborated MAGENDIE. HALDAT and ENGEL[4] maintained that it is true for the crystalline lens alone. When the crystalline lens is separated from the humors of the eye and examined in the air, its focal length is extraordinarily short, and, by ordinary optical laws, the distance of an image in it is not markedly different whether the object is at infinity or just 7 inches away. This explains the results obtained by ENGEL.[5]

On the contrary, HUECK,[6] VOLKMANN,[7] GERLING,[8] MAYER[5] and CRAMER,[9] by more accurate experiments, verified the fact that the eyes of human beings and other animals had different focuses for objects at different distances, although theoretically the matter was beyond doubt. TREVIRANUS[10] believed that he could give a theoretical explanation of the supposed fact that the position of the image is independent of the position of the object by assuming for this purpose a special law for the increase in thickness of the lens. His mathematical discussion was refuted by KOHLRAUSCH.[11]

STURM[12] believed he could utilize the fact that the refracting surfaces of the eye are not strictly accurate surfaces of revolution, to explain accommodation for different distances. To begin with, he studied the behaviour of a homocentric bundle of rays refracted at a curved surface which is not a surface of revolution, and found that the rays, instead of being united in a single focal point, have two focal planes. In each of these planes the rays meet in a focal line, the directions of the two focal lines being perpendicular to each other. Thus, if the cross section of the bundle of rays in one focal plane is a short horizontal straight line, it will change through an ellipse with its major axis horizontal into a circle as we proceed towards the other focal plane and then through an ellipse with its major axis vertical into a vertical straight line when the second focal plane is reached. STURM's idea was that between the two focal planes the cross section of the bundle of rays in the eye contracted enough to give clear images. If the luminous point is close to the eye, the two focal planes are at some distance beyond the lens, but as long as the retina lies between them, the images are perhaps distinct enough for vision.

Aberrations of the kind that STURM supposes actually do seem to occur in most human eyes, and the phenomena dependent on them will be described in §14; where, however, it will be shown that the interval between the two focal planes is by no means so important as STURM thinks and that, instead of promoting the clearness of vision, this defect in the eye tends rather to impair it.

[1] GRAEFE und WALTHERS *Journal.* 1832. Bd. VIII. S. 347.

[2] *Comptes rendus.* 1842.

[3] *Ann. d. Ch. et de Phys.* Sér. 3. Tom. XII. p. 94.

[4] J. ENGEL, *Prager Vierteljahrsschr.* 1850. Bd. I. S. 167.

[5] See refutation of above by MAYER, ibid. 1850. Bd. IV. Ausserord. Beilage.

[6] *Diss. de mutationibus oculi internis.* Dorpati 1826. p. 17. — *Die Bewegung der Kristallinse.* Leipzig 1841.

[7] *Neue Beiträge zur Physiol. d. Gesichtssinnes.* 1836. S. 109.

[8] POGGENDORFFS *Ann.* XLVI. 243.

[9] *Het Accommodatievermogen.* Haarlem 1853. S. 9.

[10] *Beiträge zur Anat. u. Physiol. der Sinneswerkzeuge.* 1828. Heft I.

[11] *Über* TREVIRANUS *Ansichten vom deutlichen Sehen in der Nähe und Ferne.* Rinteln 1836.

[12] *Comptes rendus.* XX. 554, 761 and 1238. See refutations by CRAHAY, *Bull. de Bruxelles.* XII. 2. 311. BRÜCKE, *Berl. Berichte.* I. 207.

DE LA HIRE[1] insists that there is only one distance of distinct vision, and that so long as an object is not too far from this place one way or the other, it can be seen well enough to be recognized; but there is no accommodation. HALLER[2] was of much the same opinion but supposed that the contraction of the pupil contributed also to diminish the blur circles of near objects. Quite recently the same opinion has been advocated by BESIO.[3]

All these views which deny not only the necessity but even the existence of any internal change in the eye during accommodation may be most easily refuted by the simple fact that an object at a fixed distance from the eye can be seen distinctly or indistinctly at pleasure. Moreover, SCHEINER's experiment which consists in viewing a fixed point through two holes in a card, which can be seen single or double, voluntarily, is another way of disproving them. Lastly, the observations with the ophthalmoscope as mentioned in §11 which enable the changes of the optical image on the retina to be seen objectively is an answer to such opinions.

2. *Opinions which consider the contraction of the pupil as sufficient to account for accommodation in near vision.* SCHEINER[4] discovered that the pupil is contracted in near vision. When the eye is accommodated for distant vision, the blur circles on the retina corresponding to luminous points that are near by can certainly be diminished by a contraction of the pupil. However, a very simple experiment suffices to show that the contraction of the pupil is not enough to enable the eye to accommodate for near objects. All that anybody need to do is to look through a narrow hole that is smaller than the pupil and that acts, so to speak, as an artificial, motionless pupil, in order to verify the fact that even such near objects appear indistinct in far vision, and far objects appear indistinct in near vision. Besides HALLER, who was mentioned above, LEROY,[5] HALL,[6] and MORTON[7] were supporters of this view. OLBERS,[8] DUGES,[9] HUECK and DONDERS,[10] adduced arguments in opposition thereto. J. MILE[11] proposed a curious theory of the result of the contraction of the pupil, but it is likewise disproved by the experiment mentioned above, and he himself afterwards abandoned it.[12] He believed that in far vision the peripheral rays of the cone of light which would cross the optic axis in front of the retina were deflected from it by diffraction at the edge of the pupil and, consequently, did not cross the axis until later. Diffraction of light, however, is not at all a simple deflection of entire rays in this fashion.

3. *Opinions that involve a change of curvature of the cornea.* The first person who supposed he had detected a change of curvature of the cornea appears to have been LOBE.[13] OLBERS[14] does not venture to say definitely as a result of his observations that the convexity increases in near vision. HOME,[15] ENGELFIELD and RAMSDEN, however, claimed to have definitely detected an

[1] *Journal des Sçavans.* 1685. p. 398.
[2] *Elementa Physiologiae.* 1743. Tom. V. p. 516.
[3] *Giorna'e Arcad.* CV. p. 3.
[4] *Oculus.* p. 31.
[5] *Mém. d. l'Acad. d. Sciences.* 1755. p. 594.
[6] MECKELS *Archiv.* Bd. IV. S. 611.
[7] *American Journal of med. Sciences.* 1831. Nov.
[8] *De oculi mutationibus internis.* Gotting. 1780. p. 13.
[9] *Institut* 1834. No. 73.
[10] RUETE, *Leerboek der Ophthalmologie.* 1846. bl. 110.
[11] MAGENDIE, *Journal de Physiologie.* VI. p. 166.
[12] POGGENDORFFS *Ann.* XLII.
[13] ALBINUS, *Dissert. de oculo humano.* Lugd. Bat. 1742. p. 119.
[14] *De oculi mutat. int.* p. 39.
[15] *Philosoph. Transact.* 1795. p. 13 and 1796. p. 2.

increase of the curvature. A subject who possessed a good power of accommodation was attached to a firm board with a slot in it so that he could not move his head. A little way from the eye, a plate was fastened to the board in which there was a small opening serving as fixation point. On one side of the eye an adjustable microscope, with an ocular micrometer, was also placed on the board, in order to observe the farthermost curvature of the surface of the cornea. In near vision the cornea was supposed to become more convex, so that its centre moved forward 1/800 of an English inch. Measurement of the reflex image in the cornea, carried out later by HOME, gave more doubtful results. Perhaps, in both instances he was deceived by very minute, uniform movements of the head of the subject. No such differences were found by THOMAS YOUNG[1] in the measurements he made of the reflex images in the cornea. Indeed, he completely refuted the hypothesis of alterations in the convexity of the cornea by demonstrating, as described above, that the power of accommodation remained constant even when the eye is under water. HUECK[2] obtained similar results when he repeated HOME's experiments, but he thought he had ascertained that respiratory movements produced regular to and fro movements of the head, due to inhaling usually in near vision and exhaling in far vision. When he held his breath, the movements of the middle of the cornea ceased entirely or became very irregular. These irregular movements were apparently produced by contractions of the *musculus orbicularis oculi*, as with every contact with the eyelashes the eyeball was drawn back a little. BUROW[3] repeated HOME's work very painstakingly, but failed to detect any regular movements of the corneal surface. VALENTIN,[4] obtained the same results. SENFF[5] measured the reflex images with a telescope so that his results were not affected by small displacements of the eye; and found that the radius of curvature of the cornea did not change as much as 0.01 Paris line,[6] in a range of accommodation of from 4 to 222 inches. CRAMER[7] also obtained negative results from a measurement of the reflex image in the cornea made with the help of his ophthalmoscope. Such measurements may be very easily and accurately made with the ophthalmometer,[8] and invariably these results as obtained by the author were negative also.

Recent support of the hypothesis that accommodation is effected by a change of curvature of the cornea is to be found in the works of FRIES,[9] VALLÉE[10] and PAPPENHEIM.[11] The latter assumes that the contraction of the iris causes the cornea to become more convex in near vision.

4. *The supposition that accommodation is produced by a shifting of the position of the lens.* This assumption is the oldest, having been proposed by KEPLER,[12] whose theory of vision was the first also to recognize the necessity of accommodation. This hypothesis always has had many adherents among

[1] *Philosoph. Transact.* 1801. I. p. 55.

[2] *Die Bewegung der Kristallinse.* S. 40.

[3] *Beiträge zur Physiologie und Physik des menschl. Auges.* Berlin 1842. S. 115.

[4] *Lehrbuch der Physiologie.* 1848. Bd. II. S. 122.

[5] WAGNERS *Handwörterbuch der Physiologie.* Art. Sehen. S. 303.

[6] * That is, not as much as about 0.02 mm. (J. P. C. S.)

[7] *Het Accommodatievermogen.* bl. 45.

[8] GRAEFES *Archiv für Ophthalmologie.* Bd. I. Abt. II. S. 24.

[9] *Über den optischen Mittelpunkt im menschl. Auge.* Jena 1839. S. 27.

[10] *C. R. de l'Acad. d. Sciences.* 1847. Oct. p. 501.

[11] *Spezielle Gewebelehre des Auges.* Breslau 1842.

[12] *Dioptrice.* Propos. 64.

whom were SCHEINER,[1] PLEMPIUS,[2] STURM,[3] CONRADI,[4] PORTERFIELD,[5] PLATTNER,[6] JACOBSON,[7] BREWSTER,[8] J. MÜLLER,[9] MOSER,[10] BUROW,[11] RUETE,[12] WILLIAM CLAY WALLACE,[13] and C. WEBER.[14] Most of them considered it probable that voluntary contractions of the ciliary body are able to move the lens to and fro. In order to get around the mathematical difficulty of requiring the lens to execute an impossible displacement in the act of accommodation, they had to resort to the assumption that the focal length of the cornea is greater and that of the lens less than is actually the case. This hypothesis found support in recent times especially by observations of the living eye which demonstrate that the pupil approaches the cornea in near vision. BIDLOO[15] had already observed the increased convexity of the iris in near vision in birds. This was afterwards confirmed for the human eye by HUECK,[16] BUROW[17] and RUETE. C. WEBER showed by a mechanical contrivance that in the case of dogs the anterior surface of the lens moves forward as soon as the anterior portion of the eye is stimulated by an electric current. For this purpose he made a round hole in the centre of the cornea of the eye of a living, opium-narcotized dog; and inserted a little rod in it, until it came in contact with the anterior surface of the lens. The other end of the rod was supported by the shorter arm of a sensitive lever which magnified the forward movement of the anterior surface of the lens.

On the other hand, HANNOVER[18] assumed the possibility of a forward and backward movement of the lens within its capsule, during which process the so-called *Liquor* MORGAGNII would exchange places with it. That there is no such fluid in the normal capsule of the lens, has already been stated.

5. *Hypothesis of change of form of the lens.* This theory, which has triumphed at length was likewise proposed a long time ago and had many defenders even before the actual existence of such changes was known to be a fact.[19] DESCARTES[20] originated this explanation of acommodation; and others

[1] *Oculus.* Oeniponti 1619. Lib. III. p. 163.

[2] *Ophthalmographia.* Lovanii 1648. B. III.

[3] *Dissertatio visionem ex obscurae camerae tenebris illustrans.* Altdorfii 1693. p. 172.

[4] FRORIEPS *Notizen.* Bd. 45.

[5] *On the eye.* Edinburgh 1759. Vol. I. p. 450.

[6] *De motu ligamenti ciliaris.* Lipsiae 1738. p. 5.

[7] *Suppl. ad. Ophthalm.* Copenh. 1821.

[8] *Edinb. Journal of Science.* I. 77. — POGGENDORFFS *Ann.* II. 271.

[9] *Zur vergleichenden Physiologie des Gesichtsinns.* Leipzig 1826. S. 212.

[10] *Repertor. d. Physik.* Berlin 1844. Bd. V. S. 264.

[11] *Beiträge zur Physiol. u. Physik des menschl. Auges.* Berlin 1842.

[12] *Lehrbuch der Ophthalmologie.*

[13] *The accommodation of the eye to distances.* New York 1850.

[14] *Disquisitiones quae ad facultatem oculum accommodandi spectant.* Marburgi 1850. p. 31.

[15] *Observ. de oculis et visu variorum animalium.* Lugd. Bat. 1715.

[16] *Bewegung der Kristallinse.* S. 60.

[17] *Beiträge zur Physiol. usw.* S. 136.

[18] *Bidrag til Öjets Anatomie.* Kjöbenhavn 1850. p. 111.

[19] SCHEINER suspected that, along with some slight elongation of the eyeball, there might also be a concomitant change in the form of the crystalline lens in accommodation. HUYGEN'S views on the subject seem never to have been definitely settled in his own mind, and at one time he inclined to think that the act of accommodation was effected by a change of curvature of the lens, but subsequently, in 1670, he returned again to the explanation of a forward movement of the lens as the sole responsible agency for producing this effect. (J. P. C. S.)

[20] CARTESIUS, *Dioptrice.* Lugd. Bat. 1637.

who embraced it were PEMBERTON,[1] CAMPER,[2] HUNTER,[3] YOUNG,[4] PURKINJE,[5] GRAEFE,[6] TH. SMITH,[7] HUECK,[8] STELLWAG VON CARION[9] and FORBES.[10] Some earlier anatomists, for example, LEEUWENHOEK and PEMBERTON, called the lens, perhaps for this reason, the *musculus crystallinus*, supposing that its fibres were contractile. YOUNG inclined to this view on the basis of experiments which, while they seemed to him to be completely convincing, did not succeed on every eye. If the blurred image of a luminous point is examined through a fine grill of parallel wires, it is seen to be crossed by dark straight lines which are shadows of the wires. When YOUNG's own eye was accommodated for distant vision, these lines appeared to be perfectly straight; but when he looked at near objects, the lines at the sides of the blur circle were convex outwards. The phenomenon was not changed by putting the eye under water so as to eliminate the influence of the cornea. The only explanation of the curvature of the previously straight shadows was the change of curvature of the surfaces of the lens. The pupil should be dilated in order to perform this experiment. WOLLASTON could not observe the effect (nor could the author), but KOENIG, another friend of YOUNG's, verified it. Corroboratively, YOUNG found, by looking through four parallel narrow vertical slits on his optometer, that the four images of the white horizontal line intersected at one point when the eye was accommodated for far vision, but did not do so when it was adjusted for near vision.

The variation of the reflex images in the lens during accommodation were noted first by MAX LANGENBECK,[11] who drew the correct inference that the anterior surface of the lens is, therefore, more convex in near vision. His method of observation, however, was not favourable, because he had the subject look directly at the flame, and this meant that the observer had to see the three reflex images very close together, so that the exceptionally bright corneal image makes it hard to see the other two. Perhaps, this is the reason why LANGENBECK's observation did not attract the notice of physiologists. CRAMER observed the same thing; but he improved the method of observation especially by having the rays of light fall on the eye from one side while the observer looks into it from the other. He also described an instrument which he called an *ophthalmoscope*, designed to make the observations more easily and more accurately. This instrument was essentially a stand to which was attached a microscope with a magnifying power of from 10 to 20, a lamp, cross hairs serving as a fixation mark, and a hollow conical tube conveniently shaped for the adjustment of the patient's eye. The lamp is regulated to enable the observer to watch the reflex image in the anterior surface of the lens in between the other two reflex images when he looks through the microscope at the pupil of the patient's eye. However, the essential thing here, which is the reduction of the size of the reflex image in the anterior surface of the lens, is not so convenient for observation in this way as the method with the naked eye as described above, in which the object con-

[1] *Dissert. de facultate oculi, qua ad diversas distantias se accommodat.* Lugd. Bat. 1719.

[2] *Dissert. physiol. de quibusdam oculi partibus.* Lugd. Bat. 1746. p. 23.

[3] *Philosoph. Transact.* 1794. p. 21.

[4] Ibid. 1801. P. I. p. 53.

[5] *Beobachtungen u. Versuche zur Physiol. d. Sinne.* Berlin 1825.

[6] REILS *Archiv für Physiologie.* Bd. IX. S. 231.

[7] *Philosophical Magazine.* 1833. T. V. 3. No. 13.—SCHMIDTS *Jahrbücher.* 1834. Bd. I. S. 6.

[8] *Bewegung der Kristallinse.* Leipzig 1841.

[9] *Zeitschrift der k. k. Gesellschaft der Ärzte zu Wien.* 1850. Heft 3 and 4.

[10] *Comptes rendus.* XX. p. 61.

[11] *Klinische Beiträge.* Göttingen 1849.

sists of two separated luminous points. The mere shifting of the reflex in the anterior surface of the lens, which can be easily and accurately seen in CRAMER's instrument, is not conclusive by itself on account of the asymmetry of the eye, (of which CRAMER was not then aware), unless a preliminary set of experiments that are easily carried out has shown that this reflex image invariably moves towards the centre of the pupil from any other position.

Without knowledge of the work of either of the last two investigators, and at a time when CRAMER's discovery had been published merely in brief notices[1] by himself and DONDERS, and even before the appearance of DONDERS' great book which was crowned by the Dutch Institute of Sciences, the author had discovered the same facts[2] for himself, and had discovered, besides, the behaviour of the posterior surface of the lens in accommodation,[3] as above related.

Numerous instances in which the eye appeared to accommodate after the removal of the lens in the operation for cataract, were cited to prove that the power of accommodation was not dependent on displacements and changes of form of the lens. However, it should be borne in mind here that patients even with poor accommodation may recognize objects whose images are blurred. The mere fact that a person who can read print with cataract glasses can with the same glasses distinguish people at a distance, window-frames, etc., is no reason at all for supposing that he has the power of accommodation. Anybody can easily verify for himself the fact that when his eye is focused on his finger about a foot away, he can still perceive various details of distant objects. In order to establish the existence of acommodation, the patient must be able with the same glasses to see a given object either distinctly or indistinctly, as he chooses, depending on whether he tries to focus for one distance or another. SZOKALSKY believes that he has actually observed such a case. This particular eye, however, could see objects distinctly at a distance of 17 inches without cataract glasses which is impossible without some substitute for the lens. DONDERS has suggested the use of entoptical phenomena to determine during life whether the lens has been renewed in eyes which have been operated on for cataract.

6. *The supposition that the form of the eyeball changes.* If the retina could be removed farther from the optical system of the eye, that is, if the eyeball could be elongated, this would afford the eye a means of accommodation for near vision. The supporters of this opinion usually have assumed that the form of the eyeball could be changed by pressure exerted on it by the eye muscles, either by the recti alone, or by the obliques alone, or by all acting together, or by the added action of the *orbicularis oculi*. Among these were STURM,[4] LE MOINE,[5] BUFFON,[6] BOERHAVE,[7] MOLINETTI,[8] OLBERS,[9] HAESELER,[10] WALTHER,[11] MONRO,[12] HIMLY,[13]

[1] *Tijdschrift der Maatschappij vor Geneeskunde.* 1851. W. 11. bl. 115 and *Nederlandsch Lancet.* 2. Serie. W. 1 bl. 529. 1851–52.

[2] *Monatsberichte der Berliner Akad.* 1853. Feb. S. 137.

[3] GRAEFES *Archiv für Ophthalmologie.* Bd. I. Abt. II. S. 1-74.

[4] *Dissert. de presbyopia et myopia.* Altdorfii 1697.

[5] *Quaestio an obliqui musculi retinam a crystallino removeant.* Parisiis 1743.

[6] *Histoire naturelle.* Paris 1749. T. III. p. 331.

[7] *Praelectiones academ.* Taurini 1755. Vol. III. p. 121.

[8] HALLER, *Elementa Physiologiae.* 1763. T. V. p. 511.

[9] *Dissert. de oculi mutat. int.* Gottingae 1780. §43.

[10] *Betrachtungen über das menschliche Auge.*

[11] *Dissert. de lente crystallina.* § 1.

[12] *Altenburger Annalen f. d. J.* 1801. S. 97.

[13] *Ophthalmologische Beobachtungen und Untersuchungen.* Bremen 1801.

Meckel,[1] Parrot,[2] Poppe,[3] Schroeder van der Kolk,[4] Arnold,[5] Serre,[6] Bonnet,[7] Henle,[8] Szokalsky[9] and Listing.[10] Clavel[11] assumed that the eye muscles might not only change the form of the eyeball but also might cause the cornea to become more convex and the lens to be moved forward. The reasons why such a change of the form of the eyeball appears to be improbable have been already stated above.

The various views here considered are the more important of those that have been advanced concerning this intricate subject. Many other modes of explanation have been suggested also, which rightly found less favour. Here v. Grimm[12] may be mentioned who assumed that the indices of refraction of the ocular media might vary, and Weller[13] who proposed to explain accommodation not by a change in the eye itself but by a psychic process.

Literature:

1611. Kepler, *Dioptrice.* Propos. 26.
1619. Schneider, *Oculus.* Oeniponti 1619. Lib. III. p. 163.
1637. Cartesius, *Dioptrice.* Lugd. Batav.
1648. V. F. Plempius, *Ophthalmographia.* Lovanii. B. III.
1685. De la Hire, *Journal des Sçavans.* 1685. p. 398.
1693. Sturm, *Dissertatio visionem ex obscurae camerae tenebris illustrans.* Altdorfii, p. 172.
1697. Sturm, *Dissert. de presbyopia et myopia.* Altdorfii.
1712. A. F. Walther, *Diss. de lente crystallina oculi humani.* Lipsiae. Also in Haller, *Disput. anat.* Vol. IV.
1715. Bidloo, *Observationes de oculis et visu variorum animalium.* Lugd. Batav.
1719. Pemberton, *Dissert. de facultate oculi, qua ad diversas distantias se accommodat.* Lugd. Batav.
1738. J. J. Platner, *De motu ligamenti ciliaris in oculo.* Lipsiae. p. 5.
1742. J. P. Lobé (Albinus), *Diss. de oculo humano.* Ludg. Batav. p. 119. Also in Haller, *Disput. anat.* Vol. VII.
1743. Haller, *Elementa Physiologiae.* T. V. p. 516.
Le Moine, *Quaestio an obliqui musculi retinam a crstallino removeant.* Paris.
1746. P. Camper, *Dissert. physiologica de quibusdam oculi partibus.* Lugd. Batav. p. 23. Also in Haller, *Disput. anat.* Vol. IV.
1749. Buffon, *Histoire naturelle.* Paris. T. III. p. 331.
1755. Le Roy, *Mémoires de l'Acad. de Paris.* 1755. p. 594.
Boerhave, *Praelectiones academicae.* Taurini. Vol. III. p. 121.
1758. v. Grimm, *Diss. de visu.* Gottingae.
1759. Porterfield, *On the eye.* Edinburgh. Vol. I. p. 450. — *Edinb. med. Essays.* Vol. IV. p. 124.
1763. Molinetti in Haller, *Elementa physiologiae* V. p. 511.

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[1] Cuvier, *Vorlesungen über vergl. Anat.* Trans. by Meckel. Leipzig 1809. Bd. II. S. 369.
[2] *Entretiens sur la physique.* Dorpat 1820. T. III. p. 434.
[3] *Die ganze Lehre vom Sehen.* Tübingen 1823. S. 153.
[4] Luchtmans *Diss. de mutatione axis oculi.* Traject. ad Rhenum. 1832.
[5] *Untersuchungen über das Auge des Menschen.* Heidelberg 1832. S. 38.
[6] *Bulletin de thérapie.* 1835. T. 8. L. 4.
[7] Frorieps *N. Notizen.* 1841. S. 233.
[8] Canstatts *Jahresbericht für 1849.* Bd. I. S. 71.
[9] *Archiv für physiologische Heilkunde.* VII. 1849. 7.—8. Heft.
[10] Wagners *Handwörterbuch d. Physiologie.* IV. 498.
[11] *Comptes rendus.* XXXIII. p. 259.
[12] *Dissert. de visu.* Gottingae 1758. See also Olbers, *de oculi mutationibus internis.* p. 29.
[13] *Diätetik für gesunde und schwache Augen.* Berlin 1821. S. 225.

1783. OLBERS, *Diss. de oculi mutationibus internis.* Gottingae.

1793. TH. YOUNG, *Observations on vision.* *Phil. Trans.* 1793. P. II. p. 169.

1794. HUNTER, *Phil. Trans.* 1794. p. 21.

1795. HOME, *Phil. Trans.* 1795. P. I. p. 1. (Accommodation after operation for cataract.)

1796. HOME, *Phil. Trans.* 1796. P. I. p. 1.
TH YOUNG, *De corporis humani viribus conservatricibus.* Gottingae. — *Phil. Trans.* 1800, p. 146.

1797. KLÜGEL in REILS *Archiv.* Bd. II. S. 51. (In answer to HOME.)

1801. MONRO, *Altenburger Annalen f. d. J.* 1801. S. 97.
HIMLY, *Ophthalmologische Beobachtungen und Untersuchungen.* Bremen.
*TH. YOUNG, *On the mechanism of the eye.* *Phil. Trans.* 1801. P. I. p. 23.* (This is a work of astonishing perspicacity and originality, which was qualified to settle the question as to accommodation even at that time, but, on account of its conciseness, it is often hard to follow, and, moreover, it presupposes the most thorough knowledge of mathematical optics.)

1802. HOME, *Phil. Trans.* 1802. P. I. p. 1. (Accommodation after operation for cataract.)
ALBERS, *Beiträge zur Anatomie und Physiologie der Tiere.* Heft I. Bremen.

1804. GRAEFE in REILS *Archiv für Physiologie.* Bd. IX. S. 231.

1809. CUVIER, *Vorlesungen über die vergleichende Anatomie,* translated by MECKEL. Leipzig. Bd. II. S. 369.

1811. WELLS, *Phil. Trans.* 1811. P. II. Also in GILBERTS *Annalen* XLIII. 129 and 141.

1816. MAGENDIE, *Précis élémentaire de Physiologie* I. p. 73. Paris. Translated by ELSÄSSER. Tübingen 1834. I. 54.

1820. G. PARROT, *Entretiens sur la physique.* Dorpat. T. III. p. 434.

1821. JACOBSON, *Suppl. ad Othphalm.* Copenhagen.
C. H. WELLER, *Diätetik für gesunde und schwache Augen.* Berlin. S. 225.

1823. J. POPPE, *Die ganze Lehre vom Sehen.* Tübingen. S. 153.
RUDOLPHI, *Grundriss der Physiologie.* Berlin. Bd. II. Abt. 1. S. [illegible].
LEHOT, *Nouvelle théorie de la vision.* Paris.
PURKINJE, *De examine physiologico organi visus et systematis cutanei.* Vratislaviae. (Discovery of the lens reflex.)

1824. BREWSTER, *Edinb. Journal of Science* I. p. 77.— POGGENDORFF, *Annalen* II. S. 271.
SIMONOFF in MAGENDIE, *Journal de Physiologie.* T. IV.

1825. PURKINJE, *Beobachtungen und Versuche zur Physiol. der Sinne.* Berlin. S. 128.*

1826. J. MÜLLER, *Zur vergleichenden Physiologie des Gesichtssinns.* Leipzig. S. 212.
HUECK, *Diss. de mutationibus oculi internis.* Dorpati.
MILE in MAGENDIE, *Journal de Physiologie* VI. p. 166.

1828. TREVIRANUS, *Beiträge zur Anatomie und Physiologie der Sinneswerkzeuge des Menschen und der Tiere.* Heft I.

1831. MORTON in *American Journal of med. Sciences.* 1831. Nov.

1832. RITTER in GRAEFE u. WALTHERS *Journal.* VIII. S. 347.
FR. ARNOLD, *Untersuchungen über das Auge des Menschen.* Heidelberg. S. 38.
G. J. LUCHTMANS, *Diss. de mutatione axis oculi secundum diversam distantiam objecti ejusque causa.* Traject. ad Rhenum.

1833. TH. SMITH, *Philos. Magazine* V. 3. No. 13. — SCHMIDTS *Jahrbücher der Medizin.* 1834. Bd. I. S. 6.

1834. DUGÉS, *Institut.* Nr. 73.

1835. SERRE, *Bulletin de Thérapie.* T. VIII. L. 4.

1836. VOLKMANN, *Neue Beiträge zur Physiologie des Gesichtssinns.* S. 109.
R. K. KOHLRAUSCH über TREVIRANUS' *Ansichten vom deutlichen Sehen in der Nähe und Ferne.* Rinteln.

1837. SANSON, *Leçons sur les maladies des yeux,* publiées par BARDINOT et PIGNE. Paris. (Concerning the reflexes of the crystalline lens.)
MILE in POGGENDORFFS *Annalen* XLII. S. 37 u. 235.

1838. Pasquet in Froriefs *Notizen.* Bd. VI. Nr. 2.
1839. J. F. Fries, *Über den optischen Mittelpunkt im menschlichen Auge, nebst allgemeinen Bemerkungen über die Theorie des Sehens.* Jena. S. 27.
1840. Neuber in Osanns *Zeitschrift.* Heft 7-12. S. 42.
1841. Hueck, *Die Bewegung der Kristallinse.*
Bonnet in Frorieps *Neue Notizen.* 1841. S. 233.
1842. Haldat in *Comptes rendus.* 1842.
Adda in *Annales de Chimie et de Phys.* Ser. III. T. XII. p. 94.
1842. Burow, *Beiträge zur Physiol. u. Physik des menschl. Auges.* S. 94-177.*
S. Pappenheim, *Die spezielle Gewebelehre des Auges.* Breslau.
1844. Moser, *Repertor. d. Physik* V. S. 364.
1845. Sturm, Sur la théorie de la vision. *Comptes rendus.* XX. p. 554, 761, 1238; Poggendorffs *Annalen* LXV. 116.
Forbes, *Comptes rendus.* XX. p. 61; *Institut. No.* 576. p. 15; No. 578. p. 32.
de Haldat, *Comptes rendus.* XX. p 458 and 1561; *Institut.* No. 596. p. 90 (in reply to Forbes.)
1846. Donders in Ruete, *Leerboek der Ophthalmologie.* bl. 110.
H. Meyer in Henle u. Pfeuffer, *Zeitschrift für rationelle Medizin.* Bd. V. (Origin of the lens reflex.)
Senff in R. Wagners *Handwörterbuch der Physiologie,* Art. "Sehen" by Volkmann. S. 303.
Besio, *Giorn. Arcad.* CV. 3; *Institut.* No. 666. p. 338.
J. G. Crahay, *Bulletin de Bruxelles.* XII. 2. 311; *Institut.* No. 644. p. 151.
1847. L. L. Vallée, *Comptes rendus.* XXV. p. 501.
1848. Valentin, *Lehrbuch der Physiologie.* Bd. II. Abt. 2. S. 12.
Szokalsky in Griesinger, *Archiv für physiol. Heilkunde.* VII. S. 694.
1849. Max. Langenbeck, *Klinische Beiträge aus dem Gebiete der Chirurgie und Ophthalmologie.* Göttingen.
Donders in *Nederlandsch Lancet.* 1849. bl. 146.
1850. Jos. Engel, *Prager Vierteljahrsschrift* XXV. S. 167 and 208.
H. Mayer, ibid. Bd. XXVIII. Ausserord. Beilage, and Bd. XXXII. S. 92.*
Henle in Canstatts *Jahresbericht für 1849.* Erlangen. S. 71.
William Clay Wallace, *The Accommodation of the eye to distance.* New York.
C. Weber, *Nonnullae disquisitiones quae ad facultatem oculum rebus longinquis et propinquis accommodandi spectant.* Marburgi.
C. Stellwag von Carion, *Wiener Zeitschrift der Ges. der Ärzte.* VI. S. 125. 138.
A. Hannover, *Bidrag til Öiets Anatomie, Physiologie og Pathologie.* Kjöbenhavn. p. III.
1851. H. Helmholtz, *Beschreibung eines Augenspiegels zur Untersuchung der Netzhaut im lebenden Auge.* Berlin. S. 37.*
Listing in R. Wagners *Handwörterbuch der Physiologie.* Art.: Dioptrik des Auges. Bd. IV. S. 498.*
Cramer, *Tijdschrift der Maatschappij vor Geneeskunde.* 1851. W. 11. bl. 115; *Nederlandsch Lancet.* Ser. 2. W. 1. bl. 529.
Clavel, *Comptes rendus.* XXXIII. p. 259; *Archives des sciences phys. et natur.* XIX. p. 76.
1852. Donders, *Nederl. Lancet.* 1852. Feb. bl. 529.
1853. H. Helmholtz, *Monatsberichte d. Akad. zu Berlin.* Feb. S. 137.
*A. Cramer, *Het Accommodatievermogen der Oogen physiologisch toegelicht.* Haarlem. Translated by Doden. Leer 1855.
L. and A. Fick in J. Müllers *Archiv für Anat. u. Physiol.* 1853. p. 449.*
1854. Donders in *Onderzoekingen gedaan in het Physiologisch Laborat. der Utrechtsche Hoogeschool.* Jaar VI. p. 61.
J. Czermak in *Prager Vierteljahrsschrift* XLIII. S. 109.
1855. *H. Helmholtz, Über die Accommodation des Auges in v. Graefe, *Archiv für Ophthalmologie.* Bd. I. Abt. II. S. 1.

Supplement

An experiment by BAHR should be mentioned here with respect to the phenomena connected with the mechanism of accommodation. Accommodated for near vision, he looked at a brightly illuminated rectangle, until a strong after-image had been formed in his eye. He then projected it on a distant surface, changing his accommodation accordingly, and estimated its apparent size thereon. Inasmuch as the size of the image on the retina is proportional to the distance of the retina from the posterior nodal point of the eye, and since the size of the retinal image is the same in both cases, it is possible to calculate from such an experiment how much the distance of the retina from the second nodal point has changed. By his experiments BAHR found a forward displacement of the nodal point amounting to 0.35 mm; whereas, according to the author's calculation as given on page 152, this displacement was 0.4 mm. If, instead, there had been an elongation of the eyeball, the change of distance would have had to be much more considerable; and if such an elongation were the sole basis of accommodation, it would have to amount to 3 mm, which is impossible, as BAHR's experiments also show.

KNAPP[1] determined for the eyes of four individuals the positions of the far point and near point and the curvature and position of the cornea and of the surfaces of the lens in accommodation for both far and near vision. According to these data, the accommodation as computed from the changes of curvature of the crystalline lens agrees close enough with the amplitude of accommodation that was actually found; so that any assumption of an elongation of the eye was thus ruled out.

In two cases very favourable for examination in which the lens had been removed in a cataract operation, DONDERS[2] was convinced that there is no evidence of accommodation in such eyes, though they could see distinctly with the aid of a convex lens placed before them. Nevertheless, in the attempt to see objects near at hand, convergence and contraction of the pupil occurred. If it were possible to elongate the eyeball by means of the pressure of the eye muscles, this would be able to produce a certain amplitude of accommodation even in eyes without the crystalline lens. As a result of all these facts it can no longer be doubted that *an elongation of the eyeball does not occur in accommodation for near vision.*

A much better method than any of those described above for measuring the curvatures of the crystalline lens is to use the ophthal-

[1] *Archiv für Ophthalmol.* IV, 2, p. 1-52.
[2] *On the anomalies of accommodation and refraction.* London. p. 320-321.

mometer in a dark room and to obtain the reflex images by a beam of sunlight; as was done by B. Rosow.

Concerning the muscles that produce the change of form of the lens, it may be observed in the first place that cases have been noticed in which, although the iris ceased to function, entirely satisfactory accommodation occurred. The author himself knew an astronomer, with whom naturally it was easy to perform optical experiments and who was well acquainted with the phenomena concerned in them; and although his iris was completely paralyzed, notwithstanding he was able to accommodate perfectly well. Moreover, A. v. Graefe[1] found that the power of accommodation was entirely normal in the case of a workman after recovery from an injury of his eye which had resulted in the complete separation of the iris.

There is nothing left except the ciliary muscle to which accommodation can be attributed. In this structure a circular layer of fibres has recently been discovered first by van Reeken, and more definitely by H. Mueller and Rouget. These are situated in the angle turned towards the ciliary processes; moreover, they are interwoven with meridional fibres, and in many instances the circular fibres bend so as to continue as meridional fibres. From this anatomical arrangement of the circular fibres it may be inferred in the first place that these elements of the ciliary muscle cannot act except in conjunction with the meridional fibres. Indeed, such an arrangement of muscle fibres is evidently a very favourable one for action on the zonule; for if there were nothing but radial fibres in the muscle, as represented in the old descriptions, the inner corner of the muscle would have been forced in, and the zonule would have been bent convex towards Schlemm's canal (Fig. 63*s*). The result would be that the zonule would be much less relaxed than in the existing arrangement where any such bending is avoided. The circular fibres of the muscle must pull the corresponding edge of the muscle towards the tip of the ciliary processes and towards the edge of the lens. The effect of this action is to shift the centre of the zonule towards the margin of the lens in the direction of the edges of its folds, without pulling it outwards towards Schlemm's canal.

Whether the circular fibres of the ciliary muscle exert a pressure on the ciliary processes, as H. Mueller supposes, which is transmitted to the edge of the lens, is hard to determine, because we do not know whether the ciliary processes are filled with blood in the living eye so as to be stout enough to exert an appreciable pressure on the lens. Many ophthalmologists consider it very doubtful whether they even touch the lens.

[1] *Archiv für Ophthalmologie.* VII, 2, p. 150–161.

W. HENKE has conjectured that it is only the circular fibres of the ciliary muscle that produce accommodation for near vision, and that it is the contraction of the meridional fibres that restore the accommodation for far vision. Thus he regards the two points of attachment of the meridional fibres of the muscle as fixed, and supposes that it would be curved inwards by the action of the circular fibres, and be stretched straight again, when accommodation relaxes, by active tension, the circular fibres being released. This mode of action seems to the author highly improbable, first, because there are many indications to show that there is no active accommodation for far vision, and also because the fibre layers of the ciliary muscle are too much interwoven, meridional fibres being confounded with circular fibres and *vice versa;* so that separate action of the individual fibres is scarcely to be thought of. The example of the iris which HENKE adduces against this argument is of very doubtful value in the light of recent investigations of the *musculus dilatator iridis*. Besides, in the opinion of the writer, the *ligamentum pectinatum* as an anterior point of attachment and the choroid as a posterior point of attachment for the muscle are much too elastic to permit a considerable action of the muscle, such as HENKE supposes, in a direction that is so disadvantageous. Finally, according to HENKE's description, the external surface of the muscle would have to stand out from the sclerotica in accommodation for near vision and catch hold again in far vision. But it is not very clear where a fluid can come from that could fill the empty space of this cleft, and unless there were something of the kind, the air pressure would prevent the muscle from yielding.

In conclusion, the author sees no reason to modify the theory of the mechanism of accommodation as given above on page 151 which still seems to him to give the most probable explanation. Recent experiments reported by C. VÖLCKERS and V. HENSEN tend to confirm this opinion.

1855. RUETE, De Irideremia congenita. *Progr. acad. Leipzig.* VIRCHOWS *Archiv.* XII, 342.

— VAN REEKEN. Ontleedkundig Onderzoek. van den toestel voor accommodatie van het oog. *Onderzoekingen gedaan in het Physiol. Laborat. der Utrechtsche Hoogeschool,* Jaar. VII, 248-286.

1856. J. P. MAUNOIR, Mémoire sur l'ajustement de l'œil aux différentes distances. *Arch. des sciences phys.* XXXI, 309-316.

— BRETON, Adaptation de la vue aux différentes distances, obtenue par une compression mécanique, exercée sur le globe oculaire. *C. R.* XLIII, 1161-1162. *Inst.* 1856. p. 455. *Cosmos.* IX, 690. X, 29-30.

— GOODSIR, Notice respecting recent discoveries on the adjustment of the eye to distinct vision. *Proc. of Edinb. Soc.* III, 343-345. *Edinb. J.* (2) III, 339-342

1857. STOLTZ, Accommodation artificielle ou mécanique de l'œil à toutes les distances. *C. R.* XLIV, 388-390; 618-620. *Arch. des sciences phys.* XXXV, 139. *Cimento* VI, 154 155. *Cosmos.* X, 320-321.

— BAHR, *De oculi accomodatione experimenta nova. Dissertat.* Berlin.

— H. Müller, Über einen ringförmigen Muskel am Ciliarkörper. *Archiv für Ophthalm.* III, 1, IV, 2. S. 277-285.

1859. J. Mannhardt, Bemerkungen über den Akkommodationsmuskel und die Akkommodation. Ibid. IV. 1. S. 269-285.

— Ch. Archer, On the adaptation of the human eye to varying distances. *Phil. Mag.* (4) XVII, 224-225.

— Respighi, Sull'accommodamento dell'occhio humano per la visione distinta. *Mem. di Bologna.* VIII, 355-389. *Zeitschr. für Chemie.* 1859. S. 10-18.

— Magni, Dell'addattamento dell'occhio umano alla visione distinta. *Cimento* X, 12-20.

1860. J. H. Knapp, Über die Lage und Krümmung der Oberflächen der menschlichen Kristallinse und den Einfluss ihrer Veränderungen bei der Akkommodation auf die Dioptrik des Auges. *Archiv. für Ophthalmol.* VI, 2, S. 1-52. VII, 2, S. 136-138.

1860. W. Henke, Der Mechanismus der Akkommodation für Nähe und Ferne. Ibid. VI, 2, S. 53-72.

— L. Happe, *Die Bestimmungen des Sehbereichs und dessen Korrektion, nebst Erläuterungen über den Mechanismus der Akkommodation.* Braunschweig 1860.

1861. A. v. Graefe, Fall von acquirierter Aniridie als Beitrag zur Akkommodationslehre. *Archiv für Ophthalmol.* VII, 2, S. 150-161.

1863. O. Becker, Lage und Funktion der Ciliarfortsätze im lebenden Menschenauge. *Wiener Medic. Jahrbücher.* 1863.

1864. E. Förster, Zur Kenntnis des Akkommodationsmechanismus. *Sitzungsber. d. Ophthalmol. Ges. Erlangen,* S. 75-86. *Klinische Monatsbl. für Augenheilk.* Sept. to Dec. 1864.

1865. B. Rosow, Zur Ophthalmometrie. *Archiv für Ophthalmol.* XI, 2, S 129-134.

— Mandelstamm, Zur Ophthalmometrie. Ibid. XI, 2, S. 259-265.

§ 13. Chromatic Aberration in the Eye

It is only approximately correct to suppose that rays of light issuing from a luminous point and entering the eye are reunited in a single focus after traversing the ocular media. We must now examine this affair more closely, with particular reference in this section to the so-called *chromatic aberration*; due to the fact that the different wave-lengths of light are unequally refrangible in solid and liquid transparent media. Since the magnitudes of the focal lengths of a curved refracting surface depend on the indices of refraction of the media, it follows that rays of light of different colours, after traversing an optical system of such surfaces, will generally be focused at different points. It is only be means of special combinations of refracting surfaces that the foci for light of different colours can be made to coincide, in which case the optical contrivance may be said to be *achromatic* in a certain sense.

Now the eye is not achromatic, although in ordinary vision the colour dispersion is scarcely noticeable at all. Fraunhofer demonstrated that the optical system of the eye really has different focal lengths for light of different colours in the following way. Observing a prismatic spectrum through an achromatic telescope which had a fine set of cross hairs in the eyepiece, in order to get sharp definition, he found that he had to focus the ocular lens nearer the cross hairs for the violet part of the spectrum than for the red. With the other eye

focused on an external object, he viewed the cross hairs in the telescope illuminated by light from one part of the spectrum, and by adjusting the ocular contrived to see both them and the external object distinctly at the same time. Then he measured how much the ocular had to be displaced to get the same adjustment in another part of the spectrum. Thus, having previously measured the chromatic aberration of the ocular itself, he was able to calculate the chromatic aberration of the eye with respect to a pair of colours corresponding to two different places in the spectrum. By these experiments FRAUNHOFER found that an eye which is emmetropic for light corresponding to the *C* line (in the bright red part) of the solar spectrum had to be from 18 to 24 Paris inches[1] nearer the object, in order still to see it distinctly when the object emitted light corresponding to the *G* line (in the violet part) of the solar spectrum, without changing its accommodation.

The author has obtained similar results with his own eyes. Monochromatic light from different parts of a prismatic spectrum was admitted through a fine hole in an opaque screen, and the maximum distance was found for which the small illuminated opening could be seen as a well defined point. The far point of the author's eye was about 8.5 ft for red light and 1.6 ft for violet light; whereas it was only a few inches away for the extreme violet light at the end of the solar spectrum that cannot be seen at all unless the brighter parts of the spectrum are shaded.

In an ordinary rectangular spectrum projected by a prism on a white wall some distance away, the differences of the distances of distinct vision for different colours are strikingly manifest; because, while the red end may appear fairly well in its proper form, the violet end is more or less blurred (and in the author's own case has a sort of swallow-tail shape).

The rather slight chromatic aberration of the human eye as compared with artificial optical instruments is explained by the fact that water and aqueous solutions generally have much less dispersion than glass. Since the indices of refraction of the ocular media are not very different from that of water, it is reasonable to expect that the aqueous and vitreous humors, at least, would have about the same dispersion as that of water. Accordingly, the writer has calculated the dispersion of LISTING's reduced eye, which has a single refracting surface, on the assumption that the light is refracted from air to water. For the rays used by FRAUNHOFER in his experiments, the indices of refraction of water are as follows:

For *C* line (red) 1.331705;
For *G* line (violet) 1.341285.

[1] ¶1 Paris inch = 12 Paris lines = 27.07 mm. (J. P. C. S.)

The radius of the refracting surface of the reduced eye is 5.1248 mm; and hence the second focal lengths will be found to be 20.574 mm (for red light) and 20.140 mm (for violet light.) Thus, when the eye is accommodated for parallel red rays, so that the focal point for red light is on the retina, the focal point of the violet rays will be 0.434 mm in front of it; which implies that this is accommodated to see distinctly a source of violet light for a distance of 713 mm away (or about 28 inches). FRAUNHOFER (as stated above) found this distance in his own case to be between 487 and 650 mm; which indicated that the dispersion in an eye made of distilled water would be rather less than it is in the actual human eye. But if it is supposed that the reduced eye eye is accommodated for a distance of 8.5 ft (2.6 m) in the red, corresponding to the author's experiment with his own eye, the retina would still have to be 0.123 mm beyond the focal point of red light; and at the same time the eye is accommodated for a violet source of light 22 inches away or 560 mm; whereas in the writer's case this latter distance was 19 inches. MATTHIESSEN[1] also calculated from his experiments the interval between the focal points of the human eye for red and violet light and obtained from 0.58 to 0.62 mm; which can be compared with the value 0.434 mm for an eye made of distilled water. MATTHIESSEN's method consisted in measuring the shortest distance at which a ruled glass surface could be seen distinctly when it was illuminated, first, by red light and then by violet light. All these experiments by different methods agree in showing that, so far as chromatic aberration is concerned, the human eye corresponds very closely to an eye of distilled water; the dispersion of the natural eye being if anything a little higher. Perhaps, we might conjecture that, just as the refractivity of the crystalline lens is greater than that of pure water, so also its dispersion is higher.

Certain other experiments in which the colour dispersion of the eye is noticeable may also be described here. Phenomena of this kind are generally much more striking when instead of using white light, a mixture of only two colours is employed that are as different as possible in refrangibility. The best way to obtain this effect is to transmit sunlight through ordinary violet coloured glass, which will absorb the intermediate colours almost entirely and transmit only the extreme red and violet. But if lamplight is used, which is deficient in blue and violet, ordinary blue (cobalt) glass is better, as it filters out also much of the orange, yellow and green and transmits abundantly the extreme red, and blue and violet. A coloured glass of this kind is placed right behind a narrow aperture in a dark screen and illuminated

[1] *Comptes rendus.* T. XXIV. p. 875.

from behind by a lamp whose rays after traversing the glass and the hole in the screen enter the observer's eye. Under these circumstances the aperture may be regarded as a luminous point emitting red and violet rays; which will appear differently, depending on the distance for which the observer's eye is accommodated. When it is accommodated for the red rays, the violet light produces a blur circle, and the opening appears as a red point with a violet fringe around it; and *vice versa*. There is an intermediate state of accommodation for which the focus of the violet rays is in front of the retina, and that of the red rays beyond it, in such fashion that the coloured blur circles on the retina are equal in size; and only when this is the case will the source appear to be of the same uniform colour. For this state of accommodation, there might be some light of intermediate colour, say, green, which would be focused at a point on the retina.

Incidentally, this method affords a means of finding with considerable degree of accuracy the distances for which the eye can be accommodated for the intermediate parts of the spectrum; for they are the same as the distances for which the eye can see the mixed red-violet spot as being of uniform colour. The difference of colour at the edges is very easily detected, even by an unpractised observer; and it is much easier to distinguish than the inexactness of a white image. When the eye is accommodated for light of a certain definite frequency for a distance farther than that of the luminous point, the blur circle of the red rays on the retina will be larger than that of the violet, and the spot will appear as a violet disc with a red border; whereas the reverse effect will be observed when the eye is accommodated for a distance less than that of the luminous point. Effects similar to these always occur whenever an object emits two kinds of light of very different frequencies. The phenomena are very striking, for example, in certain experiments on the mixing of spectrum colours which will be subsequently described in the theory of colour mixtures.

With white light there is, of course, also some separation of the component colours, but ordinarily it is hardly noticeable. In this connection, experience shows that a white surface which is farther away than the point of accommodation of the eye, is tinged with a faint blue border; whereas if the object is nearer than the point of accommodation, the border will be a faint reddish yellow. But when the eye is accommodated for the exact distance of the white surface, no such coloured border is seen, unless some opaque obstacle is held close to the eye so as to cover part of the pupil; in which case a coloured border appears along the opaque edge. The border between a white and black field appears yellow when a card is interposed halfway in front of the

pupil from the side of the black portion, and blue when it is interposed from the side of the white portion.

The phenomena of chromatic aberration in the human eye as above described are easily explained by the fact that the second focal point for violet light lies in front of that for red. In Fig. 68 the luminous point is supposed to be at A; the first principal plane of the eye is represented by the straight line b_1b_2; the points on the optical axis designated by r and v are the points conjugate to A with respect to the red

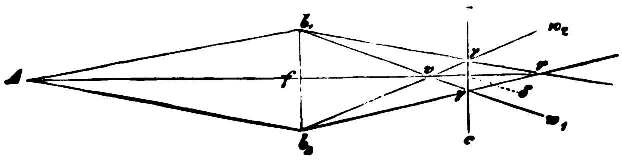

Fig. 68.

and violet rays, respectively; the straight line cc represents the plane where the outside red rays of the cone b_1b_2r intersect the outside violet rays of the cone b_1b_2v. Obviously, if the retina of the eye is in front of the plane cc, that is, when the eye is accommodated for an object farther away than A, it will be illuminated at the boundary of the cone by red light only, whereas near the axis there will be a mixture of colours. When the retina coincides with cc, so that the eye is accommodated for light of intermediate frequency coming from A, it will be illuminated all over by a uniform mixture of colours. And when the retina is beyond cc, as will be the case when the accommodation is for a point nearer than A, the edge of the illuminated area will be violet while the central part will be a mixture of colours. Now suppose that when the eye is accommodated for A, so that the retina is at cc, the lower half of the aperture b_1b_2 is covered by an opaque screen; thereby cutting off all the violet rays beyond b_2v and fv, all the red rays between b_2r and fr, and, of course, also, the prolongations of these rays. All the violet illumination disappears then from the part of the retina above the axis, and all the red from the part below the axis, the result being a small blur circle on the retina which is red above and violet below, instead of a clear-cut image of the point A.

If the source at A, instead of being a single luminous point, is a surface uniformly illuminated with both red and violet light, there will be formed on the retina simultaneously both a red and a violet image of this surface, one of which is necessarily blurred. But the blurred image of a uniform surface looks exactly the same as if it were in focus, except towards the edge where the blur circles of the separate points only partially overlap one another, and here the image fades out, encroaching on the surrounding darkness only so far as the blur circles of the border points extend. Now if red and violet images of a

luminous area are projected together on the same surface, the central portion of the resultant image where both colours are supposed to have normal brightness will appear as a blended mixture of the two; whereas the border will have that one of the two colours for which the blur circles are larger and for which, therefore, the border of the image encroaches farther on the surrounding parts. If the image of the surface is intercepted in the plane *cc* where the red and violet blur circles are of the same size, the colours will be uniformly mixed right to the edge. But when the edge of a card is gradually inserted in front of the pupil, the blurred images are partially eclipsed, as we saw in § 11; and these eclipses proceed from opposite sides when one of the blurred images is due to too near accommodation and the other to too far accommodation, as is the case here with the red and violet images. Hence, the congruence of the coloured images ceases, and coloured edges are visible. For red light the surface behaves like an object that is too near the eye, and hence the image in this case disappears from the side opposite to that on which the card is inserted in front of the pupil. With violet light it is just the other way. Hence, if the card is introduced, say, from below, the red surface vanishes from above and the violet from below with a red margin below and a violet margin above. When a small red-violet line is inspected through a narrow slit, the red image can be easily separated entirely from the violet image by moving the slit to and fro in front of the pupil.

When the light emitted by a luminous point is composed not simply of red and violet but of white light of all colours, the intermediate colours are distributed in their natural order between red and violet, and the effects of the chromatic aberration are not so striking as with two colours alone. Instead of a purple field with a violet border, we now have a white field bordered by pale blue, indigo and violet; and the inner part of this border is so nearly white that the border usually appears narrower. And instead of a purple field with a red border, there appears, likewise a white field faintly tinged at the boundary with yellow, orange and red, the pale yellow being hardly discernible against the white background.

The special case when the retina is at the place *cc* (Fig. 68) where the diameter of the bundle of rays is smallest needs to be considered. Here the blur circles of the red and violet rays are the same size. The green rays corresponding to the middle of the spectrum are united at the point where the axis meets the retina; and the rays of the rest of the colours form small blur circles. The resultant image on the retina ought therefore to be greenish at the centre, with a border of mixed red and violet, that is, purple; but it is not that way at all in the eye. For this particular position of the retina the light corresponding to the

brightest colours, yellow and green, is concentrated almost exactly at a point, and the purple border is both too narrow and too faint to be perceived.

All the phenomena above described may be perceived just as in the eye, only exaggerated, in a non-achromatized telescope, provided a higher magnifying power is used than the instrument is intended to have. The real image made by the objective is not caught on a screen, as on the retina of the eye, but is viewed through a magnifying eyepiece. The image in the objective must be magnified, otherwise the coloured border is generally too small to be seen distinctly. In this case, also, when the telescope is focused on a more distant object, a white area will appear to have a red and yellow border; whereas when the telescope is focused on a nearer object, the same area will appear to have a blue border. When the telescope is focused so as to give the sharpest image of the white area, a very narrow purple border is seen. If one half of the object glass is screened, the opposite side of the white area has a blue and yellow border, etc., in perfect analogy to the case of the eye.

In order to calculate the size of the blur circle due to chromatic aberration in the eye, Listing's reduced eye with water for the refracting medium may be used for this purpose, since, according to Fraunhofer's measurements, the dispersion of such an eye would not be very different from that of the human eye. From Fig. 68 we have:

$$\frac{\gamma\gamma}{b_1b_2}=\frac{\delta r}{fr}=\frac{\delta v}{fv},$$

and therefore

$$\gamma\gamma\cdot fr=b_1b_2\cdot\delta r,$$
$$\gamma\gamma\cdot fv=b_1b_2\cdot\delta v.$$

By addition,

$$\begin{aligned}\gamma\gamma[fr+fv]&=b_1b_2\cdot[\delta r+\delta v]\\&=b_1b_2[fr-fv]\\\gamma\gamma&=b_1b_2\frac{fr-fv}{fr+fv}.\end{aligned}$$

Taking 4 mm, the mean pupillary diameter in normal eyes, as the length of b_1b_2, and substituting the following values as previously obtained:

$$fr=20.574\text{ mm},$$
$$fv=20.140\text{ mm};$$

consequently,

$$\gamma\gamma=0.0426\text{ mm}.$$

According to the table given in § 11 for the dimensions of blur circles of objects for which the eye is not accommodated, the diameter $\gamma\gamma$ of the blur circle caused by dispersion should be equal to that given by a luminous point 1.5 metres away (or about 5 ft) in an eye accommodated for infinity. This degree of lack of accommodation

would result in a very appreciable confusion of the details in the image of an object, as may be easily verified by trying the experiment. In order to understand why the dispersion of white light in the eye which results in blur circles on the retina of these dimensions does not actually confuse vision to any sensible extent, not only the size of the blur circle but the distribution of the light in it has to be taken into account.

When the eye gets light of a definite frequency emitted from a point-source, the retina being to one side or the other of the focus of the bundle of refracted rays, the blur circle on the retina is equally illuminated all over. But if the luminous point emits white light of all sorts of frequencies, and the retina of the eye is in focus for the brightest part of the spectrum corresponding to greenish yellow light, there will be blur circles for the light of other colours, and their diameters will be greater in proportion as these colours are farther from the middle of the spectrum. Thus, whereas the centre of the affected area of the retina is uniformly illuminated by light of all kinds, especially by the most brilliant and most concentrated kind, the outlying portions get light only from the extreme parts of the spectrum, which not only have less intrinsic brilliancy, but are still further enfeebled in their effects by being spread over a larger area. Theoretically, the image is infinitely brighter at the centre than anywhere else.

As the law of the luminosity of the different parts of the spectrum has not yet been mathematically formulated, the calculation will be made on the assumption that all the colours of the spectrum are equally bright. Of course, this means that the values found for the illumination at the borders of the blur circles will be too large; but even so, the calculation will show why the blur circles due to chromatic aberration produce far less confusion than those of equal size due to imperfect accommodation.

Theoretical Distribution of Intensity in a Blur Circle due to Chromatic Aberration of Light proceeding from a Point-Source.

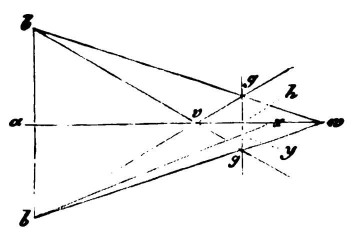

Fig. 69.

In Fig. 69 the straight line bb represents the principal plane of the reduced eye of radius R; and we may suppose that the diaphragm which limits the bundle of rays lies in this plane, as is approximately the case in the natural eye, so that bb is a diameter of the stop, whose radius will be denoted by b. The incident rays are supposed to be parallel; and the focal points of the extreme violet and extreme red rays are designated by v and w, respectively.

These extreme rays intersect each other at the two points designated by g, so that the straight line gg is the diameter of the entire blur circle, whose centre is at the point designated by h. To get the best image, the retina should be in the plane gg. Let the index of refraction corresponding to light from the middle part of the spectrum, that is focused at h, be denoted by N and put $ah = F$. Then by equation (3a), §9:

$$F = \frac{NR}{N-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1a)$$

If n denotes the index of refraction for light of some other kind which is focused at a point x, and if $f = ax$ denotes the focal length for these rays, then

$$f = \frac{nR}{n-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1b)$$

Let $\rho = hy$ denote the radius of the blur circle due to this kind of light; then its value will be given by one of the following formulae:

$$\frac{\rho}{b} = \frac{f-F}{f}, \quad \text{or} \quad \frac{\rho}{b} = \frac{F-f}{f},$$

according as $f > F$ (that is $n < N$) or $f < F$ (that is, $n > N$), respectively. When the above expressions for F and f are substituted, we have, either

$$\frac{\rho}{b} = \frac{N-n}{n(N-1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2a)$$

for $n < N$, or

$$\frac{\rho}{b} = \frac{n-N}{n(N-1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2b)$$

for $n > N$.

Now the intensity H of the illumination on the retina for light of colour corresponding to the value n of the index of refraction is:

$$H = A\frac{b^2}{\rho^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

where A denotes the intensity of the incident light over the area bb. Combining this equation with either equation (2a) or equation (2b), so as to eliminate b and ρ, we find in both cases:

$$H = A\frac{n^2(N-1)^2}{(n-N)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3a)$$

Now the intensity of illumination J at any point in the blur circle is:

$$J = \int H dn, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

the integral being taken over all values of n included in the given

range of colours. In the expression for H, the factor A is really a function of n, but its mathematical form is unknown. The factor n^2 varies very little over the range of the spectrum; and so for practical purposes, the product

$$An^2(N-1)^2=B$$

may be regarded as constant, which is equivalent to assuming that the brightness of the spectrum is practically uniform throughout its range, varying but little from red to violet. This assumption is really less favourable to our theory than the actual fact. According to equation (4):

$$J=\int\frac{Bdn}{(N-n)^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4a)$$

the integral being taken between the proper limits of n. However, every point of the blur circle is illuminated by light from both the red and violet ends of the spectrum. Let n_1, n_2 denote the extreme values of the index of refraction for the red light, so that

$$N>n_2>n_1,$$

and let n_3, n_4 denote the extreme values of the index of refraction for violet light, so that

$$n_4>n_3>N.$$

Equation (4a) becomes therefore:

$$\left.\begin{aligned} J&=B\int_{n_1}^{n_2}\frac{dn}{(N-n)^2}+B\int_{n_3}^{n_4}\frac{dn}{(N-n)^2} \\ &=B\left\{\frac{1}{N-n_2}-\frac{1}{N-n_1}+\frac{1}{N-n_4}-\frac{1}{N-n_3}\right\}\end{aligned}\right\} \quad . \quad . \quad . \quad . \qquad (4b)$$

Now if ρ_0 denotes the distance from the centre of the blur circle to the point where the intensity is to be ascertained, this point will be illuminated by light of all colours for which the radii of the blur circles are greater than ρ_0, or between ρ_0 and r. For the less refrangible colours, equation (2a) gives:

$$\frac{1}{N-n}=\frac{1}{N}+\frac{1}{N(N-1)}\cdot\frac{b}{\rho}.$$

Since $\rho=r$ for $n=n_1$, and $\rho=\rho_0$ for $n=n_2$, therefore,

$$\left.\begin{aligned}\frac{1}{N-n_1}+\frac{1}{N}+\frac{1}{(N-1)N}\cdot\frac{\rho}{r} \\ \frac{1}{N-n_2}=\frac{1}{N}+\frac{1}{(N-1)N}\cdot\frac{b}{\rho_0}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (4c)$$

Similarly, with respect to n_3 and n_4, we must employ equation (2b), which may be written:

$$\frac{1}{N-n}=\frac{1}{N}-\frac{1}{N(N-1)}\frac{b}{\rho}.$$

For $n=n_4$ put $\rho=r$, and for $n=n_3$ put $\rho=\rho_0$; accordingly,

$$\left.\begin{aligned}\frac{1}{N-n_3}&=\frac{1}{N}-\frac{1}{N(N-1)}\frac{b}{r}\\ \frac{1}{N-n_4}&=\frac{1}{N}-\frac{1}{N(N-1)}\cdot\frac{b}{\rho_0}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (4d)$$

Substituting in (4b) the values given by (4c) and (4d), we get finally:

$$J=\frac{2B}{N(N-1)}\left\{\frac{b}{\rho_0}-\frac{b}{r}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

The value of J becomes infinite at the center of the blur circle ($\rho_0=0$) and vanishes at the border where $\rho_0=r$.

Calculation of the intensity at the edge of a uniformly illuminated area. The straight line AB in Fig. 70 represents the edge of a luminous surface; and it is assumed that every point of this edge is blurred. Suppose that p is the point where the intensity is to be found, and let $r=pq$ denote the radius of the blur circle. This point p will get light from all those points of the surface that lie within the circle described around p as centre with radius r. Suppose s is one of these points, and put $sp=\rho$, and angle $spq=\omega$. Let J denote the intensity of illumination due to a single blur circle at the distance ρ from its centre. Then the total intensity at the point p will be

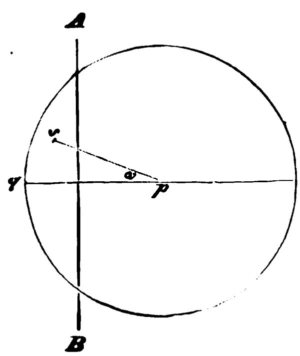

Fig. 70.

$$H=\int\int J\rho d\omega d\rho, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

where the double integral must be taken over all parts of the area within the circle described around p. Now suppose that the edge of the illuminated surface is a straight line, and let the distance of the point s from it be denoted by x; then if the centre p of the circle is on this edge,

$$\rho \cos\omega = x,$$

Integrating the expression for H, first, with respect to ω, using the limiting value of the angle ω as given by the last equation, we obtain:

$$H = \int_x^r 2J\rho \text{ arc cos } \left(\frac{x}{\rho}\right) d\rho \quad . \quad . \quad . \quad . \qquad (6a)$$

When the blur circles are due to imperfect accommodation, J may be considered as independent of ρ, and in this case:

$$H = J\left[r^2 \text{ arc cos } \left(\frac{x}{r}\right) - x\sqrt{r^2 - x^2}\right] \quad . \quad . \quad . \qquad (7)$$

This equation gives, therefore, an expression for the intensity of illumination at a point near the edge of the surface as a function of the distance from the edge. For $x = r$, we get $H = 0$; and for $x = -r$, we have $H = Jr^2\pi$, so that here the intensity is equal to the general intensity of the uniformly illuminated area.

When the blur circles are due to dispersion, the value of J given by equation (5) may be substituted in equation (6a). The result of performing the integration in this case is:

$$H = \frac{2Bb}{N(N-1)}\left\{r \text{ arc cos } \left(\frac{x}{r}\right) + \frac{x}{r}\sqrt{r^2 - x^2} + x \log \text{nat } \left(\frac{r - \sqrt{r^2 - x^2}}{r + \sqrt{r^2 - x^2}}\right)\right\} \quad (8)$$

For $x = r$, we find (as before) $H = 0$, and for $x = -r$,

$$H = \frac{2Bbr\pi}{N(N-1)};$$

so that the intensity at such a point becomes equal to the constant intensity of the central portions of the surface.

The variations of these functions are represented graphically by the curves in Fig. 71, curve A corresponding to equation (7) and curve B to equation (8). The ordinate H is shown as a function of the abscissa x. The ordinate ba represents the uniform intensity of

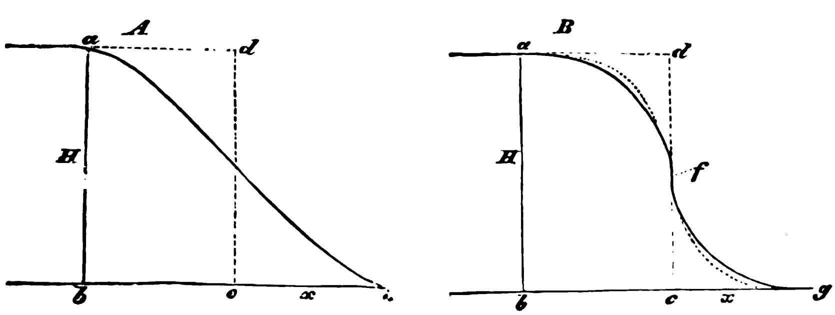

Fig. 71.

illumination of the interior portion of the surface, and the point c shows the position of the edge, so that the dotted line adc would indicate the intensity of an image that was sharply defined at the edge. The diameter of the blur circle of the point c is bg. A striking

difference between curve A and curve B is that the latter becomes perpendicular to the x-axis at the point f, corresponding to the actual position of the boundary. That is, for $x=0$, the derivative

$$\frac{dH}{db}=\frac{2Bb}{N(N-1)}\left\{\frac{2}{r}\sqrt{r^2-x^2}+\log\text{ nat }\left[\frac{r-\sqrt{r^2-x^2}}{r+\sqrt{r^2-x^2}}\right]\right\} \quad . \quad . \quad . \quad (9)$$

becomes infinite. This sudden drop in intensity enables the eye to recognize the position of the edge, even if some light does extend beyond it. In the case represented by curve A, however, the falling off is more gradual, so that there is nothing to indicate exactly where the edge is.

If it were possible to take into account the diminishing brightness of the colours towards the ends of the spectrum, the curve B would be found to follow more nearly the dotted line drawn adjacent to it. That is, the intensity would approach its normal maximum value more rapidly inside the edge, and would fall off more rapidly outside, than curve B indicates.

It will be clear from these considerations why the distinctness of visual images is so little affected by chromatic aberration. A combination of lenses intended to correct the chromatic aberration of the eye does not produce any appreciable improvement of the visual acuity, according to the writer's experience. A concave flint glass lens of 15.4 mm focal length, taken from the objective of a microscope, was found suitable for this purpose. It was combined with convex crown glass lenses so as to produce a system with a negative focal length of about 2.6 ft; which was adapted for the writer's eye so that he could discern distant objects through it clearly. On looking through this arrangement with half of the pupil covered, no coloured fringes appeared at the boundary between light and dark. The same was true even with the eye imperfectly accommodated; so that the lenses evidently rendered the eye practically achromatic. There was, however, no apparent improvement in definition.

The chromatic aberration in the eye was known to NEWTON, who mentions the coloured fringes that appear when the pupil is half covered.[1] NEWTON made the mistake of supposing that the dispersion of all transparent media is proportional to their refraction; and hence he concluded that an achromatic combination of lenses was impossible. Curiously enough, EULER,[2] starting from the false premise that the eye is achromatic, argued that NEWTON must have been wrong as to his theory of dispersion and thence deduced the correct conclusion that an achromatic combination of lenses was possible. D'ALAMBERT[3] took exception to this reasoning, by pointing out that the chromatic aberration may not necessarily become noticeable in the

[1] *Optics.* Lib. I. P. II. Prop. VIII.

[2] *Journal Encyclop.* 1765. II. p. 146. — *Mém. de l'Acad. de Berlin.* 1747.

[3] *Mém. de l'Acad. de Paris.* 1767. p. 81.

eye, even if it were just as great as it is in glass lenses. DOLLOND,[1] too, argued the same way and asserted that the eye cannot be achromatic, no matter if there are several different refracting media in it; because at every refraction the rays are bent towards the axis. The validity of DOLLOND's reasoning will be manifest when it is remembered that the violet rays are always more refracted than the red rays in going from one medium to another, so that at every refraction in the eye the violet rays must be bent towards the axis more than the red rays. MASKELYNE[2] measured the chromatic aberration of the eye, and found the interval between the focal points to be 0.61 mm corresponding to a visual angle of 15″; whereas a telescope is considered as achromatic when this angle is as much as 57″. JURIN[3] noted the coloured edges of ill-defined objects; and WOLLASTON[4] the peculiar appearance of the prismatic spectrum due to the inability of the eye to accommodate for all colours at once. MOLLWEIDE[5] gave a complete theory of the appearances observed with the pupil half covered, and TOURTUAL discussed fully all the phenomena relating to the subject. The first accurate measurements of the chromatic aberration of the eye were made by FRAUNHOFER[6] with reference to the fixed spectral lines discovered by WOLLASTON and himself; and subsequent measurements were made by MATTHIESSEN.[7]

And yet in spite of all these investigations, many natural philosophers, like FORBES[8] and VALLÉE,[9] continue to cling to the idea that the eye is absolutely perfect and that it is more or less completely achromatic.

1704. J. NEWTON, *Optics*. B. I. P. II. Prop. VIII.*
1747. L. EULER, *Mém. de Berlin*. 1747. p. 285. — 1753. p. 249. — 1754. p. 200.
1767. D'ALEMBERT, *Mém. de l'Acad. de Paris*. 1767. p. 81.*
1789. MASKELYNE. *Phil. Trans.* LXXIX. 256.*
1798. COMPARETTI, *Observationes de coloribus apparentibus*. Patavini.
1801. TH. YOUNG, *Phil. Trans.* 1801. P. I. p. 50.*
1805. MOLLWEIDE in GILBERTS *Annalen*. XVII. 328. u XXX. 220.
1814. *FRAUNHOFER in GILBERTS *Annalen*. LVI. 304. — SCHUHMACHERS *astronom. Abhandlungen*. Altona 1823. Heft II. S. 39.
1826. J. MÜLLER, *Zur vergl. Physiol. des Gesichtssinns*. Leipzig. S. 195. 414.*
1830. *TOURTUAL, Über Chromasie des Auges. MECKELS *Archiv*. 1830. S. 129.*
1837. MILE, POGGENDORFFS *Ann.* XLII. 64.
1847. A. MATTHIESSEN, *Comptes rendus*, XXIV. 875; *Institut*. No. 698. p. 162; POGGENDORFFS *Ann.* LXXI. 578*; FRORIEPS *Notizen*. III. 341; *Archive d. sciences phys. et natur.* V. 221; *Berl. Berichte*. 1847. S. 183.*
L. L. VALLÉE, *Comptes rendus*. XXIV. 1096; *Berl. Ber.* 1847. S. 184.*
1849. J. D. FORBES, *Proceed. Edinburgh Roy. Soc.* Dec. 3. 1849. p. 251; SILLIMAN'S *Journ.* (2) XIII. 413; *Berl. Ber.* 1850. p. 492.*
1852. L. L. VALLÉE, *Comptes rendus*. XXXIV. 321; *Berl. Ber.* 1852. S. 308.*
1853. L. L. VALLÉE, *Sur l'achromatisme de l'œil* *C. R.* XXXVI. 142–144; 480–482.
1855. CZERMAK, Zur Chromasie des Auges. *Wiener Sitzungsber.* XVII. 563.
1856. A. FICK, Einige Versuche über die chromatische Abweichung des menschlichen Auges. *Archiv für Ophthalm.* II. 2. 70–76.

[1] *Philos. Trans.* LXXIX. p. 256.
[2] *Philos. Trans.* LXXIX. 258.
[3] SMITH's *Optics*. 96.
[4] *Philos. Trans.* 1801. P. I. p. 50.
[5] GILBERTS *Annalen*. XVII. 328. XXX. 220.
[6] GILBERTS *Annalen*. LVI. 304. — SCHUMACHERS *astronom. Abhandlungen*. Heft II. S. 39.
[7] *Comptes rendus*. XXIV. 875.
[8] *Proc. Roy. Edinb. Soc.* Dec. 3. 1849. p. 251.
[9] *Comptes rendus*. XXIV. 1096. XXXIV. 321.

1862. F. P. LEROUX, Expériences destinées à mettre en evidence le défaut d'achromatisme de l'œil. *Ann. de chimie.* 3. LXVI. 173-182. *Cosmos.* XX. 638-639.
— TROUESSART, *Défaut d'achromatisme de l'œil. Presse scientifique.* p. 72-74.

Note by A. Gullstrand

With the further development of the theory of optical imagery and of the so-called aberrations, for the sake of accuracy, a distinction should be made between the phenomena that were formerly grouped together under the head of chromatic aberration. Since all the magnitudes that determine the nature of the imagery may have different values for different indices of refraction, the only exact way of describing the phenomena of chromatic aberration, as it is called, is in terms of the *chromatic differences* of these parameters. Along the axis of a centered optical system, for example, we must first take into account the *chromatic difference of focus* and the *chromatic difference of magnification*. By the former is meant the distance between the two focal points due to two different kinds of light; and the latter is proportional to the difference between the primary focal lengths of the system. The difference of magnification cannot be found directly from the difference of focus except in the special case when the position of the second nodal point is independent of the index of refraction, as is the case in the reduced eye, but as is not the case in the human eye. For the same reason, both differences vanish simultaneously in the achromatization of an infinitely thin optical system; but as a rule this does not happen.

The focus difference is what HELMHOLTZ has in mind above in his discussion of the distribution of light within the blur circles. As to the difference of magnification, perhaps it might be neutralized by just the same mechanism as for the focus difference, provided the images are very small. The reason why a bright point is seen without a coloured border, is because, as HELMHOLTZ has proved, the dimensions of the blur circles of light of the longest and shortest wave-lengths are about equal and also because the coloured borders are faint. However, two adjacent bright points cannot produce concentric blur circles at the same time for light of both short and long wave-lengths, since one of them must necessarily lie outside the axis. The centre of the blur circle due to light of short wave-length is nearer the axis for this latter point than that of the blur circle due to light of long wave-length; and hence the two sides of this point cannot be seen in one and the same colour at the same time, on account of the chromatic difference of magnification. Since the chromatic difference of the size of the image is to the size of the image for a given kind of light in the same ratio as the chromatic difference of focal length is to the focal length for the given kind of light, the value of this ratio can be obtained for

the reduced eye from HELMHOLTZ's numerical data. In round numbers it is 3 per cent. While it may be somewhat larger than this in the human eye, obviously the image must be fairly large before the chromatic difference of magnification would be noticeable; and considering the distribution of light in the chromatic blur circles, it is hardly to be supposed that this difference is appreciable when it is measured in minutes and is of the same order of magnitude as the angular size of the place of most distinct vision.

The chromatic difference of magnification increases with the size of the image, but at the same time the ability to see the coloured borders diminishes, because they fall on parts of the retina that are farther removed from the place of most distinct vision. Now if this decrease of the visual acuity were proportional to the increase of the width of the coloured border, the phenomena of the chromatic difference of magnification, although physically present on the retina, would not be physiologically revealed by the visible appearance of coloured fringes. This physiological mechanism is adequate to mask the chromatic differences of magnification in the outlying portions of a large retinal image. If, however, the object consists of two bright, narrow lines whose angular separation amounts to 3°, and a point is fixated symmetrically situated between them, the lines will appear more than 5′ nearer together when they are illuminated by violet light than when they are illuminated by red light; this difference being divided equally between both lines, so that each of them appears like an impure spectrum of about 2.5 minutes in apparent width. Without making the statement that the eye is able to distinguish between a spectrum of this kind and a bright line at an angular distance of 1.5° from the point of fixation (because at present exact determinations of this question have not been made and perhaps they would be difficult), it may be noted that there is in the eye a mechanism tending to neutralize the chromatic difference of magnification in the case of objects of this kind.

As the writer has shown by entoptical investigations,[1] considerable chromatic dispersion occurs in the passage of light from the vitreous humor to the retina. The reason of this is that the shadows due to refraction of light in the most curved central portion of the fovea—the entoptical fovea, as the writer has called it—cannot be seen with light of long wave-length, but only with light of short wave-length, and is undoubtedly due to the luteous ingredients in the lymph of the retina. On leaving the vitreous humor, rays belonging to image-points near the axis are refracted in the fovea away from the axis, and since

[1] Die Farbe der Macula centralis retinae. *Arch. f. Ophth.* LXII, 1. S. 1. 1905. Zur Maculafrage. Ibid., LXVI, 1. S. 141. 1907.

this effect is greater for the rays of shorter wave-length, the result is a comparative magnification of the images due to these rays. Just how much this contributes toward neutralizing the effect of the chromatic difference of magnification, it would be difficult to say.

Since the centres of the refracting surfaces of the eye do not all lie on the line of sight, the centre of the exit-pupil of the eye has different positions for light of different wave-lengths. Strictly speaking, therefore, a *chromatic difference of the directions of the lines of sight* in the vitreous humor has to be taken into account. However, this effect is hidden by the unsymmetrical form of the blur circle due to the monochromatic aberrations. This is made manifest by viewing a small artificial source of light through cobalt glass with the eye adjusted for red; the bluish blur circle that most people see appears broader on the temporal than on the nasal side.

§ 14. Monochromatic Aberrations[1]

Besides the inexactness of the image due to unequal refrangibility of light of different frequencies, optical instruments made of glass lenses with spherical surfaces are subject to another kind of aberration, namely, *defects on account of spherical form* or so-called *spherical aberration*. This aberration is due to the fact that, in general, a homocentric bundle of rays even of one homogeneous kind of light after being refracted at curved surfaces will be only approximately homocentric. There are, indeed, certain curved surfaces (aplanatic surfaces) which may, under proper conditions, reunite the rays exactly at one point. They are surfaces of revolution generated, as a rule, by a curve of the fourth degree; though in certain cases, for example, when the point source is at infinity, the genetratrix is an ellipse.[2] Moreover, by a suitable choice of constants (indices of refraction, curvatures and distances apart) it is possible to design centered systems of spherical refracting surfaces in which for a given point on the optical axis the spherical aberration is more or less inappreciable. Systems of this kind are called aplanatic also.[3] Of course, the blurred image of a luminous point on the axis of a centered system of spherical refracting surfaces is symmetrical with respect to the axis. It forms a round spot which is brightest at the centre and fades rapidly in all directions from that point.

[1] See Appendix V at end of Part I. G.

[2] ¶ Or an hyperbola, in case the first medium is more highly refracting than the second. (J. P. C. S.)

[3] ¶ This is the original meaning of the term "aplanatism" in geometrical optics; but since the time of ABBE the word has become more restricted and is applied to an optical system which is not only free from spherical aberration along the axis but also satisfies the so-called "sine condition." (J. P. C. S.)

The monochromatic aberrations in the optical system of the eye are not, like the spherical aberration of glass lenses, symmetrical about an axis. They are much more unsymmetrical and of a kind that is not permissible in well constructed optical instruments. Neither the term "spherical aberration" as applied to spherical surfaces nor the awkward expression "aberration on account of the form of the refracting surface," which is used with respect to other curved refracting surfaces, is sufficiently general to describe this particular fault in the case of the optical system of the eye; and the term that will be used here is *monochromatic aberration*, since this aberration is concerned simply with homogeneous (monochromatic) light, and also in order to distinguish it from the *chromatic aberration*, which was the subject of the preceding section.

The phenomena are as follows:

1. Suppose the object is a very small luminous point (for example, a pinhole in a piece of opaque black paper illuminated from behind), which is situated rather farther from the eye than the greatest distance of accommodation; so that its image on the retina will be indistinct. In place of the bright point, what is seen is not a circular spot, as in the case of a telescope out of focus, but a star-shaped pattern with from four to eight irregular points or rays, which is usually different in the two eyes, and different also for different individuals. Fig. 72, *a* and *b*, show the patterns as they look in the writer's right and left eye, respectively. When blurred images of this kind are produced by white light, the outer edges of the bright parts have a blue border, and the edges towards the center are orange coloured. For most people the vertical dimension of the figure is greater than the horizontal. When the illumination is faint, only the brightest parts of the pattern are visible, and it then appears as a small cluster of spots, one of which is usually more conspicuous than the others. With very intense light, on the other hand, as when the pinhole is illuminated by direct sunlight, the points of the star seem to merge together, while all around it an immense number of exceedingly fine, brilliantly coloured, radiating lines form a sort of corona of much greater extent. The name *hair corona* will be used to distinguish this phenomenon from the star-shaped blurred image.

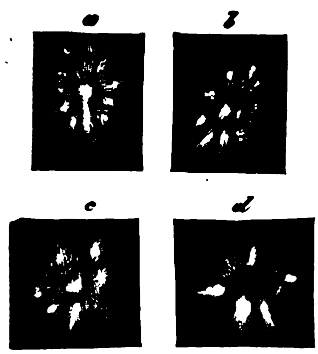

Fig. 72.

Now as he is watching the star-shaped figure, or the group of spots, in case of feeble illumination, suppose the observer gradually inserts

a card from below in front of his eye; then he will see the lower part of the blurred image eclipsed first, which corresponds, of course, to the upper part of the retinal image; and if the card is interposed from above, or from the right or left, the part of the image that disappears first will be the upper part or right or left part, respectively. But the behaviour of the more extended hair corona, produced by very bright illumination, is different. When the pupil is gradually covered from below, the lower part of the central, star-shaped figure disappears as before, but not the lower part of the corona. However, the appearance of the corona will be disturbed and altered, and very brilliant diffraction patterns will develop, depending on the reduction and deformation of the pupillary opening. The radiant appearance of stars and distant lights is an example of these phenomena.

2. When the eye is accommodated for a distance greater than that of the luminous point (which may be accomplished in case the source is far away by inserting a weak concave lens before the eye), there appears another kind of star-shaped figure, whose longer dimension is generally horizontal. Fig. 72, *c* and *d*, represent these appearances in the writer's right and left eye, respectively. Now when the pupil is gradually covered from one side, it is the opposite side of the figure that disappears first, and hence the same side of the retinal image. Accordingly, this figure is made by rays that have not yet crossed the axis of the eye. If the cornea is moist with tear-fluid, or has been covered with oil-drops from the MEIBOMian glands by vigourous blinking, the star pattern is usually larger and more irregular and considerably altered by blinking, and when the pupil is covered from one side, the effect is not a simple disappearance of one side of the figure.

3. When the luminous point is situated where the eye can accommodate for it, moderate illumination shows it as a small, round, bright spot without any irregularities. The radiant appearance again becomes evident, however, when the illumination is increased, whatever the state of accommodation. With gradual changes of accommodation, all we find is that the starry pattern which is vertically elongated for shorter accommodation becomes smaller and rounder and then elongates horizontally into a star again, when the distance of accommodation exceeds that of the source of light.

4. When the source of light is a narrow line, the resulting appearance can be found by constructing the blurred starry images for each separate point of the luminous source, which will partly overlap each other. The brighter portions of these figures unite to form stripes, which appear as multiple images of the bright line. Most people see two of them, whereas in certain positions many persons see five or six of these double images.

In order to demonstrate experimentally also the connection between these double images of lines and the star-shaped images of points, cut a narrow slit in a piece of black paper, and make a small pinhole a little way from one end of it, as shown in Fig. 73, *a*. Viewed from a distance, the double images of the slit will be seen to be just as far apart as the brightest parts of the star, and to be exactly in line with them, as shown in Fig. 73, *b*; where, in the blurred image of the bright point, only the brightest parts of the star-shaped spot in Fig. 72, *a* are reproduced.

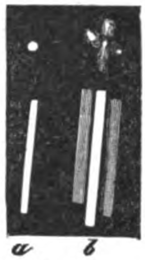

Fig. 73.

An example of this phenomenon is afforded by the multiple images of the horns of the crescent moon as seen by most persons. Another illustration is the effect that is occasionally seen at the edge of a bright area for which the eye is not exactly accommodated. The transition at the edge from light to darkness appears to be made in two or three stages. Other related phenomena are to be described later in the theory of irradiation.

5. In general, the accommodation of the eye is not the same for horizontal and for vertical lines at the same distance. Suppose a person looks intently at a pencil of rays all radiating from one point, as shown in Fig. 74, the eye being at a suitable distance for easy accommodation. Now it will be noticed that each of these lines can be seen separately clear and distinct one after another, but only one at a time can be made to stand out sharp and black, and all the others will appear more or less blurred. If the person is accustomed to noting changes of accommodation in his own eyes, he will observe that it takes less accommodation to focus the horizontal line distinctly, and more accommodation to focus the vertical line distinctly. That is, a vertical line must be held farther from the eye than a horizontal one, in order to see them both with equal clearness at the same time. A. Fick saw vertical lines distinctly at a distance of 4.6 m, and at the same time horizontal lines at a distance of 3 m. In the writer's case the distances were 65 cm and 54 cm for horizontal and vertical lines, respectively.

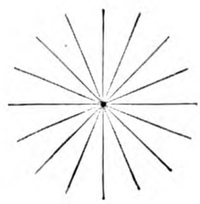
Fig. 74.

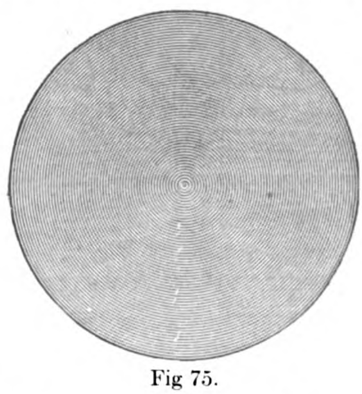
Fig 75.

Consider a white card on which a large number of concentric circles is drawn very close together but at equal distances apart, as repre-

sented in Fig. 75. When this is inspected at a convenient distance for good accommodation, peculiar radiating sectors will be seen, which on closer examination will be found to be due to the fact that along the brighter radii the black and white lines are sharply differentiated, while in between them are bright grey blurred places where the black lines are fainter. By varying the accommodation a little or the distance of the diagram from the eye, other portions of the figure become clear, and we get an impression as if the sharply defined radii were rapidly rotating. When the card is held fairly close to the eye, and the eye accommodated for a farther distance, some 8 or 10 sectors will be seen with clear-cut lines; and where these join each other, they are nebulous, but it is evident that the black lines of one sector do not harmonize with those of the adjacent sectors. The result is the most interior circles have a queer distorted appearance. Obviously, these phenomena are all due to some sort of asymmetry of the eye. An optical instrument that is symmetrical around its axis may certainly produce a blurred image of a point on the axis; but it would have to be symmetrical and therefore circular.

The next thing to be ascertained is what features of the small, star-shaped, blurred image are permanent and occur regularly in good eyes; and, on the other hand, how it is modified by such transient influences as the presence of tear-fluid and other impurities on the cornea from the lids and eyelashes. A. Fick has shown that these latter effects may be imitated artificially by obtaining the image of a luminous point in a glass lens with a film of water on its surface.

These evanescent phenomena are of rare occurrence in the writer's own eyes. Ordinarily the same patterns as those shown in Fig. 72, which somehow recall the star-shaped structure of the crystalline lens as depicted in Fig. 20, constantly recur. In fact, the author might be persuaded that the most essential characteristics of these star-shaped patterns were due in some way to the irregularities of the crystalline lens, because when the illuminated hole is very close to the eye, the so-called entoptical phenomena, to be described in the next section, become manifest in the blurred image. There also we shall see how the place in the eye can be determined where the little object is that causes this phenomenon. What was found was, that as the distance of the luminous point from the eye was gradually increased, certain alternately bright and dark bands produced by the entoptical image of the crystalline lens were transformed into the bright and dark specks and streamers of the star patterns shown in Fig. 72, *c* and *d*. Sketches illustrating the process were made by Young.[1]

[1] *Philos. Transact.* 1801. I. pl. VI.

As to the second class of phenomena above mentioned, namely, differences of focal length for horizontal and vertical lines, the explanation as yet is not quite so certain. Such effects are of course to be expected wherever light is refracted by surfaces with different curvatures in different directions, or even by spherical surfaces, when the rays are incident obliquely; either of which might occur in the eye. Horizontal and vertical meridian sections of the refracting surfaces of the eye have not quite the same curvature; and, besides, we know that the human eye is not exactly centered, and that the place of direct vision is not on the straight line that comes nearest to meeting the requirements of an axis of the eye.

The focal lengths of YOUNG's eye were quite different in the horizontal and vertical meridians, and he found by experiment that this difference was not in the cornea.[1] Thus, when his eye was immersed in water so as to neutralize almost completely the refraction of the cornea, there was practically the same difference of power of accommodation in the two meridians as before. This fault of the eye can be corrected, as YOUNG likewise remarked, by using spectacles inclined at a certain angle to the axis of the eye; as has been verified by the writer, who found that a weak concave lens could be held before the eye and adjusted at the right angle to enable him to see equally distinctly both horizontal and vertical lines at the same time.

And, lastly, imperfect transparency of the ocular media may also be partly responsible for monochromatic aberrations in the eye. The fibres of the cornea and lens certainly seem to be bound together by another substance of pretty nearly the same index of refraction, so that with moderate illumination these parts appear perfectly homogeneous and transparent. But when they are strongly illuminated by concentrating light on them with a lens, the light reflected from the edges of the elementary parts is strong enough to produce a faint cloudy glow. A part of the light transmitted through them is scattered and must fall on parts of the retina outside the place where the regularly refracted light goes. In fact, when a very brilliant light is seen against a dark background, there is a nebulous white glow over the surroundings, brightest near the light. As soon as the light is cut off, the luminosity of the background vanishes also. It would seem that this phenomenon must be explained by diffused refraction.[2]

The theory of refraction at aspherical surfaces and at spherical surfaces at oblique incidence will not be fully treated here, because, without more accurate information as to the forms of the refracting

[1] *Philos. Transact.* 1801. I. p. 40.

[2] HELMHOLTZ in POGGENDORFFS *Ann.* LXXXVI. 509.

surfaces of the eye, it would be of little use for investigations of ocular refractions. It will suffice to examine two cases of this sort of refraction, in which the conditions are mathematically simple.

First, consider the refraction at the vertex of an ellipsoid with unequal axes. Let the axis corresponding to the vertex be represented by the straight line gb, Fig. 76; the luminous point p being supposed to lie on it. The plane of the diagram is a principal section of the ellipsoid and contains therefore another axis gh. Since the normals to the ellipsoid at points in a principal section are also contained in that section, the normals to the curve bch must therefore lie in the plane of the diagram. A ray pc will be refracted at c in a direction lying in the plane through p and the normal at c, that is, in the plane of the diagram, and hence will intersect the axis bg at some point q. This would not be so if the plane of the diagram were not a principal section of the ellipsoid. Draw ad normal to the surface at c; then by the law of refraction

$$\sin \angle pcd = n \sin \angle acq,$$

where n denotes the relative index of refraction of the two media. So far the condition is the same as for a surface of revolution. The rays that are incident on the surface nearly normally in the vicinity of the vertex b will therefore all be refracted to a point on the axis whose position depends on the radius of curvature ($r_{\prime}$) of the curve bch at the point b. When the point p is at infinity, the focal length of the surface in this principal section is equal to $\dfrac{n\, r_{\prime}}{n-1}$. A perfectly similar expression will be obtained in the same way for rays issuing from p in the other principal section (which is the plane containing bq and the third axis of the ellipsoid), except that the radius of curvature at the vertex of the ellipsoid has a different value ($r_{\prime\prime}$) in this principal section; and hence the focal length of the rays in the second principal section is $\dfrac{n\, r_{\prime\prime}}{n-1}$. That is, the ray pq will be intersected by those immediately adjacent to it in the plane of the diagram at some point q; but it will be intersected by the adjacent rays lying in a plane perpendicular to that of the diagram, not at the point q, but at some other point s. Now let us suppose that the rays from p all pass through a circular opening perpendicular to the axis pq at b. The cross section of this bundle of rays will be circular

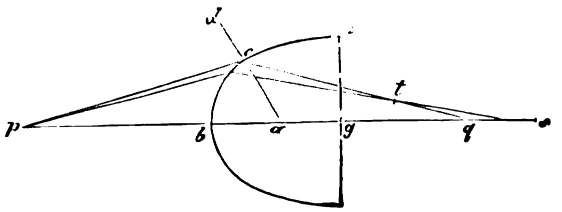

Fig. 76.

at b; but between b and q it will be elliptical with the major axis of the ellipse perpendicular to the plane of the diagram. The ellipse gets smaller and more elongated, the nearer the cross section is to q; and at q the cross section of the bundle of rays is a horizontal straight line perpendicular to the plane of the diagram. Beyond this point the cross section is again an ellipse with its major axis perpendicular to the plane of the diagram, which soon becomes more and more round and is actually circular about midway between q and s; then it becomes an ellipse again with its major axis in the plane of the diagram, which collapses into a vertical line at s. Beyond this point it gradually gets broader again and becomes more and more circular.

The results are similar when a narrow bundle of incident rays encounters a spherical refracting surface obliquely. Suppose that the curve *bch* in Fig. 76 is the section of a spherical surface and that the straight line *pc* represents a ray incident on it at a finite angle of incidence. We know (cf. Fig. 32) that rays proceeding in the plane of the diagram and incident on the surface in the immediate vicinity of the point c do not intersect each other in a point on the axis *pq* after refraction, but at a point off the axis which lies on the caustic surface. Suppose t is this point. Conceive now that the whole figure is rotated around *ap* as an axis; so that the ray *pc* will come in succession into the positions of all rays from p that cross the axis at the same angle. The corresponding refracted ray *cq* will also describe a ray-cone with its vertex at the point q. Consequently, whereas the rays immediately adjacent to *pc* in the plane of the diagram intersect it at t, those adjacent to it on either side of that plane intersect at it q. Finally, it may be added that other adjacent rays do not intersect *pc* at all.

Another question to be asked about this subject is, What is the effect of diffraction of light at the edge of the pupil on the monochromatic aberrations? And this suggests a further question as to whether the star-like form of the small blurred patterns may not be due to the minute indentations in the edge of the pupil. In fact, if a luminous point is viewed through an opening smaller than the pupil, the edge of which is not perfectly smooth, a more extensive star-shaped figure will be seen; although, as a rule, such patterns consist rather of very fine hair-like brightly coloured rays, similar to the hair corona of the eye which was described above and which is seen around any brilliant point-source even without using an artificial opening. When the opening is rotated around its centre, the entire corona turns with it; which shows that the effect is due to the contour of the aperture.

The writer could not be sure of perceiving in his own eye the indication of any diffraction of light due to the fine fibrous structure of the crystalline lens. When a small luminous point is observed through a

smooth round hole in a metal disc, invariably the whole diffraction pattern turns with the disc. Any features of this pattern that might be due to the fibres of the cornea or lens would necessarily have remained stationary. However, BEER[1] describes diffraction phenomena in his eye which he attributes to a fibrous structure of the ocular media.

Still these diffraction phenomena are essentially differentiated from the appearance of the small blur circles by the circumstance that the latter are eclipsed on one side, without affecting the appearance of the opposite side, when the pupil is screened on one side. On the contrary, when a diffraction pattern is caused by a fine thread or a narrow slit, it never extends merely in one direction, but always in two opposite directions, because every interruption of a wave of light is invariably manifested in opposite, and usually in all, directions. Now the hair-line figures have this very characteristic; as soon as an obstacle begins to encroach upon the pupil, the whole figure becomes more or less altered and distorted.

But apart from the diffraction effects due to irregularities in the contour of the pupil, it should be remembered that the pupil as a whole being a small round opening, may also cause diffraction. Whenever light from a point-source is refracted at one or more surfaces of limited aperture, even if the optical system is otherwise achromatic and aplanatic, the focus of the refracted rays is not a sharply defined point, but a small light pattern, due to diffraction at the edge of the aperture, with alternately bright and dark places, whose form and position depend generally on the size and shape of the aperture. If the aperture is circular, as in the eye and in most optical instruments, the diffraction figure consists of a bright central disc surrounded by several dark and bright rings whose brightness falls off rapidly. Let d denote the diameter of the aperture, r its distance from the image, and l the wave-length of the light; then by both theory and experiment it is found that the diameter of the central bright circle is

$$\delta = 2.440 \frac{lr}{d}.$$

If we put $l = 0.0005$ mm (as representing the average value of a wave-length of light) and $r = 20$ mm for the eye, the above formula becomes:

$$\delta = 0.0244 \cdot \frac{1}{d},$$

where the magnitudes are all expressed in millimetres. For the minimum diameter of the pupil ($d = 2$ mm), $\delta = 0.0122$ mm. A blur circle of this size on the retina corresponds to a visual angle of 2′6″, and is

[1] POGGENDORFFS *Ann.* LXXXIV. 518.

equal to that corresponding to a point 25 metres away, when the eye is accommodated for an infinite distance. Since the smallest visible object consisting of two separate points subtends an angle of about 1′, diffraction effects begin to impair visual acuity when the pupil has its least diameter.

Another instance of monochromatic aberration is found in the streaks of light that appear to proceed upwards and downwards from a bright object when the eyelids are half closed. These are due to refraction by the concave boundaries of the areas of moisture left upon the eyeball by the lids. These curved boundaries act like a small prism of variable angle, or a series of small prisms, and cause considerable deviations in the direction of the incident light.

Measurements made by various physicists as to the inequality of the focal lengths of the eye in the horizontal and vertical meridians indicate that much divergence exists among individuals in this respect. Some observers, as Brücke,[1] fail to detect any difference at all, or obtain conflicting results.

Young reports that his eye will bring to the same focus rays in the vertical meridian diverging from a luminous point 304 mm away and rays in the horizontal meridian from one 213 mm away. In order to express this difference independently of the accommodation distance, he computes the focal length of a cylindrical lens which, used as a spectacle glass, would make the two distances identical, and finds that it would be 700 mm. That is, a convex lens of this focal length with the cylindrical axis horizontal, or a concave one with the cylindrical axis vertical, would have neutralized the defect in his case. The two corresponding distances found by A. Fick for his eyes were 3 m and 4.6 m, respectively, while the writer finds 0.54 m and 0.65 m. These latter figures will be seen to differ in the opposite sense from those of Young, and to be much smaller in amount. Expressed in terms of the focal length of a cylindrical lens, the difference of the two meridians in Fick's case corresponds to a focal length of 8.6 m and in the writer's case to one of 3.19 m. Data of this kind are easily obtained by means of two pins stuck, one vertically and the other horizontally, in a long bar; the observer looks from one end of the bar at one of the pins and adjusts the other at such a distance that both are clearly visible at once.

A. Fick observes that an eye looking casually at anything is usually accommodated for vertical lines. In order to make an approximate calculation of the distance between the two focal planes, let us suppose that Listing's schematic eye is accommodated for vertical lines; and that the difference of focus in the horizontal and vertical meridians is the same for this eye as it was in each of the three cases cited above. We find that the focal point of rays in the horizontal meridian would be, according to the data of:

Th. Young, 0.422 mm in front of the other;
A. Fick, 0.035 mm beyond the other;
H. Helmholtz, 0.094 mm beyond the other.

It is to be noted that these differences amount to less than the chromatic aberration of the focal points of red and violet rays (0.6 mm). Moreover, they have very little effect on the visual acuity so long as it is simply a question of distinguishing separate lines that all have the same general direction; and they are not detrimental to any extent except when it comes to seeing crossed lines distinctly at the same time.

[1] *Fortschritte der Physik im Jahre* 1845. Bd. I. S. 211.

The multiple images of a point or a line arising from imperfect accommodation had been noted by DE LA HIRE[1] and JURIN,[2] neither of whom, however, found the correct explanation. Subsequently, YOUNG[3] described and sketched the form of the blurred figures for various distances of the luminous point; and expressed the opinion that the star-like shapes might be due to slight irregularities in the anterior surface of the crystalline lens. HASSENFRATZ[4] referred to them also and attributed them to the same cause, regarding them as lines of intersection of two caustic surfaces. PURKINJE[5] describes the phenomena of multiple images, and also those observed on viewing fine parallel lines, and constructs the star figure, which he considers as most probably due to facets of the cornea. PÉCLET[6] notes the multiple images of a bright line, and ascribes it to peculiarities of the refracting surfaces, as do also NIEDT,[7] GUÉRARD[8] and FLIEDNER.[9] The last named gives a very complete description of all these phenomena. TROUESSART[10] thinks that a network must be assumed to be beyond the refracting surfaces of the eye, and that the multiple images are produced by its manifold meshes, on the same principle as SCHEINER'S experiment. FICK'S[11] view of this matter has been mentioned already. These same phenomena have been discussed also by AIMÉE[12] and CRANMORE.[13] An entirely unique theory as to the origin of the multiple images, known to oculists as *polycopia monophthalmica*, was advanced by STELLWAG VON CARION.[14] He claims to have observed that the different images are formed by light polarised in different planes. But this is a mistake; and he was probably misled by using in his experiments a poorly polished tourmalin crystal with slightly curved surfaces or striations in the interior. A plate of this kind with a slightly cylindrical surface, rotated in front of the eye, would bring the horizontal and vertical rays to a focus alternately, thus eliminating one set of double images at a time. In order to get rid of the effect of such a fault in the crystal, it should be placed between the source of light and a small pinhole in a screen, so that the light is polarised before it comes through the opening. If now the opening is observed from a sufficient distance to get the star-shaped figure, it will be found that rotating the tourmalin, and therefore also the plane of polarisation of the light, does not have the slightest influence on the double images. Besides, the results said to have been obtained by STELLWAG are not in accordance with the laws of double refraction. His theory has been disproved by GUT.[15] STELLWAG'S paper includes a summary of the medical literature on the pathological occurrence of abnormal *diplopia monophthalmica*.

[1] *Accidens de la vue.* p. 400.

[2] SMITH'S *Optics.* "Essay on distinct and indistinct vision." p. 156.

[3] *Philos. Transactions.* 1801. I. p. 43. Pl. VI.

[4] *Ann. de Chimie.* 1809. T. LXXII. p. 5.

[5] *Beiträge zur Kenntnis des Sehens.* S. 113—119. *Neue Beiträge z. Kenntnis d. Sehens.* S. 139-146. 173.

[6] *Ann. d. Chimie et d. Phys.* LIV. 379. — POGGENDORFF'S *Ann.* XXXIV. 557.

[7] *De dioptricis oculi coloribus ejusque Polyopia.* Dissert. Berolini 1842.

[8] *Institut.* 1845. No. 581. p. 64.

[9] POGGENDORFFS *Ann.* LXXXV. 321. 460. LXXXVI. 336. *Cosmos.* I, 333.

[10] *C. R. de l'Acad. d. sciences.* XXXV. 134 136, 398. *Archive de Genève.* XX. 305. *Institut.* 1852. p. 304.

[11] HENLE u. PFEUFFER, *Zeitschrift.* N. Folge V. S. 277.

[12] *Ann. d. Chimie et d. Physique.* LVI. 108. — POGGENDORFFS *Ann.* XXXIII. S. 479.

[13] *Philos. Magazine.* (3) XXXVI, 485.

[14] *Wiener Sitzungsberichte.* VIII. 82. *Denkschriften d. k. k. Akad.* V. 2. p. 172.

[15] *Über Diplopia monophthalmica.* Dissert. Zürich 1854.

The phenomena of diffraction in the eye have been investigated by BAUDRIMONT,[1] WALLMARK[2] and BEER.[3] The streaks of light, produced by the concave films of tear-fluid when the eyelids are half closed, were noted by MEYER[4] of Leipzig.

YOUNG[5] seems to have been the first to mention the asymmetry of the eye in different meridian planes. He quotes the statement of a Mr. CARY, to the effect that many persons find it necessary to wear their glasses close against the eyes, in order to see well with them. Further observations on the subject were made by AIRY,[6] FISCHER,[7] CHALLIS,[8] HEINEKEN,[9] HAMILTON,[10] SCHNYDER,[11] and, finally, by A. FICK.[12] SCHNYDER employed cylindrical spectacle lenses to correct this defect. A more complete summary of these investigations will be found in FECHNERS *Zentralblatt* for 1853, pp. 73-85; 96-99; 374-379; 558-561.

The question of the spherical aberration of the eye, in the sense in which that term is applied to artificial optical instruments, loses its practical importance as compared with the much more serious defects that have been described. In YOUNG's experiments with his optometer, described in the preceding section, a single thread viewed through four openings appeared as four intersecting lines which did not meet all in one point when the eye was accommodated for near vision. VOLKMANN[13] likewise endeavoured to determine by experiment whether the optical system of the eye shows spherical aberration. Through four small openings in a curved line in a screen, he looked at a needle, which could be set at different distances from the eye. Evidently, if the axial portion of the eye brought the rays to a nearer focus than the peripheral parts, this fact would be proved by getting distinct images through the inner apertures before getting them through the outer apertures, when the needle was moved away from the eye towards the position of distinct vision; and, on the contrary, if the images in the outer holes are seen distinctly before those in the inner holes, it means that the edge rays are brought to a focus first.[14] VOLKMANN's results were different for different individuals. For regular refracting surfaces of revolution these experiments of YOUNG and VOLKMANN would undoubtedly have indicated the nature and magnitude of the spherical aberration of the eye. But in most meridians of the ordinary eye the points of intersection of the refracted rays with the central ray do not form a continuous series at all, so that here the conception of spherical aberration really does not apply.

[1] *C. R. d. l'Acad. d. sc.* XXXIII. 496; *Institut.* No. 931; *Phil. Mag.* (4) II. 575.

[2] POGGENDORFFS *Ann.* LXXXII. 129.

[3] POGGENDORFFS *Ann.* LXXXIV. 518.

[4] POGGENDORFFS *Ann.* LXXXIX. 429.

[5] *Phil. Transact.* 1801. I. p. 39.

[6] *Edinb. Journal of Sc.* XIV. p. 322.

[7] *Berl. Denkschriften* 1818 and 1819. S. 46.

[8] *Transact. of the Cambridge Phil. Soc.* II.: *Phil. Mag.* (3) XXX. 366.

[9] *Phil. Mag.* XXXII. 318

[10] FRORIEPS *Notizien.* VII. 219.

[11] *Verhandl. d. schweizer. naturf. Ges.* 1848. S. 15; FRORIEPS *Notizen.* X. 346; *Arch. de Genève.* X. 302.

[12] *De errore quodam optico asymmetria bulbi effecto.* Marburgi 1851; HENLE u. PFEUFFER *Zeitschrift.* N. Folge. Bd. II. S. 83.

[13] R. WAGNERS *Handwörterbuch für Physiol.* Article: "Sehen."

[14] See description of "annulus" method of testing for spherical aberration, similar to VOLKMANN's method for the eye, in paper by L. D. WELD in *School Science and Mathematics.* XVIII (1918), 547. (J. P. C. S.)

1694. DE LA HIRE, *Accidens de la vue* in *Mém. de l'Acad. de Paris.* 1694. p. 400.
1738. JURIN, *Essay on distinct and indistinct vision.* p. 156 in R. SMITH'S *Optics.*
1801. TH. YOUNG in *Philos. Transactions for* 1801. I. p. 43.*
1809. HASSENFRATZ, *Ann. de Chimie.* T. LXXII. p. 5.
1818. FISCHER, *Berliner Denkschriften für* 1818 *u.* 1819. S. 46.
1819. PURKINJE, *Beiträge zur Kenntnis des Sehens.* Prag. S. 113-119.*
1824. PÉCLET, *Ann. d. Chimie et d. Phys.* LIV. 379; POGGENDORFFS *Ann.* XXXIV. 557.
AIMÉE, *Ann. d. Chim. et d. Phys.* LVI. p. 108.
1825. PURKINJE, *Neue Beiträge zur Kenntnis des Sehens.* Berlin. S. 139-146. 173.*
BREWSTER, *Edinb. Journal of Science.* XIV. p. 322. (Concerning AIRY'S eye).
1842. NIEDT, *De dioptricis oculi coloribus ejusque Polyopia.* Dissert. Berolini.*
1845. GUÉRARD, *Institut.* No. 581. p. 64.
1846. VOLKMANN, Article: "Sehen," in R. WAGNERS *Handwörterbuch für Physiologie.*
1847. CHALLIS in *Philos. Magazine.* (3) XXX. p. 366; *Trans. Cambridge Phil. Soc.* II.
1848. H. MEYER in HENLE u. PFEUFFER, *Zft. für rat. Med.* V. 368.
HEINEKEN, *Philos. Magazine.* (3) XXXII. p. 318.
HAMILTON in FRORIEPS *Notizen.* VII. 219.
SCHNYDER, *Verhandl. d. schweizer. naturf. Gesellsch.* 1848. p. 15.
1849. WALLMARK, *Öfvers af Akad. förhandlingar.* 1849. p. 41; POGGENDORFFS *Ann.* LXXXII. 129.
1850. CRANMORE in *Philos. Mag.* (3) XXXVI. p. 485.
BAUDRIMONT, *Comptes rend. de l'Acad. d. sc.* XXXIII. 496; *Institut.* No. 931; *Philos. Mag.* (4) II. 575.
1851. BEER, POGGENDORFFS *Ann.* LXXXIV. S. 518.
A. FICK, *De errore optico quodam asymmetria bulbi oculi affecto.* Marburg. Summary in HENLE u. PFEUFFER, *Zft. für rat. Med.* Neue Folge. II S. 83.
1852. *FLIEDNER, Beobachtungen über Zerstreuungsbilder im Auge, sowie über die Theorie des Sehens. POGGENDORFFS *Ann.* LXXXV. 321.* 460.* LXXXVI. 336*; MOIGNO, *Cosmos.* I. 333.
1852. TROUESSART, *Comptes rend. d. l'Acad. d. sc.* XXXV. p. 134-136. 398; *Archive de Genève.* XX. 305; *Institut.* 1852. p. 304.
STELLWAG VON CARION, *Wiener Sitzungsber.* VIII. 82; *Denkschr. d. k. k. Akad.* V. 2. S. 172; *Zeitschrift d. Ärzte zu Wien.* 1853. Heft 10 u. 11; FECHNERS *Zentralblatt.* 1854. 281-292.
1853. MEYER (of Leipzig), POGGENDORFFS *Ann.* LXXXIX. 429.
1854. A. FICK in HENLE u. PFEUFFER, *Zft.* N. Folge. V. 277.
GUT, *Über Diplopia monophthalmica.* Dissert. Zürich.

Supplement

Since the publication of the preceding section, these particular forms of aberration have been investigated more from the medical standpoint, especially by DONDERS and KNAPP. The convenient term *astigmatism* (derived from the Greek στίγμα, a mark, and ἀ privative, meaning "without focus") has been suggested by WHEWELL to describe these defects. He makes a distinction between *regular* and *irregular astigmatism.* The former includes those phenomena described above under No. 5, and in the remainder of the section, due to the fact that the curvatures of the refracting surfaces of the eye, especially of the anterior surface of the cornea, are different in different meridians. On the other hand, *irregular astigmatism,* as manifested in the phenomena of *polyopia monocularis,* includes those

effects that are produced as the result of the fact that even when the rays all lie in a single meridian plane they do not come to an exact focus.

Apart from those cases in which there are conical protuberances, ulcers, and similar injuries on the surface of the cornea, irregular astigmatism is usually traceable to the crystalline lens, as explained above. The best evidence of this is the fact that eyes without the crystalline lens exhibit no symptoms whatever of *polyopia*, whereas the manifestations of regular astigmatism, especially the linear or oval form of the blur circle, are much more regular and pronounced in such eyes than in normal eyes.

The separate effects due to single sectors of the crystalline lens were carefully investigated by DONDERS, by moving about in front of the eye a small perforated screen so as to let the light go sometimes through one sector of the lens and sometimes through another. His results showed, first, that while each sector of the lens converged the rays approximately to a focus, the focus is not the same for different sectors. And, moreover, the focusing is not quite accurate even for a single sector, the rays near the axis having apparently a longer focal length than those traversing the peripheral parts of the lens. Consequently, in the blur circle due to a single sector of the lens the rays traversing the peripheral portion are more concentrated before arriving at the narrowest section of the bundle, and the central rays are more crowded together beyond this place.

Nearly all human eyes exhibit at least some slight degree of regular astigmatism, which may be measured in a manner similar to that used for determining the amplitude of accommodation. As already explained, in astigmatic eyes the distances of distinct vision are different for lines lying in different azimuths of the field of view (see Fig. 74). If the greatest of these distances of distinct vision is denoted by P, and if, for the same state of accommodation, the least distance is denoted by p, the astigmatism is measured by the difference of the reciprocals of these lengths, that is,

$$As = \frac{1}{p} - \frac{1}{P}\,.$$

As long as the amount of astigmatism (As) is less than 1/40,[1] there is no appreciable impairment of visual acuity, but if it exceeds this value, good definition becomes difficult, and cylindrical lenses will be of aid to the patient. The focal length of the cylindrical surface should be equal to the quantity As, and the lens may be either convex

[1] ¶The distances are supposed here to be measured in inches. The limiting value of the so-called "physiological astigmatism," as given in the text, is equivalent to one dioptry. (J. P. C. S.)

with the axis of the cylinder parallel to the line that is in focus at the distance P, or concave with the axis of the cylinder perpendicular to that line. The other surface of the lens may be ground spherical (spherocylindrical lens) so as to correct at the same time any ametropia that may need correction.

The best way of ascertaining easily whether there is any astigmatism, the amount of it, and the meridians of greatest and least refraction, is by means of a set of cylindrical lenses. STOKES has proposed the use of an astigmatic lens of variable degree of astigmatism composed of two equal cylindrical lenses in contact. Placed with their cylindrical axes at right angles, the combination is not astigmatic but equivalent to an ordinary spherical lens; but by rotating one of the lenses relatively to the other, the cylindrical axes can be inclined to each other at various angles, so as to increase the amount of astigmatism to its maximum value when the cylindrical axes are parallel.

E. JAVAL has devised a convenient apparatus, made by Messrs. NACHET of Paris, for quick measurement of astigmatism. Two charts each consisting of 24 radial lines are observed through a pair of convex lenses with their optical axes parallel. The charts are adjusted at such a distance that at first only one of the lines can be seen distinctly; and then cylindrical lenses, either single or combined in pairs, and set in circular mountings capable of being rotated, are inserted and adjusted until all the lines in both figures stand out clearly. The cylinders can be rotated around the optical axis so as to bring the cylindrical axes in the right azimuths for most distinct vision for each eye.

Corneal measurements of astigmatic eyes made by DONDERS and KNAPP show that almost invariably regular astigmatism is a case of corneal astigmatism, and that the higher degrees of corneal astigmatism are frequently masked to a slight extent by an opposite astigmatism of the crystalline lens.

The azimuth in which the distance of distinct vision is greatest is, as a rule, nearer the vertical meridian than the horizontal, as, for example, in the cases of both A. FICK and the writer, as above stated; but the contrary condition is also not uncommon, as illustrated by YOUNG's vision in this respect.

1852. A. MÜLLER, Über das Beschauen der Landschaften mit normaler und abgeänderter Augenstellung. (Supposedly due to astigmatism.) POGGENDORFFS *Ann.* LXXXVI. 147-152. *Cosmos.* I. 336.
1852. A. BEER, Über den optischen Versuch des Herrn LIBRI. POGGENDORFFS *Ann.* LXXXVII. 115-120.
— J. HIPPESLEY, Phenomena of light. *Athen.* 1852. p. 1069-1070; 1368.

— R. W. H. Hardy, Phenomena of light. Ibid. p. 1306.

1853. Fechner, Über einige Verschiedenheiten des Sehens in vertikalem und horizontalem Sinne nach verschiedenen Beobachtungen. Fechners *Zentralblatt.* S. 73-85; 96-99; 374-379; 558-561.

— L. L. Vallée, Théorie de l'œil. *C. R.* XXXVI. 769-773; 865-867.

— Fliedner, Zur Theorie des Sehens. Poggendorffs *Ann.* LXXXVIII, 29-44.

— H. Meyer, Über die sphärische Abweichung des menschlichen Auges. Ibid. LXXXIX. 540-568.

— Beer, Über den Hof um Kerzenflammen. Ibid. LXXXVIII. 595-597.

— Powell, On a peculiarity of vision. *Rep. of Brit. Assoc.* 1852, 2. p. 11.

1854. J. P. Depigny, Hof um Kerzenflammen. *Arch. des sciences phys.* XXVI. 166-172.

— J. Gut, Über Doppeltsehen mit einem Auge. Henle und Pfeuffer, *Zft.* (2) IV. 395-400.

1855. Über den Gang der Lichtstrahlen im Auge. *Verhandl. der naturforsch. Ges. in Basel.* I. 269-282. *Arch. des sciences phys.* XXXII. 145-146.

— H. Meyer, Über den die Flamme eines Lichts umgebenden Hof usw. Poggendorffs *Ann.* XCVI, 235-262; 603-607; 607-609.

1856. Idem, Über die Strahlen, die ein leuchtender Punkt im Auge erzeugt. Ibid. XCVII 233-260. XCVIII, 214-242.

1857. van der Willigen, Eine Lichterscheinung im Auge. Poggendorffs *Ann.* CII. 175-176.

— J. Tyndall in *Phil. Mag.* (4) XI. 332. (An interference phenomenon.)

1858. G. M. Cavallieri, Sulla cagione del redere le stelle e i punti luminosi affetti da raggi. *Cimento.* VIII. 321-360.

1860. F. Zöllner, Beiträge zur Kenntnis der chromatischen und monochromatischen Abweichung des menschlichen Auges. Poggendorffs *Ann.* CXI. 329-336. *Ann. de chimie.* (3) LX. 506-509.

— Wharton Jones, Analysis of my sight, with a view to ascertain the focal power of my eyes for horizontal and for vertical rays, and to determine whether they possess a power of adjustment for different distances. *Proc. of Roy. Soc.* X. 380-385. *Phil. Mag.* (4) XX. 480-483.

1861. Donders, Beiträge zur Kenntnis der Refraktions- und Akkommodationsanomalien. *Arch. für Ophthalm.* VII. 1 S. 155-204.

1862. J. H. Knapp, Über die Asymmetrie des Auges in seinen verschiedenen Meridianen. *Arch. für Ophthalm.* VIII, 2 S. 185-241.

— Giraud Teulon, Causes et mécanisme de certains phénomènes de polyopie monoculaire. *C. R.* LIV. 904-906; 1130-1131. *Inst.* 1862. p. 138-139; 173.

— F. C. Donders, *Astigmatismus und zylindrische Gläser.* Berlin.

1863. B. A. Pope, Beiträge zur Optik des Auges. *Archiv für Ophthalm.* IX, 1 S. 41-63.

— C. Kugel, Über die Wirkung schief vor das Auge gestellter sphärischer Brillengläser beim regelmässigen astigmatismus. Ibid. X, 1 S. 89-96.

— Middelburg, *De Zidplaats van het Astigmatisme.* Utrecht.

— Ph. H. Knauthe, *Über Astigmatismus.* Dissert. Leipzig.

1864. F. C. Donders, Der Sitz des Astigmatismus (nach Middelburgs Resultaten). *Archiv für Ophthalm.* X, 2. S 83-108.

— J. H. Knapp, Über die Diagnose des irregulären Astigmatismus. *Monatsbl. für Augenheilkunde.* 1864. S. 304-316.

— Donders, *Anomalies of accommodation and refraction.* London. 1864. p. 449—556.

1865. L. Kugel, Über die Sehschärfe bei Astigmatikern. *Archiv für Ophthalm.* XI, 1. S. 106—113.

— H. Kaiser, Zur Theorie des Astigmatismus. Ibid. XI, 3. S. 186-229.

— X. Galezowski, Étude sur la diplopie monophthalmique. *Ann. d'oculistique.* LIV. p. 199-208.

1866. E. Javal, Sur le choix des verres cylindriques. *Ann. d'oculist.* LV. p. 5-29.

— Idem, Histoire et bibliographie de l'astigmatisme. Ibid. LV. p. 105-127.

§ 15. Entoptical Phenomena

Under suitable conditions light falling on the eye may render visible certain objects within the eye itself. These perceptions are called *entoptical*. Ordinarily the small opaque particles suspended in the vitreous or aqueous humors or in the crystalline lens do not cast any visible shadows on the retina and are therefore not noticed. This is because the intensity of the light entering the pupil is generally uniformly distributed over every part of it, and therefore, so far as the illumination of the posterior chamber of the eye is concerned, the entire pupil acts like a luminous surface. But when a source of light is a broad surface, no perceptible shadows are produced unless the opaque object is large or else is very near the screen. Now there are certainly some objects in the eye, particularly the blood vessels of the retina, which fulfil the latter condition by being very close to the sensitive membrane behind them, and which are therefore, in position to cast shadows on the retina. But the areas of the retina lying behind the blood vessels are always thus shaded, so that this is their normal condition, and it is only under special circumstances, to be discussed later, that these shadows become visible.

But at present suppose we consider the tiny, opaque bodies in the transparent ocular media. In order to perceive them, it is necessary to use light from a very small source held close in front of the eye. This source may be the real image of a distant lamp produced by a small convex lens, or the virtual image of the sun or of a flame in a highly polished small metal ball, or the customary small illuminated opening in an opaque screen. The best arrangement is to use a convex lens of large aperture and short focus, mounted as shown at *a* in Fig. 77, whereby a reduced image of a flame *b* at some distance in front of it is focused on a small opening in the opaque screen *c*. This results in a wide cone of rays emerging from the opening. The eye at *o*, very close to the opening, looks through it at the broad, uniformly illuminated aperture of the lens, where the objects that are entoptically perceived will appear now with great clearness. If, as in Fig. 78, the luminous point *a* lies between the eye and its anterior focal point *f*, the image of it in the optical system of the eye will be virtual and situated at a point $\mathfrak{a}$ in front of *a*, so that the rays reaching the retina seem to diverge from $\mathfrak{a}$. A small opaque object *b* that happens to be in the

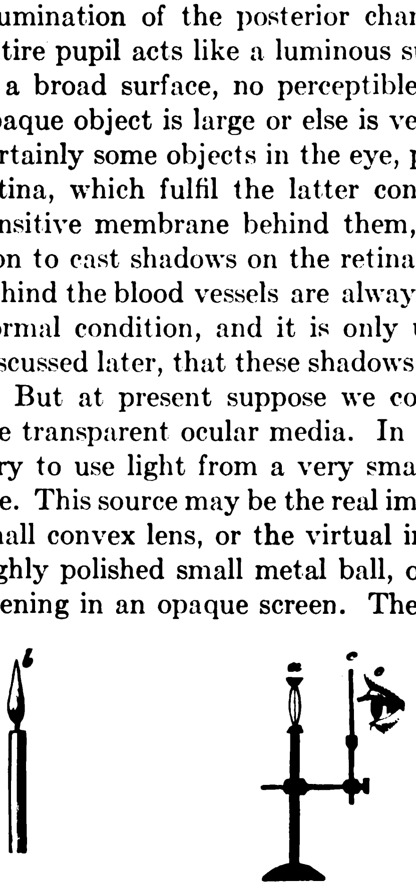

Fig. 77.

vitreous humor will then cast on the retina a shadow β, which will be larger than the obstacle itself. If the point a coincides with the first focal point of the eye, the rays in the vitreous humor will be parallel, as shown in Fig. 79, and then the shadow β will be of the same size as the obstacle b; whereas if the focal point f is between the eye and the luminous point a, as in Fig. 80, the image α will lie beyond the retina, the rays in the vitreous humor being, therefore, convergent, and in this case the shadow will be smaller than the obstacle. Consequently, these entoptical appearances are magnified in size, the nearer the eye is to the luminous point; and *vice versa*.

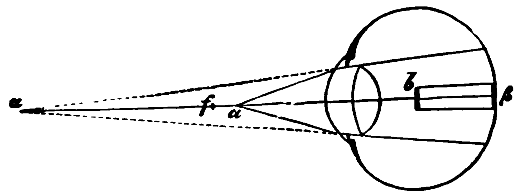

Fig. 78.

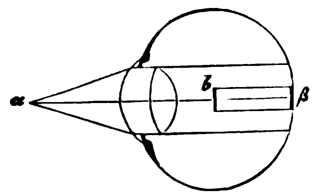

Fig 79.

The area of the retina that is illuminated in these experiments is really the blur circle of the luminous point, and it is within this area that the shadows are cast of the opaque particles. While these shadows are sufficiently distinct to enable us to get some idea of the form of the obstacle, they are always somewhat

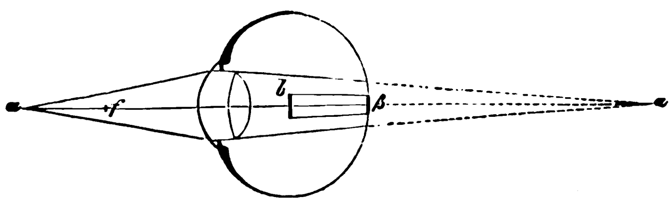

Fig. 80.

blurred. This is because the source of light is not strictly a point but has a more or less appreciable area, and it is the image of this area as produced by the optical system of the eye that acts here as the source so far as the shadow on the retina is concerned, and naturally this image will always have a certain extent. Now any shadow depending on a source of light that is not a mere point will have a penumbra where the illumination of the shadow gradually merges into the general bright illumination of the environment. Consequently, the distinctness of the perception of an entoptical phenomenon will be greater in proportion as the illuminated aperture is smaller, and also in proportion as the little opaque obstacle that casts the shadow is closer to the retina. However, of course, a small aperture involves a correspondingly more

intense light for its illumination. And, besides, with a very narrow aperture another phenomenon has to be taken into account that affects the sharpness of delineation of the shadow, namely, the diffraction fringes due to diffraction around the edges of the obstacle consisting of bright and dark bands that appear on the contours of all shadows produced by an extremely small, intense source of light. With ordinary sources of light of larger size these diffraction fringes vanish in the penumbra.

When the position either of the eye or of the luminous point is varied, the shadows of opaque particles at different distances from the retina will be shifted differently, forming new configurations. This effect can be employed, as LISTING showed, to determine the approximate position of a particle within the eye. The entoptical field of view is circumscribed by the round shadow of the iris. As various points of this are observed one after the other, the shadows of all the opaque particles that do not lie in the pupillary plane will be shifted with respect to the contour of the field. This motion of the shadows in the entoptical field is what LISTING called the *relative entoptical parallax*. He counts this parallax as positive when the movement of the shadow is in the same sense as that of the point of fixation, and negative when it is in the contrary sense.[1] Accordingly, for objects in the pupillary plane, the parallax is zero; for objects beyond the pupil, it is positive; and for objects in front of the pupil, it is negative. For objects which are very close to the retina, the movement of the shadow is almost the same as that of the fixation point, so that the shadow seems to follow the fixation point unless the opaque particle is itself actually in motion in the fluid vitreous humor. The shadow on the retina is, of course, on the same side of the eye as the opaque particle; but since an image on the retina is projected in the field of view on the opposite side of the optical axis of the eye, objects that are perceived entoptically always appear to be inverted.

Entoptical phenomena are as follows:

1. The illuminated field is bounded by the shadow of the iris, and is therefore almost round corresponding to the form of the pupil. If there are notches, creases or salients in the edge of the pupil, as is often the case, they will all be duplicated in the entoptical image. Even the dilatation and contraction of the pupil can be observed entoptically, most easily when the other eye is alternately covered and uncovered with the hand. This causes the pupils of both eyes to dilate and contract in unison, which can be easily seen in the similar variation of the size of the entoptical image.

[1] ¶That is, positive or negative according as the movement of the eye is "with" or "against" the shadow movement, respectively. (J. P. C. S.)

2. There are usually visible in the entoptical field stripes, cloud-like or more luminous places, and circular areas resembling drops of liquid with bright nuclei, which come and go and move about as the eye is opened and shut. These are due to the moisture secreted by the tear-glands and spread over the surface of the cornea by winking the eyelid. These appearances are depicted in Fig. 81. The spots usually run together, and have an independent downward motion. The stripes are most pronounced near the edge of the eyelid when the lid arrives in front of the pupil, and are due to the thin concave film of moisture stretched from the cornea to the edge of the eyelid. The droplets are probably drawn together by capillary action about particles of dust, mucus, etc. The bright spot in the middle of each droplet seems to be an imperfect optical image of the source of light; for example, if the light comes through a triangular opening, the bright spot is triangular.


Fig. 81.


Fig. 82.

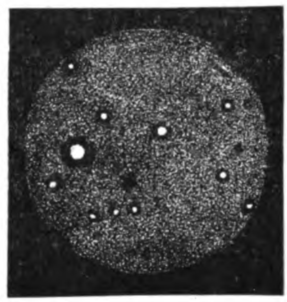
Fig. 83.

This image appears erect in the entoptical field, while it must be inverted on the retina. The accumulations of liquid on the cornea thus act as small convex lenses, beyond each of which is formed an inverted image of whatever objects lie before it. The fact that these images move downward across the field is evidently due to the actual settling of the viscous masses of mucus which the opening eyelid has just dragged upwards with it.

3. The irregularities on the anterior surface of the cornea after pressing or rubbing the closed eye for a time with the fingers. For from a quarter of an hour to an hour afterwards there appears an indistinct array of wavy or reticulated lines and scattered spots, as shown in Fig. 82. Certain areas, however, may sometimes remain unchanged, thus indicating that the cornea is not quite the same here as elsewhere.

In some cases there are special dark spots and lines, also originating in the cornea, which seem to be permanent and are nearly always the result of some previous inflammation or injury.

4. Various entoptical phenomena originate in the anterior wall of the lens-sac or capsule, or in the anterior portion of the substance of the lens. Four of these are described by LISTING, as follows:

(a) *Pearl specks,* more or less round discs, bright in the centre, with clear-cut dark borders. They look sometimes like little air-bubbles or drops of oil or microscopic crystals (Fig. 83). LISTING believes them to be small particles of mucus in the *liquor Morgagnii.*

(b) *Dark specks,* distinguished from the foregoing by the absence of a bright nucleus and also by their greater diversity of form. They seem to be due to local opacity of the lens or lens capsule (Fig. 84.)

(c) *Bright patches,* usually having a few radiating arms like an irregular star, and located near the middle of the field (Fig. 85). LISTING regards these as the images of umbilicate structures with ridge-like branches or seams in the anterior capsule membrane, probably dating from the pre-natal separation of this membrane from the inner surface of the cornea.

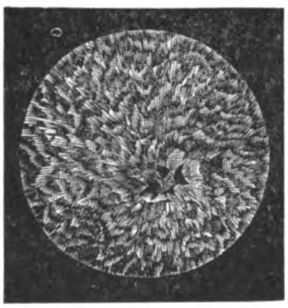

Fig. 84.

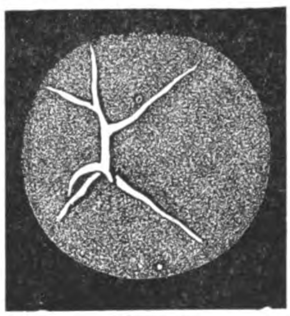

Fig. 85.

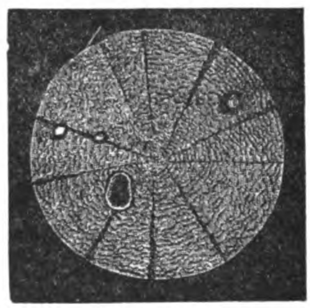

Fig. 86.

(d) *Dark radiating lines* (Fig. 86), which may be regarded as evidence of radial structure of the lens.

Practically all eyes exhibit one or more of these phenomena; rarely is one found that does not.

5. *Moving objects* in the vitreous humor, commonly called "flying gnats" (*mouches volantes* or *muscae volitantes*). Sometimes these look like strings of pearls or groups of circles with bright nuclei, or irregular masses of minute globules, or faint lines like creases in a transparent membrane. Many of them are close enough to the retina to be visible without any special arrangement, by merely looking at some broad uniformly illuminated background like the bright sky. That these motions are real and not simply apparent is easily proved by holding the head erect and looking towards the sky through a window pane, at the same time noting some fixed mark on the glass; then, as a rule, the entoptical appearances will be seen to move slowly downwards, across the field of view. If the observer glances downwards, and then quickly up again, the floating spots follow the movement of the fixa-

tion point, but they are usually thrown a little higher than this point and then begin to settle down again. But for a simple downward or lateral movement of the eye, the case is different, and the object does not seem to waver over the point of fixation. Moreover, in looking vertically up or down, these peculiar effects are fairly stationary. In making such observations there is a strong temptation to try to look at one of these "gnats" that is near the point of fixation so as to see it better; which makes it seem to fly away without ever being overtaken. This curious effect is probably the explanation of the name *mouches volantes*. This apparent motion is not to be confused with a real motion, and in order to investigate the latter, it is necessary to have a steadfast external point of fixation. The best way to compare a movable object of this kind with something which is at rest is to select a position for the head such that it is possible for the eye to look vertically either upwards or downwards with comfort, because in this case the apparent movements of the floating particles cease almost entirely, and the real movements can be watched. Any one of the little objects that is out to one side in the field of view can be brought to the place where it can be seen most distinctly by simply turning the eye suddenly towards it, and then bringing it slowly back again.

The various forms of these objects are classified by DONDERS and DONCAN[1] as follows:

(a) *Rather large, isolated circles*, with bright centres and darker or paler borders, usually surrounded also by a small halo of light. They vary from 1/28 to 1/120 mm in diameter and are usually found from 1/3 to 3 or 4 mm in front of the retina, though they sometimes occur much nearer the lens. When the eye has been in repose a long time, there are only a few of these appearances. They become particularly manifest, apparently from below, on a quick upward movement of the eye, coming to a sudden standstill and then sinking slowly back again. The movements of the darkest of them may be followed directly for a distance of 1.5 mm, and probably extend much farther. DONCAN found the lateral movement of these spots, produced by a sidewise movement of the eye, to be much less; but the writer has not been able to observe any such difference. By putting his head on one side, he can make these spots appear to sink just as readily and as far towards the floor as with the head upright; their actual motions being, of course, upwards towards the highest corner of the eye. In the upright position the lateral movements of the spots is certainly not so great as the vertical, because laterally they participate simply

[1] ANDREAS DONCAN, *Dissert. de corporis vitrei struct.* Trajecti ad Rhenum 1854.—*Onderzoekingen gedaan in het physiologisch Laborat. der Utrechsche Hoogeschool.* Jaar VI. 171.

in the movements of the point of fixation. There is no conclusive evidence of any movement parallel to the visual axis. In many cases groups of such spots, apparently distinct from each other, are observed to be moving along together and maintaining the same distance apart, as if there were some invisible connection between them. By microscopic examination of carefully prepared specimens of the vitreous humor, DONCAN found that these aggregations correspond to pale cells in this substance not far from the surface that are apparently in process of transformation into mucus (see Fig. 87).

Fig. 87.

(b) "*Strings of pearls*" are observed in most eyes; but DONCAN could not see them. They are from 1/33 to 1/190 mm in width and from 1 to 4 mm long. The smallest of them are usually found closer to the retina, while the wider and darker ones are farther away, from 1/4 to 3 mm from the retina. Their mode of motion is generally the same as that of the circles described above, but sometimes they are stationary also. Some of them are isolated and others are connected with other appearances. They evidently correspond to certain granulated filaments which the microscope reveals in the vitreous humor (Fig. 88).

Fig. 88.

(c) The *clusters of circles* of various sizes, part pale and part dark, which are seen in the microscope as granular aggregations (Fig. 89) and are usually more opaque than the other types, because several particles lie along the line of sight. It is these clusters that most frequently produce the effect of *mouches volantes* in ordinary vision. It is not unusual for some of them to appear to be suspended in equilibrium near the visual axis; but on moving the eye, they behave after the manner of the "strings of pearls," appearing and passing out of view in the same way, only in greater numbers.

Fig. 89.

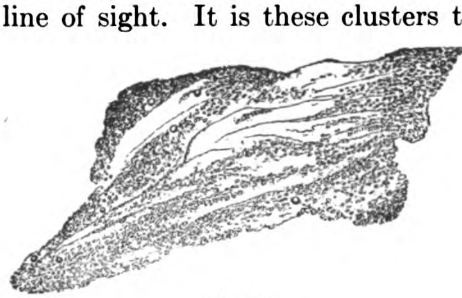

Fig. 90.

(d) *Folds or wrinkles*, appearing as bright streaks bounded by

rather indistinct dark lines. Doncan distinguishes two kinds of these also, namely: very crooked threads or separate, slender bands, very close together and connected in some invisible way; and an irregularly curled membrane crumpled in the most diverse fashion, of permanent shape, and appearing under the microscope as shown in Fig. 90. These also move like the "string of pearls" and are not more than from 2.5 to 4 mm in front of the retina. Another variety consists of very extended membranes which are either a very little way beyond the lens or from 2 to 4 mm in front of the retina, none being found between 4 and 10 mm from the retina. The folds in the former type of membrane are as broad as 1/23 mm; those in the latter are seldom broader than 1/60 mm. They appear when the visual axis is moved to one side, but especially also in case of a swift sudden downward movement. In the latter case, those lying close to the lens apparently move upwards, whereas those near the retina drop downwards, so that they are seen passing each other as they cross the visual axis. Then these wrinkled membranes get more and more indistinct, without, however, passing out of the field of view; until, perhaps, on repeating the movement, they come plainly into view again. Doncan concludes that these membranes do not actually move as they appear to do, but that what is observed is the propagation of the wrinkles which, being formed at the periphery when the movement of the eye is suddenly terminated, extend to the other side of the membrane, where they lose their individuality and are less perceptible. The explanation of the opposite movements of the membrane and the wrinkles in it is that they are on opposite sides of the centre of rotation of the eye. When the pupil is dilated by atropin, or if the luminous source is held very near the eye, so that the field of view is quite wide, it will be found, particularly with abrupt and vigourous lateral movements of the eye, that other membranes besides will be revealed in the region immediately beyond the lens, which usually do not extend as far inwards as the visual axis, and which terminate here in an irregular or torn edge.

The nature of the movements of the objects suspended in the vitreous humor leaves little room for doubt that they are small bodies of less specific gravity than the perfectly fluid medium in which they are floating. Since they are often to be seen traversing the whole entoptical field of view, not only horizontally but vertically as well, so far at least as the writer's own experience goes, and since with divergent incident light this field is larger than that determined by the pupil, it appears that the region in which they are floating is certainly of larger lateral dimensions than the pupil itself. But the drifting particles seem unable to move in a direction away from the retina; for even when the visual axis is pointed vertically upwards,

these same objects, which on account of their specific lightness might be expected to rise to the top of the vitreous humor next the lens, are seen to be in motion across the retina, but do not appear to recede from it. Perhaps, the obstacle may be the membranes whose folds are visible in the entoptical field, and which seem to be parallel to the retina. Some particles of this kind appear also to be attached to the hyaloid membrane. DONDERS reports having discovered one in his left eye that was in equilibrium on the visual axis. It may have had a tendency to descend (that is, apparently ascend) from this position, but could not really do so, as if it were tethered to the hyaloid membrane from below by a filament of some kind.

By taking heed of these entoptical phenomena one may learn to recognize individually those peculiar to his own eyes, and he will see then that the same succession of appearances is continually reproduced, and that, according to DONDERS' observations, they remain unaltered for years. Microscopic examination indicates that they are in the nature of debris from the embryonic structure of the vitreous humor. In the embryo, the humor is contained in cells, which are subsequently nearly all transformed into mucus; but some parts of the cell walls and nuclei, or the filaments into which they are resolved, still remain. Just what residue from the structure of the vitreous humor is left in the adult is not quite certain.

We come now to the perception of the blood vessels of the retina. These require somewhat different methods of observation from those employed for the perception of the entoptical objects so far described. What is common to both methods is that the position or width of the shadow of the blood vessels on the rear surface of the retina is something out of the ordinary, and moreover the shadow is kept in continual motion. The three principal methods of observing the blood vessels are as follows:

1. With a convex lens of short focus an intense beam of light (preferably sunlight) is concentrated on a point of the external surface of the sclerotica, as far as possible from the cornea, so as to form a small but exceedingly bright image of the source on the sclerotica.[1] Now when the eye looks at a dark background, the latter will appear illuminated with a reddish yellow glow, and against this will be seen the dark retinal blood vessels ramifying in various directions like the branches of a tree. Fig. 91 shows these blood vessels as they appear after the injection of a suitable stain. As the focus of the lens is moved to and fro over the sclerotica, the branched figure accompanies

[1] The best way to make this experiment nowadays is with the illumination lamps that have been introduced in the practice of ophthalmology. G.

the motion proceeding always "with" the illumination. By such movements the vascular "tree" is more clearly visible than with the light kept stationary on one spot; in which case, indeed, it gradually disappears altogether. However, movement of the appearances is not so essential in this as in other methods of observation. It might be noted that the smaller the bright spot on the sclerotica, the more sharply defined will be also the smaller branches of the vascular figure, so that by a suitable adjustment it is possible to make out even the most minute capillaries. In the centre of the field, corresponding to the point of fixation, there is a place devoid of all blood vessels. The capillaries form a ring of elongated meshes around this area. Both H. MÜLLER and the writer have noticed the peculiar appearance of this spot, quite different from the rest of the fundus of the eye. While the latter is uniformly illuminated (except for the dark, vascular figure), the place of direct vision has more lustre and looks something like shagreened leather. Another thing to be noticed in observing this spot is that when the eye is fixed on some external object, and at the same time the focus of the lens is shifted upwards over the sclerotica, the vascular figure likewise moves upwards, as already stated; but the shagreen lustre has a slight downward displacement towards the point of fixation of the eye. MEISSNER likewise observed by this same method that this place was brighter, and he mentioned a dark crescent-shaped shadow on the edge of it, such as is seen by the second method of observation. But the writer has not been able to see this with light incident through the sclerotica.

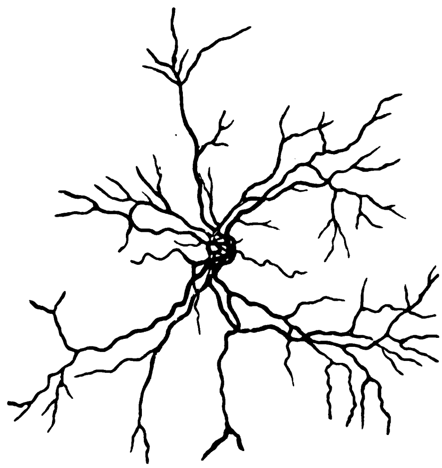

Fig. 91.

In this experiment the light enters the eye through the sclerotica and choroid. The former is translucent; the latter is covered with black pigment, but not so heavily in the back part of the eye as to absorb all the light falling on it. In front, where the eye is lined with the cilary processes, the pigment is more dense, so that when the bright focus of the lens falls on the anterior portion of the sclerotica near the cornea, the retina gets only a relatively feeble illumination. This illuminated place on the eyeball constitutes the source of light, so far as the interior of the eye is concerned; for the sclerotica, being only

translucent, does not produce regular refraction, so that the transmitted rays are scattered from the place of illumination more or less uniformly in all directions. Whereas in ordinary vision the light that reaches the retina all comes through the pupil, in these experiments it comes from a point on one side, and thus the shadows of the blood vessels in the anterior layers of the retina fall on parts of the sensitive membrane behind it that are altogether different from the places where they usually are.

That the vascular figure must have an apparent motion in the same sense as the focus of the lens can be seen from Fig. 92. Suppose that v indicates the cross section of one of the blood vessels, and that k designates the position of the nodal point of the eye. When the focus of the incident light is at a on the sclerotica, the shadow of the blood vessel falls at α, and is therefore projected by the eye in the direction αA as a dark line in the field of view. If the focus is at b, the shadow falls at β and the dark line appears on the field at B. That is, as the bright focus moves from a to b, the figure of the blood vessel apparently shifts in the field of view from A to B, so that the movement is "with" the movement of the source. The shagreened area around the centre of direct vision exhibits the opposite kind of motion, and hence it is certainly not produced in the same way as the motion of the shadows of the blood vessels; but the structure of the yellow spot is not yet well enough understood to enable us to interpret the reason of this phenomenon. As viewed against the dark background, the vascular tree appears to encroach upon the boundary of the shagreened area somewhat on the side opposite the source of light; above and below it merely touches it; and on the side towards the light there is an interval between them. The appearance is as if the light came from one corner of the eye. This would indicate that the ramifications of the blood vessels are more towards the front than the layer that by refraction or reflection of the light is responsible for the shagreened appearance of the central spot; and hence when the light is incident obliquely, the shadows of the blood vessels on the rear surface of the retina are not perpendicularly under the vessels. Consequently, that structure which produces the shagreened appearance seems to have almost exactly the same extent as the place in the retina where there are no blood vessels.

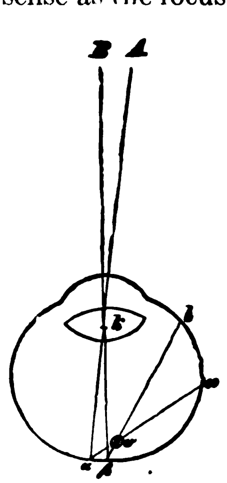

Fig. 92.

2. The second method of observing the vascular system of the retina is as follows: The observer looks at a dark background and at

the same time moves a brilliant light to and fro a little below or to one side of the eye. Under these circumstances the back-ground presently appears suffused with a pale grey glow, against which may be seen the dark outlines of the vascular figure. It remains visible only so long as the light is kept moving. When the light is moved from right to left, the vertical branches are most distinctly seen; and when it is moved up and down, the horizontal ones are more prominent. As the light moves, the whole vascular figure moves also, but not uniformly in all its parts. MEISSNER very aptly compares its motion with that of an image reflected from the surface of water which is agitated by ripples. Closer examination shows that as the light alternately approaches and recedes from the visual axis, the motion of the vascular figure against the background proceeds in the same sense ("with the light"); but if the light describes an arc about a point on the visual axis, the motion of the figure occurs in the opposite sense ("against the light"). Thus, for example, when the light is held below the eye and moved vertically upwards and downwards, the vascular tree is seen to move likewise upwards and downwards with the light; but if it is moved horizontally from right to left, the figure moves at the same time from left to right, and *vice versa*. The more central branches of the vascular tree do not appear as distinctly outlined in this method as in the other two.

In the centre of the field, corresponding to the point of fixation, is what several observers describe as a bright circular or elliptical disc, a drawing of which as made by BUROW is reproduced in Fig. 93. It is brightest in the middle, while on the edge next the source of light there is a dark, crescent-shaped shadow. H. MÜLLER is unable to see this disc at all, and all that the writer can see is the crescent-shaped shadow on the side of the contour next the light, without any distinct boundary on the opposite side. This central disc also moves when the light is moved, as may be easily verified by fixing the eye on some external point while the experiment is in progress. In the writer's own case the point of fixation always lies on the edge next the light, supposing the crescent-shaped outline is rounded out into a complete bright disc.

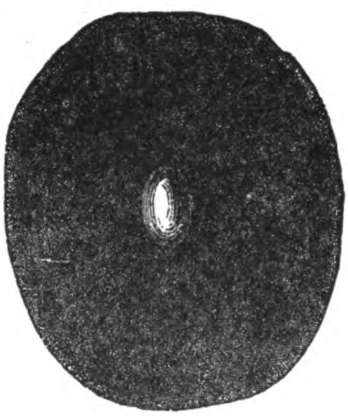

Fig. 93

The entire theory of these phenomena has been worked out by H. MÜLLER, and is as follows: The illumination for the interior of the

eye in this experiment comes from the bright retinal image of the flame, which is far around on one side of the retina, since the lamp is held at some distance from the centre of the field of view. Being also very near the eye, the lamp forms a fairly large image on the retina, and the amount of light thus diffusely reflected into the vitreous humor is sufficient to stimulate a visual sensation over the entire retina. This mode of illumination, therefore, is similar to that of the first method, but differs from it in the fact that the luminous spot on the eye-wall gets its light directly from in front through the pupil instead of from outside through the sclerotica. Images formed on the peripheral parts of the retina do not make very distinct impressions, and in order to give sufficient light, the image used here must be fairly large. This is the reason why the shadows of the finer blood vessels are not so clear-cut in this as in the preceding method. The mode of motion of the vascular figure is completely explained by MÜLLER'S theory. In Fig. 94 the points designated by *k* and *v* represent the positions of the

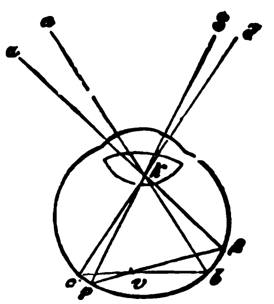

Fig. 94.

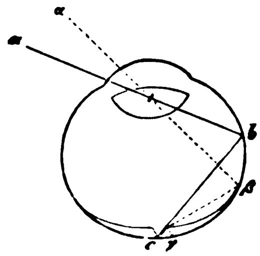

Fig. 95.

nodal point of the eye and one of the blood vessels on the retina, respectively. When the source of light is at *a*, its retinal image falls at *b*; and the light reflected from *b* casts a shadow at *c* of the blood-vessel *v*. This shadow will be visible in the field of view along the prolongation *kd* of the straight line connecting the points *k* and *c*. Now if the light is moved from *a* to α, the points *b*, *c* and *d* take the new positions β, γ, and δ, respectively; and hence *d* is seen to move in the same sense as *a*. But if *a* is moved at right angles to the plane of the diagram, the reverse happens. When *a* lies in front of this plane, *b* lies behind it, *c* in front of it, and *d* behind it. Thus when *a* is moved out from the plane of the diagram, *d* moves the other way, in exact accord with the observations.

MÜLLER gives a reasonable explanation of the appearance of the bright central disc with the crescent shadow as being the shadow of the *macula lutea*. Suppose the *macula lutea* is at *c* in Fig. 95, with the *fovea centralis* at the bottom of it. The point *b* on the retina is illu-

minated by the light at a, and the shadow of the elevated edge of the little concavity on the side towards b will fall right in the fovea. The shadow of the whole area as it actually lies on the retina itself will be turned away from the point of fixation and towards the light, and will appear therefore as projected in the field of view away from the light, in agreement with the results of observation. The writer has noticed that when the source of light a is brought nearer the visual axis so that the point b is closer to the fovea c, the crescent shadow appears to have a bright outer margin. This is doubtless due to light which has been reflected to the external surface of the *macula lutea* from behind the retina, as indicated by the dotted line $\alpha\beta\gamma$ in Fig. 95. In cases where the *macula lutea* has less steep sides, no such shadow is visible at all.

3. The third method of observing the retinal blood vessels consists in looking through a narrow aperture at a bright background, like the sky, at the same time moving the aperture rapidly to and fro in front of the pupil. The blood vessels are seen very sharply delineated as dark lines on a bright background, and appear to move in the field of view in the same sense as the aperture moves. In the centre, corresponding to the point of fixation, the non-vascular area is visible, which, so far as the writer is concerned, has a finely granulated appearance and seems to be traversed by a round shadow as the aperture is moved to and fro. With horizontal movements of the aperture, we see the blood vessels that run vertically; and *vice versa*. The same vascular figure may be seen also by looking through a compound microscope with nothing upon the stage, the background being the uniformly bright circular aperture of the diaphragm. When the eye is moved to and fro a little at the ocular, the slender retinal blood vessels appear sharply delineated in the field, particularly those running at right angles to the direction of the motion; whereas the others vanish that are parallel to this direction.

In the first two methods, the light is incident on the retina from an unusual direction, and hence also the shadows of the blood vessels fall upon parts of the retina that are not shaded in normal vision. The shadows therefore constitute an unaccustomed stimulus. On the contrary, in the third method just described the light enters the eye through the pupil in the ordinary way. If the pupil is perfectly free, and the eye is turned towards the bright sky, every point of the pupillary plane may be considered as a source of light sending rays in all directions to the fundus of the eye, just as if the pupil itself were a luminous surface. The result is that the blood vessels of the retina project broad hazy shadows on the parts of the retina immediately behind them, the length of the umbra being only about four or five

times the diameter of the blood vessel. According to E. H. WEBER, the diameter of the thickest branch of the *vena centralis* is 0.038 mm; and KÖLLIKER estimates the thickness of the retina at the back of the eye as 0.22 mm. Hence it may be assumed that the umbra of the vascular shadow does not reach the posterior surface of the retina at all. But when the light enters the eye through a narrow aperture in front of the pupil, the shadow of the blood vessel is necessarily smaller and more sharply defined, and since the umbra is longer, parts of the retina that were formerly partially shaded are now completely shaded, while other adjacent parts are not shaded at all.

The reason why we do not perceive the shadow of the vascular system under ordinary visual conditions is because the regions covered by the shadow, being accustomed to the darkness, are less fatigued than the rest of the retina. But as soon as the position or extent of the shadow varies, it falls on fatigued and less sensitive portions, and therefore becomes visible as a feeble illumination; and at the same time the more sensitive areas that are normally in the shadow are exposed to the full light, and therefore more strongly stimulated than usual. This is the reason why sometimes, and especially at the outset of the experiment, the appearance of the vascular figure may be momentarily that of a bright object on a dark background; and, indeed, many persons pay more attention to this bright aspect than to the dark permanent phase. However, when the shadow in our experiments has taken its permanent position, the shaded area begins to recover its sensitivity, whereas the unusual illumination of the parts that are normally in shadow soon fatigues and benumbs them, so that the temporary effect mentioned above rapidly dies away. In order to make it permanent, the shadow must be kept changing from place to place. This is done by moving the source to and fro, so that we continue to see those blood vessels whose shadows are altered thus. These changes of the sensitivity of the retina will be studied more fully in §25.

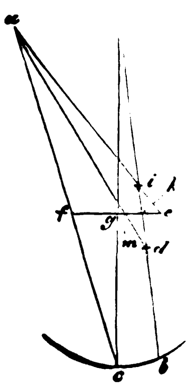

Fig. 96.

LISTING's parallax method is sufficient to enable us to tell whether an object seen entoptically is in front of the pupil or beyond it or perhaps near the retina. In Fig. 96 suppose that the point marked *a* is the image of the luminous point in the optical system of the eye, and that *c* is the position of the fovea on the retina. Let the straight line *fe* represent the plane of the pupil or rather of its image in the crystalline lens; which, however, is almost the same thing. And, finally, let *d* designate the position of some dark object behind the pupil. The shadow of the point *g* where the straight line *ac* crosses the pupil falls on the retina

right at the fovea, and hence the point g corresponds to the place in the pupil that is seen in direct (foveal) vision in the entoptical field. The shadow of the object at d is projected on the retina at the point b where the straight line ad meets the retina; and this point b corresponds also to the point h where the straight line ad crosses the pupillary plane; that is, d and h are seen at the same place in the entoptical field. Moreover, if there is another object situated at a point i on the straight line ab, but lying in front of the pupil, it will appear to be at the same place as the object d. But now suppose that either the eye itself or the source of light is displaced, so that some other point in the pupillary plane, for example, the point f, is seen entoptically in direct vision; then the source will lie somewhere on the straight line cf, say, at the point a. Accompanying this movement, the shadows of the objects d and i will shift their apparent positions with respect to the pupil, proceeding from h to m and from h to e, respectively, where m and e designate the points where the straight lines ad and ai cross the plane of the pupil. Thus, in the entoptical field, while the place of direct vision has been shifted from g to f, the image of the object d behind the pupil has been shifted in the same sense from h to m, and that of the object i in front of the pupil has been shifted in the opposite sense from h to e; in other words, using LISTING's mode of expression, the parallax of d is positive and that of i negative. With a little practice, it is always easy to tell whether the entoptical phenomenon in the circular field of view is displaced one way or the other ("with" or "against" the motion of the light), and so to determine on which side of the pupil the object lies.

BREWSTER first proposed a method of measuring more accurately the position of an object suspended in the vitreous humor. Essentially, it depends on using two sources of light and thus obtaining two shadows of the same object. Knowing the distance of the shadows from each other, we can find then the distance of the object from the retina. BREWSTER's arrangement for this purpose consisted in looking through a lens in front of the eye at two flames which were near together. DONDERS changed the details of this method, by causing the eye to look through a pair of apertures in a metal plate, 1.5 mm apart, at a brightly illuminated white area on which the entoptical phenomena appeared to be projected. Then he got the distance between the centres of the two partially overlapping circular images of the pupil, which amounts simply to measuring the diameter of the uncovered portion of one of these discs; and he measured also the distance between the two images of the entoptical object. These two measured distances are in the same ratio to each other as the required distance of the object from the retina is to the apparent distance of the pupil from the retina (18 mm); so that the distance of the object can be easily computed.

DONDERS' procedure was modified by DONCAN merely by making the measurements on the principle of the microscope method *à double vue*. One eye looks through a tiny aperture or through a pair of them towards a small concave mirror illuminated by skylight, while the other eye looks at a card placed at the distance of distinct vision. The size of the entoptical object and the distance between its two images are measured on this card with a pair of dividers, and likewise the distance between corresponding points on the edge of the iris. In order to calculate the real size of the entoptical object from its apparent size, it is necessary to know also the distance of the peep hole from the cornea. It is best to adjust this aperture at the anterior focal point of the eye, that is, 12 mm from the cornea, because then the entoptical shadow is just the same size as the object itself. The apparent size of the object in the field of view as measured by the dividers is in the same ratio to its real size as the distance of the card from the eye is to the anterior focal length of the eye (15 mm). The aperture can be adjusted approximately in the first focal

plane of the eye by attaching the metal plate at the end of a little tube of suitable dimensions.

The apparent extent of the movement of the vascular figure in the field of view, as seen by the first of the methods described above, was measured by MÜLLER; the corresponding movement of the luminous focus on the sclerotica being measured at the same time with a pair of dividers. Thence the distance of the blood vessel, that casts the shadow, from the light-senstitive layer of the retina can be found, approximately at any rate, either by geometrical construction or arithmetical computation. A section of the eye, like that in Fig. 92, is drawn in its actual size. Suppose the focus moves to and fro over the sclerotica between the points marked a and b; and let v designate the position of a blood vessel whose shadow falls at α in the vicinity of the yellow spot when the focus is at a, so that the blood vessel v whose apparent size is to be measured is on the straight line $a\alpha$. Let $\alpha\beta$ represent the real movement on the retina which is calculated from the apparent, displacement of the blood vessel in the field of view; where β, therefore, designates the position of the shadow when the focus is at b. The point v where the straight lines $a\alpha$ and $b\beta$ intersect must be the position of the blood vessel, and its distance from the retina can be found by measurement or by calculation. Thus, in several experiments, MÜLLER found for the distances of the blood vessels from the sensitive layer of the retina the following values (given in mm): 0.17; from 0.19 to 0.21; 0.22; from 0.25 to 0.29; and from 0.29 to 0.32. With three other observers he got: 0.19; 0.26; and 0.32 mm. By anatomical measurements he found between 0.2 and 0.3 mm for the distance of the blood vessels from the layer of rods and cones in the region of the yellow spot; which indicates that the cones are the elements on which the shadows are cast; as other considerations do also, which will be treated in §18.

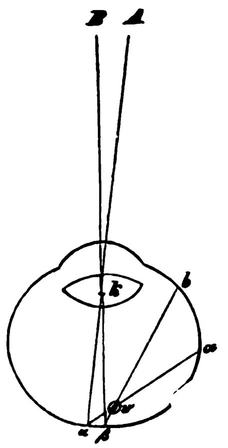

Fig. 92.

A Jesuit named DECHALES,[1] who flourished in the seventeenth century, was the first to advance a theory as to the origin of the *mouches volantes*, and to attribute them correctly to shadows of little particles suspended in the eye in the region of the retina. On the other hand, PITCAIRN[2] supposed they were on the retina itself. MORGAGNI[3] thought that they were in all the ocular media, but that those more to the front perhaps could not be seen without using small sources of light. DE LA HIRE[4] was likewise mistaken in supposing that the permanent "gnats" were confined to the retina, and that the flitting ones were in the aqueous humor. An experiment was described by LE CAT[5] which contains in principle the whole method of entoptical investigation, and which consisted in erecting a needle close in front of the eye and observing its inverted shadow in the blur circle of a small source of light. About this same epoch AEPINUS[6] also observed entoptically the shadow of the iris and the contraction and dilatation of the pupil, and gave a correct explanation thereof.

[1] *Cursus seu mundus mathematicus.* Lugduni 1690. T. III. p. 402.

[2] PITCAIRNII *Opera.* Lugd. Bat. p. 203. 206.

[3] *Adversaria anatomica* VI. Anim. LXXV. p. 94. Lugd. Bat. 1722.

[4] *Accidens de la vue.* p. 358.

[5] *Traité des sens.* Rouen 1740. p. 298.

[6] *Novi Comment.* Petropol. Vol. VII. p. 303.

But it was not until 1760[1] that small apertures and strong lenses began to be used for seeing these phenomena distinctly, although even DECHALES was not entirely ignorant of this method.

Long afterwards LISTING[2] and BREWSTER[3] developed a more exact theory of the entoptical phenomena and methods of locating the particles in the eye; and subsequently DONDERS[4] pursued the same study. Then one of his pupils, DONCAN,[5] established the agreement between the entoptical effects and the miscroscopical structure of the vitreous humor; and similar investigations were made by JAGO.[6] In addition to the authors already named, STEIFENSAND,[7] MACKENZIE[8] and APPIA[9] described various forms of entoptical objects.

The subjective appearance of the central blood vessels was discovered first by PURKINJE,[10] who showed how to see them by the three methods described above. He obtained them also by stimulating the eye by pressure and congestion of the blood. The significance of the shadow movement which is so important in the theory of these effects was pointed out by GUDDEN.[11] With homocentric light emanating from the pupil or with a focus on the sclerotica, there was apparently no difficulty about the theory of the phenomenon. But MEISSNER[12] showed that the behaviour was different when the light moved inside the eye, which led him to entertain some doubts as to previous explanations. These doubts were confirmed by H. MÜLLER,[13] to whom we are indebted for the theory of this method of experiment as given above. PURKINJE had already observed that there is a bright spot in the centre of the field of view that looks like a little pit or cavity; and BUROW[14] described more accurately the appearance of the yellow spot, but interpreted it as looking more like a protuberance than a little depression; which was due to the former incorrect theory of the experiment that MÜLLER perfected.

1690. DECHALES, *Cursus seu mundus mathematicus.* Lugduni. T. III. p. 402.
1694. DE LA HIRE, *Accidens de la vue* in *Mém. de l'Acad. d. sc.* p. 358.
PITCAIRNII *Opera.* Lugd. Bat. p. 203. 206.
1722. MORGAGNI *Adversaria anatomica* VI. Anim. LXXV. p. 94. Lugd. Bat.
1740. LE CAT, *Traité des sens.* Rouen. p. 298.
AEPINUS, *Novi Comment.* Petrop. VII. p. 303.
1760. *Histoire de l'Acad. d. sc. pour l'an 1760.* p. 57.
1819. PURKINJE, *Beiträge zur Kenntnis des Sehens.* S. 89.*
1825. Idem. *Neue Beiträge.* S. 115. 117.*
1842. STEIFENSAND in POGGENDORFFS *Ann.* LV. p. 134*; v. AMMONS *Monatsschrift für Medizin.* I. 203.
1845. *LISTING, *Beitrag zur physiologischen Optik.* Göttingen.*

[1] *Histoire de l'Acad. d. sciences.* 1760. p. 57. Paris 1766.
[2] *Beitrag zur physiologischen Optik.* Göttingen 1845.
[3] *Transactions of the Roy. Soc. of Edinb.* XV. 377.
[4] *Nederl. Lancet.* 1846-47. 2. Serie. D. II. bl. 345. 432. 537.
[5] *De corporis vitrei structura. Dis.* Utrecht 1854; *Onderzoekingen ged. in het Physiol. Laborat. d. Utrechtsche Hoogeschool.* Jaar VI. p. 171.
[6] *Proceed. Roy. Soc.* 18 Jan. 1855.
[7] POGGENDORFFS *Ann.* LV. p. 134; v. AMMONS *Monatschrift f. Med.* I. 203.
[8] *Edinburgh Medical and Surgical Journal.* July 1845.
[9] *De l'oeil vu par lui-même.* Genève 1853.
[10] *Beiträge zur Kenntnis des Sehens.* 1819. S. 89. Neue Beiträge. 1825. S. 115. 117.
[11] J. MÜLLERS *Archiv für Anat. u. Physiol.* 1849. S. 522.
[12] *Beiträge zur Physiologie des Sehorgans.* 1854.
[13] *Verhandl. der med.-physik. Ges. zu Würzburg.* IV. 100. V. Lief. 3.
[14] J. MÜLLERS *Archiv.* 1854. S. 166.

1845. BREWSTER in *Transactions of the Roy. Soc. of Edinb.* XV. 377.
MACKENZIE, *Edinb. Medical and Surgical Journal.* July 1845.
1846. DONDERS in *Nederlandsch Lancet.* 1846–47. 2. Serie. D. II. bl. 345. 432. 537.
1848. BREWSTER in *Phil. Mag.* XXXII. 1; *Arch.d. sc. phys. et natur. de Genève.* VIII. 299.
1849. GUDDEN in J. MÜLLERS *Archiv.* 1849. S. 522.*
1853. APPIA, *De l'oeil vu par lui-même.* Genève.
1854. *A. DONCAN, *De corporis vitrei structura. Dissert.* Trajecti ad Rhenum; *Onderzoekingen ged. in het Physiol. Laborat. d. Utrechtsche Hoogeschool. Jaar* VI. p. 171.
BUROW in J. MÜLLERS *Archiv.* 1854. S. 166.
1855. JAMES JAGO in *Proceedings of the Roy. Soc.* 18 Jan. 1855.

Supplement

On bright surfaces intermittently illuminated (for example, by moving the hand to and fro in front of the eye, with the fingers spread out) VIERODT observed a stream movement which he took for the motion of the blood in the retinal vessels; but neither MEISSNER nor the writer could verify this phenomenon as having the appearance of a current flowing between banks; and therefore the writer ventured to doubt VIEDERODT's explanation of it. However, he may have been able to see it better and more definitely, and in his case it may have really been a picture of the flow of blood.

Besides, PURKINJE and J. MÜLLER, looking at an extended bright surface, had seen the appearance of bright points in the field of view with a line proceeding from them, which continually reappeared at irregular intervals, always following the same path with the same fairly great velocity. O. N. ROOD has recently observed that this phenomenon can be seen very much better by looking at the sky through a dark blue glass. The way the writer does it is to look at a point in the window-pane so as to see the little moving particles in the eye always at the same place, and so to compare their paths with the vascular figure projected also on the glass.

Now after having repeated these observations, the writer is disposed to believe that they do depend on a movement of the blood in some such way that one of the larger particles gets jammed in one of the narrower vessels. In this case the vessel usually gets comparatively empty on the far side of a particle of this kind, while great numbers of blood corpuscles are congested on the other side. As soon as the obstruction gives way, the whole congregation flows past quickly. This procedure is often witnessed in watching the capillary circulation through a microscope. In the case of the experiment mentioned above, a brighter, more elongated band precedes in the visual field, corresponding to the empty place in the blood vessel beyond the obstruction; and following it, there is a darker shadow which, in the writer's opinion, corresponds to the congested corpuscles.

In the author's right eye this phenomenon is very distinct in two parallel vessels to the left of the point of fixation; and frequently recurs, occasionally in both vessels at the same time. Apparently the motion is upwards, and the figure vanishes by wriggling through a sinuous curve with considerably increased speed. The interesting thing is that in the entoptical image of the vascular figure not only are the two parallel vessels found at the place specified above, but also the S-shaped curve at their junction point leading into a larger ramification of veins; that is, the two methods of observation agree perfectly. Of course, the two blood vessels here mentioned are not the only ones which exhibit this movement, and there are numerous other places in the same eye; but they are farther from the point of fixation, and the appearances are not so characteristic.

Accordingly, the phenomenon here described should be regarded as an optical representation of little obstructions in the circulation of the blood, occurring ordinarily only at certain constricted places of the vascular tract, and only when a rather larger corpuscle tries to get past.

1853. Trouessart, Suite des recherches concernant la vision. *C. R.* XXXVI, 144–146.
1856. Vierordt, Wahrnehmung des Blutlaufs den Netzhautgefässen. *Archiv für physiol. Heilkunde.* 1856. Heft II.
— Meissner in *Jahresbericht für 1856.* Henle und Pfeuffer *Zeitschr.* (3) I, 565–566.
1857. J. Jago, Ocular spectres, structures and functions as mutual exponents. *Proc. Roy. Soc.* VIII, 603–610. *Phil. Mag.* (4) XV, 545–550.
1860. O. N. Rood, On a probable means of rendering visible the circulation in the eye. Silliman's J. (2) XXX, 264–265; 385–386.
1861. L. Reuben, On normal quasi-vision of the moving blood-corpuscles within the retina of the human eye. Silliman's J. (2) XXXI, 325–388; 417.

Note by A. Gullstrand

There are two entoptical phenomena belonging in the domain of the dioptrics of the eye that have this in common, namely, that although no notice was taken of them when they were first discovered, attention has again been directed to both of them.

When a candle flame in an otherwise dark room is brought near the visual axis of the passive eye from the temporal side, often a faint spot of light is seen to move in the opposite direction. This phenomenon was first explained by Becker[1]; and it has been described again by Tscherning[2] who recommends causing a candle flame to pass below the visual axis. Different individuals do not see the spot of light with equal distinctness. The writer cannot succeed in seeing it except under special circumstances; as it is mostly hidden by the vascular figure, and he does not know how to produce the conditions

[1] O. Becker, Über Wahrnehmung eines Reflexbildes im eigenen Auge. *Wiener Med. Wochenschrift* 1860, S. 670 and 684.

[2] Tscherning, *Optique physiologique.* Paris 1898. p. 43.

that are conducive to its apparition. Other people see it so distinctly that they recognize the inverted image of the flame in it. This phenomenon is akin to what is known as the "Lichtfleck" among opticians, due to harmful reflection of light at the refracting surfaces of an optical instrument. So far as twofold reflections are concerned, there is no refracting surface in the eye that needs to be considered except the anterior surface of the cornea, as this is the only surface where there is a sufficient difference between the indices of refraction of the adjacent media. Now calculation shows that light which is reflected again back into the eye from the anterior surface of the cornea, produces an erect image of the flame not far from the retina; which consequently appears to be inverted, so that its apparent displacement must be "against" the motion of the light. The light reflected at the anterior surface of the lens, and then back again at the anterior surface of the cornea, forms an image near the posterior surface of the lens; which must be projected on the retina in a very large blur circle that cannot be differentiated from the diffusely reflected light in the ocular media.

In a dark room most people with dilated pupils discern coloured rings around small sources of light which are particularly distinct against a dark background. The rings comprise the colours of the spectrum, red being on the outside, and the angular diameter of the yellow ring amounting to some 6° or 7°. Artificial light is usually deficient in radiations of short wave-lengths; and under such circumstances the spectrum is not apt to be very distinct beyond the blue-green. On looking through a small hole, the ring vanishes entirely as soon as the hole is centered on the pupil. But when the hole is decentered with respect to the pupil, the instant it reaches the edge of the dilated pupil two tiny spectra flash forth. The straight line joining them passes through the flame—at right angles to the direction in which the hole was moved. Thus, any two parts of the coloured ring at opposite ends of a diameter can be produced at will. The disposition of the colours in the ring is characteristic of an interference-spectrum, and the experiment with the hole shows that the phenomenon is due to a radial grating that comes into action only in the vicinity of the edge of the pupil. But owing to the ray-formed structure of the lens, the fibres of the lens are not exactly radial, and therefore cannot be responsible for any exact ring-shaped type. In confirmation of this, it may be shown that the ring seems to be composed of several small pieces not entirely similar to one another. That the ring is produced by a radial grating, may be proved also by interposing in front of it a piece of opaque paper with a straight edge. For instance, if the paper is inserted from the temporal side with its edge vertical, until only a

small segment of the pupil on the nasal side is left exposed, the two sides of the ring are eclipsed, and only the upper and lower parts remain. In this case only a portion of the grating contained in the lens is left uncovered, and this portion contains horizontal fibres together with some that are not quite horizontal. Hence, no parts of the ring can be seen except those for which the tangent is parallel to the direction along which the uncovered fibres lie.

With relaxed accommodation, or under conditions of excitement, the pupil of the eye is often dilated enough to see the coloured ring immediately. But with some persons the experiment with the hole is not successful without artificial dilatation of the pupil unless the other eye is screened; whereas other people still are not able to see the coloured ring at all without a mydriatic. The phenomenon was described and correctly explained by both DONDERS[1] and BEER,[2] and was subsequently investigated more fully by DRUAULT[3] and SALOMONSOHN.[4] The latter, by the way, appended to his paper a summary of the literature on the subject; but he was mistaken in supposing that the coloured rings that occur with glaucoma are produced in the same way. They vanish gradually, and all over at the same time, when a card is interposed in front of the eye, which shows that their origin is due to round or polygonal elements, in the same way that coloured rings are produced by a clouding of the cornea. Their diameter, too, is larger. On the contrary, in cases where the glaucoma has produced the necessary dilatation of the pupil without concomitant cloudiness of the cornea, it sometimes happens that the physiological coloured rings, originating in the lens, become visible, and may be confused with purely glaucomatous phenomena of a similar appearance.

In the same manner, and with the same behaviour with respect to an interposed card, under certain physiological conditions, coloured rings can be seen that are due to some secretion on the surface of the cornea or to some irregular evaporation of moisture there. Many persons habitually see a ring of this sort marking the outer boundary of the light diffused around a bright source; the larger ring originating in the lens being separated from it by a dark gap.

[1] See J. H. A. HAFFMANS, Beiträge zur Kenntnis des Glaukoms. *Arch. f. Ophth.* VIII. 2. S. 124. 1862.

[2] BEER, Über den Hof um Kerzenflammen. POGGENDORFFS *Ann.* Bd. 84. S. 518. 1851; Bd. 88. S. 595. 1853.

[3] A. DRUAULT, Sur la production des anneaux colorés autour des flammes. *Arch. d'opht.* 18. p. 312. 1898.

[4] H. SALOMONSOHN, Über Lichtbeugung an Hornhaut und Linse. *Arch. f. Physiologie.* Jahrg. 1898. S. 187.

§ 16. The Illumination of the Eye and the Ophthalmoscope

A part of the light that reaches the retina of the eye is absorbed, especially by the black pigment of the choroid, and another part is diffusely reflected and returns through the pupil and out of the eye altogether. Ordinarily, we are absolutely unaware of this latter portion; so much so, indeed, that the pupil of another person's eye looks entirely black. The explanation of this is to be found mainly in the peculiar refraction conditions in the eye, but to some extent also because, as a matter of fact, comparatively little light is reflected back from most parts of the black fundus of the eye.

According to the principle of the reversibility of the light path, light will traverse precisely the same route through an optical instrument from one end to the other in either direction. Thus, for example, so far as this principle is concerned, it makes no difference which one of a pair of conjugate points is regarded as the source of the light and which one as the image, because in this respect object and image are interchangeable. Suppose, therefore, that the eye is accommodated to focus on the retina an exact image of an external luminous object; and suppose that we regard the illuminated part of the retina as being itself a luminous object: then its image projected by the ocular media will coincide exactly with the external object. In other words, all the light proceeding from the retina through the eye will return punctually to the outside luminous body. In order to get some of this light, an observer would have to insert his eye between the luminous body and the illuminated eye, which, of course, cannot be done without some auxiliary contrivance to prevent the illuminating light from being intercepted.

There is the same difficulty about seeing the light that returns from a person's eye when his eye is accommodated to see distinctly the pupil of the observer's eye. In this case there is an accurate dark image of the pupil of the observer's eye projected on the retina of the other person's eye. Conversely, there is an image of this dark place on the retina projected on the pupil of the observer's eye; so that all the observer can do is to see the reflex of the black of his own eye in the other person's eye.

Thus, under ordinary circumstances, even the parts of the fundus of the eye that reflect light better than the other places, as, for example, the white area where the optic nerve enters the eye and the blood vessels, are not visible to an outside observer. Although the choroid of the eye of an albino is almost devoid of pigment, his pupil will look black too, provided proper precautions are taken to prevent light from getting into the eye through the sclerotica; as can be done by holding

a dark screen in front of the eye, with a hole made in it the size of the pupil, so as to enable the observer to see through it.[1] It is the light that penetrates through the sclerotica that causes the customary red appearance of the pupil of an albino's eye. Similarly, viewed from in front, the object-glass of a *camera obscura*, that is used to project the image of a single light in a dark room, looks black; even if the image is received on a piece of white paper.

But if the illuminated eye is not accommodated exactly either for the luminous object or for the pupil of the observer's eye, it may be possible for the observer to get some of the light returning from the pupil of the other person's eye; and then the pupil will look bright to him.

It is not hard to understand how the observer can get light from all those parts of the retina of another person's eye, that are comprised within the area covered by the blurred image of the pupil of his own eye. Suppose, for a moment, that the pupil of the observer's eye is replaced by a luminous disc whose blurred image in the other person's eye coincided exactly with what was there before. Now in this case rays of light proceed from one or more points of the illuminated disc to all the points of its blurred image. Accordingly, rays of light may come back from all parts of the illuminated area on the retina to the corresponding point or points on the disc; that is, to the place where the observer's pupil is. Thus the observer will see the other person's eye as luminous, whenever the blurred image of his own pupil in that eye partly overlaps the blurred image of a luminous object.

Suppose, therefore, that the observer looks right past the edge of a light, that is screened from his own eye so as not to blind him, into another person's eye; and suppose that this latter eye is accommodated for a point between the two eyes or for a point much farther away than the opposite eye: under these circumstances, the observer will see the pupil of the other eye shining red. The arrangement of this experiment is indicated

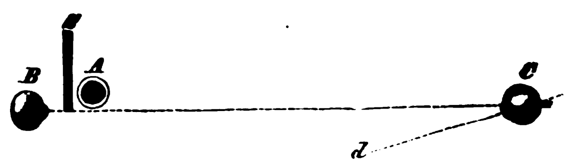

Fig. 97.

in the accompanying diagram (Fig. 97); where B represents the observer's eye, S the screen which protects it from the direct rays of the lamp-light shown in plan at A, and C the illuminated eye. The straight line BC is the visual axis of the observer's eye, whereas that of the other eye may have any direction as, for example, along the straight

[1] F. C. Donders in *Onderzoekingen gedaan in het Physiologisch Laborat. der Utrechtsche Hoogeschool.* Jaar VI. p. 153. — Van Trigt in *Nederlandsch Lancet.* 3. Ser. D. II. bl. 419.

line Cd. The experiment is usually successful also without taking account of the accommodation of the illuminated eye, provided the observer is far away, or provided the patient looks to one side, as in Fig. 97; because then the image of the light and that of the pupil of the observer's eye are projected on the lateral parts of the retina where the images are generally not clear-cut. The illumination is brightest when the light falls at the place where the optic nerve enters the eye, because the light is highly reflected from this white substance, and also because its translucency is such that it does not offer any definite surface for the projection of a clear-cut image.

It may be noted that with sufficiently strong illumination enough light goes through the choroid to the sclerotica to be perceptible when it is diffusely reflected back again. This light behaves like that of the blur circle. Hence, with strong illumination, even when the patient's eye is exactly accommodated for the pupil of the observer's eye, the luminosity may be feeble, especially if there is not much pigment in the eye, as was explained above.

The luminosity of the eye can be observed still better, provided the light from the flame is not permitted to fall directly on the eye, but is reflected on it from a glass plate, through which the observer can look into it. In Fig. 98, A shows the position of the light and S that of the transparent glass plate, which reflects the light into the eye at C as if it came from a source a behind the observer's eye at B. A small retinal image of the light is projected on the fundus of the eye C; thence the light returns towards a and falls again on the surface of the glass plate, where a part of it is reflected back to the real source at A; but another part traverses the plate and proceeds onward toward the reflected image at a. The observer with his eye at B is in a position to intercept these rays, and thus to perceive the luminosity of the eye at C. If a small hole is bored in the plate to enable the observer to see through it, the reflecting surface may be silvered.

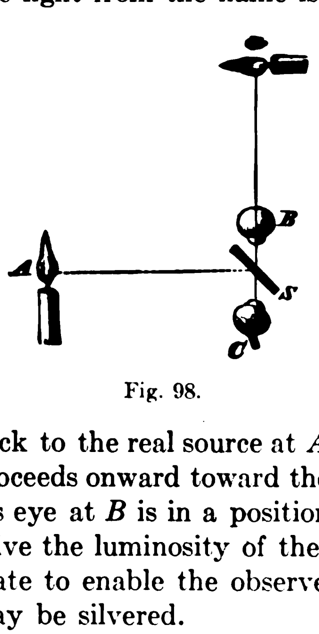

Fig. 98.

Now although under these circumstances the observer beholds the illuminated fundus of the eye, as a rule, perhaps, he cannot make out any of its details, because he is unable to accommodate for the image of the fundus that is produced by the ocular media. To do this, he must use appropriate glass lenses. The combination of an illumination apparatus with glass lenses in this way constitutes an instrument

called an *ophthalmoscope* (or "ocular mirror"), which enables the observer to see distinctly the images on the retina and the details of the retina of another person's eye, and to investigate them.

Brücke called attention to a peculiar advantage which must be afforded by the bacillary layer of rods and cones in the reflection of light from the retina. These elements are small cylinders, 0.030 mm long and 0.0018 mm thick, made of some highly refracting substance. Packed close together like palisades, they constitute the last layer of the retina. The axes of those which cover the retina in the back of the eye are pointed towards the pupil, so that all light that falls on these elements penetrates them nearly parallel to their axes. Now when light, proceeding in a denser medium, arrives at the boundary of a less dense medium at a very large angle of incidence, it is totally reflected there. So we may infer that light which has once been refracted into a retinal rod is mostly retained there, and if it should fall on the curved cylindrical surface anywhere, it is nearly all internally reflected. For example, suppose the index of refraction of the substance of the rods is equal to that of oil (1.47), and the index of refraction of the intervening substance is equal to that of water (1.33); then any rays that make an angle with the surface less than 25° will be totally reflected. But the rays that arrive through the pupil must fall on the rod-walls at angles that are never more than about 8°. If the light has at last reached the farther (outer) end of the rod, and if here part of it is diffusely reflected from the choroid, nearly all of this portion must be sent back again through the same rod. Such light as proceeds in a direction more inclined to the axis, may, of course, succeed in escaping from the rod; but not until it has been repeatedly reflected at the surface of the adjacent rod, will it contrive to get into the vitreous humor. On the other hand, the light that comes back nearly parallel to the axis of the rod suffers only a few total reflections at most, and hence will not have lost much when it emerges. Besides, this light will be directed towards the pupil and will issue through it. This function of the rods appears to be of importance, especially in the case of those animals that have a highly reflecting surface or tapetum instead of the layer of black pigment cells on the choroid. The first effect of it is that the light which was originally incident on the sensitive elements of the retina meets and stimulates them again on the rebound. And, secondly, on its return, it affects only the same elements of the retina or possibly in some measure those right next to them. Thus, the light that arrives at any one place is practically confined to a very minute region of the retina; a circumstance that must have an important bearing on the accuracy of vision. When the retinal image is sufficiently bright, diffusely scattered light of this sort may be noticeable in the

field of view. An instance of this was given in the preceding section in the description of the method of observing the vascular figure by moving a light to and fro below the eye. As the basis of the mathematical theory of the luminosity of the eye and the ophthalmoscope, a series of general propositions will now be stated and discussed. The special cases to be considered afterwards are much simplified by formulating these general laws.

I

If two rays of light traverse a system of isotropic media from opposite directions, and if in any one of these media their paths coincide, they must coincide in all the media.

Suppose that the straight line AB in Fig. 99 is the path of the two rays which we know to be common to both of them; and that the first ray, proceeding along the straight line EB, is refracted at B in the direction BA. The other ray, proceeding from A to B along the straight line AB, is likewise refracted at B, say, in the direction $BE_{,}$. First, of all, it must be proved that $E_{,}B$ and EB coincide. Draw DBC normal to the refracting surface at the point B, and put angle $EBD = \alpha$, angle $E_{,}BD = \alpha_{,}$, and angle $ABC = \beta$; and let m and n denote the indices of refraction of the media containing the points E ($E_{,}$) and A, respectively. By the law of refraction, the first ray BA must lie in the plane containing BD and EB; and, moreover,

$$m \sin \alpha = n \sin \beta.$$

Likewise, the second ray $BE_{,}$ must be in the plane determined by BD and AB, that is, in the same plane with BE; and also:

$$m \sin \alpha_{,} = n \sin \beta.$$

Fig. 99.

Consequently, $\sin \alpha = \sin \alpha_{,}$

or $\alpha = \alpha_{,}$,

since the angles here are both acute angles. Hence, the ray $BE_{,}$ coincides with the ray EB, and the two rays pursue the same path in opposite directions as far as this medium extends. At the next refracting surface the same proof applies, and so on throughout the entire system.

Notes: 1. Evidently, the proposition is true also when the light is reflected at a surface.

2. In the case of the eye, if a ray on its way through the vitreous humor coincides with another ray proceeding in the opposite direction in the same medium, the paths of these rays outside the eye coincide also.

3. According to the general enunciation of this proposition, as given here, the case can be supposed in which for certain directions of polarisation and for

certain angles of incidence, the rays might be entirely extinguished by a refraction or reflection. But such circumstances do not occur in the applications of the theorem to the illumination of the eye. The light is incident nearly normally on the refracting surfaces of the eye, and hence any polarisation effects due to either refraction or reflection will be practically negligible. Moreover, the losses of light by reflection and absorption in the eye may be left out of account. It is only where the glass plate reflector is placed very obliquely to the rays of light that there is any considerable loss of light.

There is another corresponding proposition of wide application concerning the intensity of the light that proceeds to and fro over the same path, which, however, may be given here without proof because anybody who is at all familiar with the laws of optics can easily prove it for himself, and because also the principle is not needed for the purposes of the present discussion. This more general rule may be stated as follows.

Suppose light proceeds by any path whatever from a point A to another point B, undergoing any number of reflections or refractions *en route*. Consider a pair of rectangular planes a_1 and a_2 whose line of intersection is along the initial path of the ray at A; and another pair of rectangular planes b_1 and b_2 intersecting along the path of the ray when it comes to B. The components of the vibrations of the aether particles in these two pairs of planes may be imagined. Now suppose that a certain amount of light J leaving the point A in the given direction is polarised in the plane a_1, and that of this light the amount K arrives at the point B polarised in the plane b_1; then it can be proved that, when the light returns over the same path, and the quantity of light J polarised in the plane b_1 proceeds from the point B, the amount of this light that arrives at the point A polarised in the plane a_1 will be equal to K.

Apparently the above proposition is true no matter what happens to the light in the way of single or double refraction, reflection, absorption, ordinary dispersion, and diffraction, provided there is no change of its refrangibility, and provided it does not traverse any magnetic medium that affects the position of the plane of polarisation, as FARADAY found to be the case.

II

In order to see the luminous pupil of another person's eye, the retinal image of the source of light in his eye must overlap the retinal image of the pupil of the observer's eye, at least to some extent.

If the observer is to get light from any part of the retina of another person's eye, it is necessary, in the first place, for this part of the retina to be illuminated by the source of light, so that it must be comprised in the image of the source of light. In the second place, assuming, for the sake of argument, that light proceeds from the pupil of the observer's eye, then by the preceding proposition, it could go just as well to the given part of the retina of the other person's eye as it could come from it; and therefore this part of the retina must likewise be

contained in the retinal image of the pupil of the observer's eye, no matter whether this image is clear-cut or blurred.

Notes: 1. This proposition is true not only when the light proceeds directly from the source to the illuminated eye and thence into the observer's eye, but also when any lenses or mirrors are interposed along the way. Incidentally, this fact affords a convenient way of showing experimentally how an ophthalmoscope acts in one's own eye. The light used for illumination is adjusted and the instrument arranged in the same position in front of the observer's eye as it would be in front of the patient's eye; then the part of the field that is bright corresponds to the part of the retina that is illuminated. It is possible to tell whether this bright field is large or small, and whether it is uniformly illuminated or whether there are dark places in it, and how dark they are. Then the source of light is transferred behind the instrument where the observer's eye would naturally be, so that the light shines through the peephole. Whatever is illuminated now in the field of view is comprised in the part of the retina that the observer will be able to see. This is the simplest and easiest way of getting a clear idea of the effects of various combinations of flat and curved mirrors and of convex and concave lenses in ophthalmoscopes; without having to make complicated geometrical constructions that are ofttimes more confusing than helpful to the unintiated.

2. The effect of the mode of illumination described in this section is easily regulated by the above rule. Everyday experience teaches us (as can be proved also by a simple construction of the procedure of the light) that the blurred image of a distant object cannot cover the sharp image of a nearer object that is seen distinctly, but that the blurred image of a near object may cover the sharp image of a more distant one. In the experiment with the perforated mirror the blurred image of the peephole which must be adjusted as near as possible in front of the patient's eye overlaps the image of the more distant source of light which is perhaps sharply in focus. If no mirror is employed, so that the observer looks past the light into the patient's eye, the flame and the observer's eye seem to the patient to be near together, and when his eye is not nicely accommodated for them, their blurred images are fused together. With illumination by a transparent plate of glass, both the image of the light and that of the pupil of the observer's eye may be sharply in focus. The former is seen reflected in the plate and the latter is seen through the plate, so that they come together on each other. Accordingly, it is best for the patient himself to adjust the glass plate so that his eye looks luminous to the observer. All that he has to do is to be careful that the observer's eye appears to be covered by the reflected image of the source of light.

A reciprocity law similar to that given above for light proceeding in opposite ways over the same path may also be formulated for the amount of light transmitted to and fro. In this connection, let us state here, first, the following

Fundamental Law of Photometry.

Suppose a and b are the areas of two tiny elements of surface in a transparent medium at the distance r apart; and let the brightness of the luminous element at a be denoted by H; then the quantity of light received by b will be

$$L = \frac{H . ab \cos \alpha \cos \beta}{r^2}; \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

where α, β denote the angles made by the normals to a and b, respectively, with the straight line connecting these elements. Similarly, the quantity of light which would proceed from b to a would be the same as before, supposing the brightness of the element b were the same as that of a in the first case.

III

Let n_1, n_2 denote the indices of refraction of the first and last media, respectively, of a centered system of spherical refracting surfaces; and let α and β designate two surface-elements near the axis and perpendicular to it, in the first and last media, respectively. *Assuming that the brightness of the surface-element α is n_1^2H and that of β is n_2^2H, the quantity of light that goes from β to α is equal to that that goes from α to β.*

Not to make the proof more complicated than is necessary for the applications we have here in view, the losses of light due to reflection at the refracting surfaces may be neglected; and it may be also assumed that the rays are all incident on the refracting surfaces at angles so small that the cosine of any such angle may be put equal to unity; although the proposition is true when this is not the case.

1. *Case when β does not coincide with the image of α.*

In Fig. 100 the straight line AC represents the axis of the optical system, with its first and second principal points at F and G, respectively. The element of surface α in the first medium may be represented by a point in the diagram, since it is infinitely small; and the

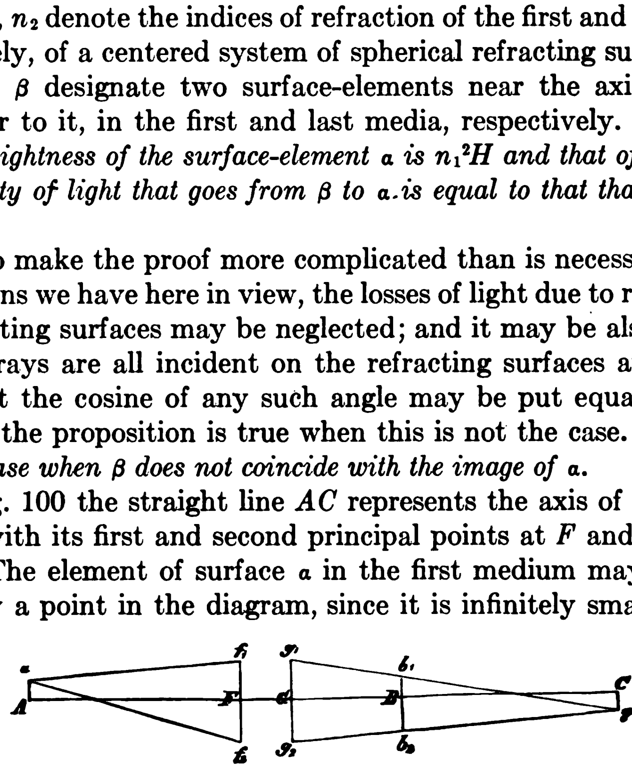

Fig. 100.

image of α in the last medium is designated by γ. Consider the bundle of rays proceeding from α; its cross section in the first principal plane is represented by f_1f_2, and the corresponding cross section in the second principal plane by g_1g_2; so that $g_1g_2 = f_1f_2 = \Phi$, say. The other surface-element β is supposed to be in a plane perpendicular to the axis at the point designated by B; and b_1b_2 represents the cross section of the bundle of rays in this plane. Let A and C designate the feet of the perpendiculars on the axis drawn from the conjugate points α and γ, respectively. According to equation (1), the quantity of light coming from α that falls on f_1f_2 is

$$\frac{n_1^2H.\alpha.\Phi}{AF^2},$$

supposing that the brightness of α is $n_1^2 H$. The more distant cross sections $g_1 g_2$ and $b_1 b_2$ receive this same amount of light. In the latter section the quantity of light that is delivered to the element β is in the same ratio to the entire quantity of light as the area of β is to that of the cross section $b_1 b_2$; so that if this ratio is denoted by Ξ, the quantity of light that comes from α to β is

$$X = \frac{\Phi}{\Xi} \cdot \frac{n_1^2 H \alpha\beta}{AF^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Moreover,

$$\frac{\Phi}{\Xi} = \frac{(g_1 g_2)^2}{(b_1 b_2)^2} = \frac{CG^2}{BC^2}.$$

Hence, by substitution in equation (2):

$$X = n_1^2 H \alpha\beta \frac{CG^2}{BC^2 . AF^2}.$$

But by equation (8a) of § 9:

$$\frac{GC}{AF} = \frac{F_2}{AF - F_1},$$

where F_1, F_2 denote the two focal lengths of the optical system; hence

$$X = H\alpha\beta \cdot \frac{N_1^2 F_2^2}{[AF \cdot F_2 + BG \cdot F_1 - AF \cdot BG]^2} \quad . \quad . \quad . \qquad (2a)$$

. Similarly, for the quantity of light sent from β to α, on the assumption that the brightness of β is $n_2^2 H$, we obtain:

$$Y = H\alpha\beta \cdot \frac{n_2^2 F_1^2}{[AF \cdot F_2 + BG \cdot F_1 - AF \cdot BG]^2} \quad . \quad . \quad . \qquad (2b)$$

Since everything is symmetrical on the two sides, all that is necessary in order to derive this last expression is to substitute Y for X in equation (2a), and to interchange the pairs of magnitudes according to the following scheme:

AF and BG
F_1 and F_2
α and β
$n_1^2 H$ and $n_2^2 H$.

Since, by equation (9c) of § 9,

$$n_1 F_2 = n_2 F_1,$$

we derive from equations (2a) of § 9,

$$X = Y,$$

which was to be proved.

2. *Case when β coincides with the image of α.*

Suppose first that β coincides with the image of α, in both size and position; in which case also α coincides exactly with the image of β. Accordingly, all light coming from α falls on β, and *vice versa*. In Fig. 100 nothing is changed here except that now we must think of β as being at γ. If the brightness of the element α is n_1^2H, the total quantity of light which it sends to β is

$$X = n_1^2 H \frac{\alpha\Phi}{AF^2}; \quad \ldots \ldots \quad (3a)$$

and, similarly, if the brightness of the element β is n_2^2H, the total quantity of light which it sends to α is

$$Y = n_2^2 H \frac{\beta\Phi}{GC^2} \quad \ldots \ldots \quad (3b)$$

Now as β here has to be the image of α, according to equation (8b), §9, we must have:

$$\frac{\alpha}{\beta} = \frac{F_2^2}{(GC - F_2)^2},$$

since the areas of similar figures are proportional to the squares of their corresponding linear dimensions. Moreover, by equation (8a), § 9:

$$GC - F_2 = \frac{GC.F_1}{AF},$$

and hence:

$$\frac{\alpha F_1^2}{AF^2} = \frac{\beta F_2^2}{GC^2},$$

But $F_1 : F_2 = n_1 : n_2$, consequently,

$$\frac{\alpha n_1^2}{AF^2} = \frac{\beta n_2^2}{GC^2} \quad \ldots \ldots \quad (3c)$$

Combining equations (3a), (3b) and (3c), we derive finally the result that $X = Y$, which was to be proved.

Should one of the two elements, α say, be greater than the image of β, then the parts of α that did not belong to the image of β would neither send light to β nor get light from it; and neither X nor Y would be changed on this account, so that the proposition would still be valid.

Notes: 1. The above proof is equally applicable to a centered optical system composed of both refracting and reflecting surfaces.

2. The condition that the two surfaces α, β shall be infinitely small is not essential, provided simply they are not so extended that the cosines of the angles of incidence of the rays at the various refracting surfaces are not very different from unity. For inasmuch as the proposition is true for each pair of infinitesimal portions of the two areas, it applies also to the entire areas.

Suppose the above proposition is applied to the case of the luminosity of the eye, one element of surface being in the retina of the illuminated eye and the other element being formed by the pupil of the observer's eye. If the difference in the indices of refraction of the aqueous and vitreous humors is neglected, and if any arbitrary symmetrical optical system of lenses or mirrors is supposed to be inserted between the two eyes, the proposition is as follows:

IIIa

The quantity of light proceeding from an element of the retina of the illuminated eye and entering the observer's eye is equal to the product of the brightness of the illumination of the retina and the quantity of light which would come from the pupil of the observer's eye and fall on the retina of the illuminated eye, on the supposition that the brightness of illumination of the pupil of the observer's eye were equal to unity.

Suppose the brightness of illumination of the element of the retina by the source of light is denoted by H; and let k denote the amount of light that is sent from the pupil of the observer's eye to the element of the retina on the supposition that the brightness of illumination of the pupil is equal to unity; then, by the proposition which has just been proved, k would likewise be equal to the quantity of light that would proceed from the element of the retina to the pupil of the observer's eye, provided the brightness of illumination of the retinal element were equal to unity. But this latter is not equal to unity but is equal to H; and hence the quantity of light that is actually communicated from the element of the retina to the pupil of the observer's eye is equal to Hk, in agreement therefore with the above statement.

In a sense this proposition is a sort of sequel to proposition II, because the quantitative relations which were lacking in that statement are given here. The proof as first given applied simply to an ophthalmoscope in which the rays of light met the surfaces of the mirrors and lenses at nearly normal incidence, with no appreciable loss of light. But evidently the proposition is true also when the eye is illuminated by oblique reflection from a glass plate, because when the light is not polarised the losses incurred in traversing such a plate from one eye to the other are just as great when the light proceeds through it one way as when it goes through it the opposite way.

IV

When an observer sees a clear-cut image of a luminous object in a symmetrical optical system of lenses and mirrors, then, on the supposition that the losses of light by reflection and refraction are negligible, each place in the image appears just as bright as the corresponding place in the

object would appear without the optical instrument; provided the entire pupil of the observer's eye is filled with light coming from one single point of that part of the object. But if this latter condition is not satisfied, the brightness of the optical image is in the same ratio to that of the object seen directly as the area of the part of the pupil of the observer's eye that is filled with light is to the area of the entire pupil.

When the eye sees an object directly and distinctly, or its image in a symmetrical optical system, the eye by itself, or the eye and the optical system together, may be regarded as an optical system that projects an image of the object on the retina. Let a denote the area of a surface-element of the object, and b the area of the corresponding part of the image on the retina. According to proposition III above, the same quantity of light that goes from a to b would go from b to a, provided the brightness of illumination of the retinal element b were equal to $\frac{(n_2)^2}{(n_1)^2}H$, where H denotes the brightness of illumination of the element a, supposed to be in a medium of index of refraction n_1, and n_2 denotes the index of refraction of the vitreous humor. The quantity of light that would go from b to a under these circumstances can easily be calculated. If the cross section of the bundle of rays going from a point in b to a point in a as made by the plane of the pupil has an area denoted by q, the quantity of light M going from b to a is equal to that going from b to q, namely,

$$M = \frac{n_2^2}{n_1^2} H \cdot \frac{qb}{R^2},$$

where R denotes the distance of the pupil of the observer's eye from the retina of the illuminated eye. To be perfectly accurate, q denotes here the cross section of the bundle of rays made by the plane of the exit-pupil of the observer's eye; and R should be measured from this plane. In the above expression for the quantity of light coming to the observer from the illuminated element b there are two magnitudes that depend on the properties of the optical system which is interposed in front of the eye, namely, the cross section q in the pupil of the observer's eye, and the size of the image b on the retina of the other person's eye. But the brightness of this image depends not only on the quantity of light that it receives, but also on the area b over which this light is spread, being inversely proportional to b. If the quantity of light per unit of area is taken as the unit of the intensity of illumination, the intensity J of illumination of the retinal element b is

$$J = \frac{M}{b} = \frac{n_2^2}{n_1^2} H \cdot \frac{q}{R^2},$$

where now q is the only magnitude that depends on the peculiarity of the artificial optical system. When the eye looks directly at the luminous object, the entire pupil, of cross section Q, is filled with light, and the intensity of illumination of the light coming from b will be

$$J = \frac{n_2^2}{n_1^2} H \cdot \frac{Q}{R^2}.$$

This is the maximum intensity, because q can never be greater than Q. It is the natural brightness of the luminous area. The intensity of illumination of a surface as seen through an optical instrument can never be greater than its natural brightness, and is to the latter always in the ratio of q to Q.

Notes: 1. It is only when the luminous point is infinitesimally small, that its image as seen through an optical instrument, even with the highest magnifying power, extends over an element of the retina that it is no larger than the smallest blur circle, so that therefore it always retains the same size; and only under these circumstances can an optical instrument increase the apparent brightness of an object. This is the explanation of how the stars can be made visible in the day-time by looking at them through a powerful telescope with a large aperture; because the apparent luminosity of the star increases in proportion to the quantity of light coming from it that is brought to a focus by the instrument; whereas the telescope does not increase the brightness of the background of the sky.

2. Moreover, when the blurred image of a uniformly illuminated surface is projected in the eye, the intensity of illumination of the retina can never exceed that obtained by looking at the surface directly without any artificial optical contrivance. The proof of this statement is exactly the same as for the case when the image is not blurred, since proposition III applies equally to clear-cut images and blurred images. Here, likewise, the brightness of illumination is proportional to the cross section made by the plane of the pupil with the bundle of rays that can be sent from the corresponding point of the retina to the illuminated surface.

The author ventures to add that optical constructions of lenses and mirrors frequently trespass against the fundamental laws here stated. Many persons still believe that by concentrating light in the eye or in a microscope, etc., by means of convex lenses or concave mirrors, not only the apparent size of the illuminated surface, but its apparent brightness also, can be magnified. With the increased amount of light that can be obtained by such means there is invariably an accompanying magnification of the image, so that the image is merely bigger, not brighter. There is no optical instrument whatever that can make a luminous surface of any appreciable dimensions appear brighter to the eye than it does to the naked eye. It is just as impossible for an illuminated surface ever to be brighter than the illuminating source.

V

General method of estimating the brightness of illumination of a place in the retina of a person's eye as seen through an ophthalmoscope.

(a) *Case when the loss of light by refraction and reflection is assumed to be negligible.* Let x designate the position of a point at the place in the retina under consideration; the problem is to find out how the

bundle of rays proceeds from this point x to the pupil of the patient's eye. According to propositions I and II, some of this light must return to the illuminating object, and some of it must go to the pupil of the observer's eye. Suppose P denotes the area of the pupil of the patient's eye, and p denotes the area of the cross section in this same plane of that part of the bundle of rays which returns to the illuminating object. Moreover, let H denote the brightness of illumination that would exist at the given place on the retina if the patient looked directly at the source of light and focused it on his retina. This magnitude may be called the *normal* brightness of illumination. It depends essentially on the structure of the retina itself, and, of course, also on the brightness of the illuminating object and on the size (P) of the pupil of the patient's eye. When an ophthalmoscope is employed, the actual brightness of illumination of this part of the retina is necessarily less than this; that is, it is equal to

$$\frac{p}{P}H.$$

Now let Q denote the area of the pupil of the observer's eye; and let q denote the area of the cross section in this plane of the part of the bundle of rays coming from x that enters the pupil of the observer's eye; then the brightness of this part of the retina as it looks to the observer is

$$\frac{q \cdot p}{Q \cdot P}H.$$

(b) *Case when there is appreciable loss of light by reflection or refraction.* In all the types of ophthalmoscope hitherto constructed the only one in which there is such a loss of light is in the case of the construction proposed by the author where unsilvered glass plates are used. In this case, and in all similar contrivances, the losses of the bundle of rays going from the eye to the illuminating object are precisely the same as those of the rays that actually go from the light to the eye. Suppose that light of unit intensity proceeds from the source to the illuminated eye, and produces there an illumination α; and that light of the same unit intensity proceeds from the patient's eye to the observer's eye and produces there an intensity β: then the above expression has to be multiplied both by α and β, so that it becomes:

$$\frac{\alpha \cdot \beta \cdot p \cdot q}{P \cdot Q}H.$$

This complete reciprocity in the problem of the illumination of the eye as contained in the preceding propositions has enabled us to investigate the brightness of illumination of the images in every case by reducing the matter to the determination of the procedure of a single bundle of rays. Otherwise, the brightness at any particular place on the retina would have had to be found

by summation of the effects of all the blur circles superposed there, that are made by the separate points of the source of light. Besides, in the writer's opinion, the subject is made clearer by the method used here. It is easy to follow the route of the rays from a point on the retina separately through each of the comparatively simple optical systems of the opththalmoscope, one of which is for illumination, and the other for observation; but a complete view of the entire procedure from the source of light to the eye of the spectator is hard to follow, mainly because there is, so to speak, a sort of mosaic made of innumerable blur circles on the retina belonging to the various points in the source of light and in the pupil of the observer's eye.

VI

Method of obtaining a distinct image of the fundus of the eye

The illuminated eye is represented at A in Fig. 101. The image of a point a on the retina is produced at the point designated by b; the location of which will depend on the existing state of accommodation of the illuminated eye. The two arrows drawn through a and b correspond to the sizes of the two figures there. In the illustration the image of the place on the retina is shown magnified and inverted. An observer who, without using glasses or apparatus of any kind, desires to see the image of the retina that is formed at b, must, therefore, adjust his eye at some place C where he is able to accommodate so as to see

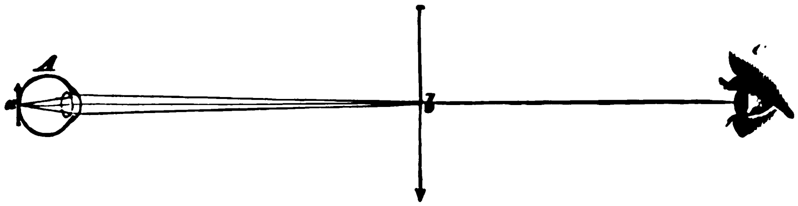

Fig. 101.

the image distinctly. However, under such circumstances, the field of view of the observer, being limited by the pupil of the patient's eye, would be so small that he might not be able to make out anything.

Heretofore two main methods have been employed to render the position of the image at b more convenient for the observer. In one case the image of the retina is virtual and erect, and in the other case it is real and inverted.

A. *Method of erect (virtual) image.*

In this method (Fig. 102) a concave lens B is employed, whose focal length B is less than the distance of the point b. The effect of this is to make the rays converging from A to b again divergent, so that they appear to come from a point d back of the patient's eye. In this figure also the arrows indicate the comparative sizes of the object at a and the images at b and d. Putting $\alpha = Bb$, $\gamma = dB$, then

$$\frac{1}{a}+\frac{1}{\gamma}=\frac{1}{p};$$

where p denotes the negative focal length of the interposed lens. The observer must be able to accommodate for the distance γ if he is to see distinctly the image of the retina at d. The distance a depends on the state of accommodation of the patient's eye and on the distance between A and B. If both a and γ are prescribed, the value of p may be calculated, which gives the power of the lens that must be used to get a distinct image under the given conditions. Thus, if both eyes were accommodated for parallel rays, then $a=\gamma=\infty$, and hence also $p=\infty$, that is, no lens would be necessary at all. Ordinarily, also a lens is not needed to see the more lateral parts of the retina, because

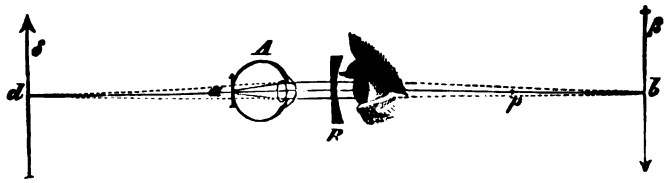

Fig. 102.

they appear to be far away, and the ocular media by themselves can form images of them that are easy for the observer to see. In this mode of observation the image of the retina at d is erect.

As to the magnification, suppose that b were a luminous object whose image would be formed on the retina at a. The rays reversed produce an image of this retinal image, which according to the preceding propositions is congruent with the luminous object at b. Let β denote the size of the luminous object and of the image at d that is equal to it; and let δ denote the size of the image seen by the observer; then

$$\frac{\beta}{\delta}=\frac{a}{\gamma}.$$

If the *apparent size* of the image as seen by the observer is defined to be the ratio of its actual size to its distance from the observer's eye, then when the observer looks through the concave lens the apparent size of the image is

$$\frac{\delta}{\gamma}=\frac{\beta}{a}.$$

The apparent size of the object at b for the eye A is

$$\frac{\beta}{a+q},$$

where q denotes the distance AB. It is therefore rather smaller than the apparent size of the image δ. Now on the supposition that the distance for which the eye A is accommodated is very much greater than q, the latter is negligible in comparison with a, and in this case the apparent size of the luminous object for the eye A is equal to β/a. Thus, with this arrangement, the image of the retina of the patient's eye appears to the observer just as large as, or rather larger than, the corresponding object appears to the patient. Hence, the magnification of the place on the retina of the patient's eye may now be easily found. If the size of the image of β formed on the retina at a is denoted by x, and if the distance of the retina from the second nodal point of the eye is denoted by y, then

$$\frac{x}{\beta}=\frac{y}{a+q}$$

$$\frac{\beta}{\delta}=\frac{a}{\gamma}.$$

Multiplying these equatations, we have:

$$\frac{x}{\delta}=\frac{y\cdot a}{\gamma(a+q)}.$$

In LISTING's schematic eye $y=15.0072$ mm; and if γ is put equal to 25 cm (which is the so-called distance of distinct vision), we obtain for the magnification:

$$\frac{\delta}{x}=16.67\,\frac{a+q}{a}.$$

Since q is usually very small as compared with a, the magnification may be put equal to $16\frac{2}{3}$.[1]

With this method the line of demarcation of the field of view as determined by the vaguely seen contour of the pupil of the patient's eye is not sharp. For practical purposes the field may be considered as limited by the lines of sight (see §11) drawn from the centre of the pupil of the patient's eye. If these lines are treated as rays of light proceeding from the observer's eye, the part of the retina of the other person's eye that is in the field of view is found to correspond to the blurred image of the centre of the pupil of the observer's eye. When this centre, or rather its image in the concave lens, is at the first focal point of the patient's eye, its blur circle, as was proved in the preceding section on entoptical phenomena, will be of the same size as the pupil of the patient's eye. Usually, however, the observer's eye cannot be as close as this to the patient's eye; and the blur circle corresponding to

[1] "In the text the distance γ is assumed to be equal to 8 (Prussian) inches; so that the value of the magnification is given as $14\frac{1}{3}$ instead of $16\frac{2}{3}$. (J. P. C. S.)

the field of view is smaller than the pupil of the patient's eye in proportion as the observer is farther way.

B. *Method of the inverted image* (*indirect method*)

The other method of enabling an observer to inspect the retina of the eye consists in inserting a convex lens of short focus (from 1 to 3 inches) close in front of the illuminated eye (Fig. 103). As before, let a designate an illuminated place on the retina, and b the position of its image outside the patient's eye A. The rays coming from a fall on the convex lens at B before they are converged to the image at b; and the lens produces an image at d which is nearer and smaller than the image at b, but inverted like b. The observer puts his eye at C, as far off as necessary to accommodate to see the image. If the distances Bb and Bd are denoted by α and γ, respectively, and if the positive focal length of the lens is denoted by p, then

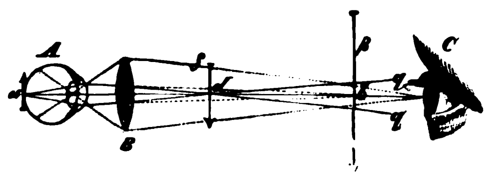

Fig. 103.

$$\frac{1}{\gamma} - \frac{1}{\alpha} = \frac{1}{p}.$$

As α is usually very much greater than p, γ is nearly equal to p, but always somewhat smaller.

Let x denote the size of a part of the retina at the point a; and let β and δ denote the sizes of the images at b and d, respectively. Finally, let y denote the distance of the retina from the second nodal point of the eye, and q denote the distance of the first principal point of the lens from the first nodal point of the eye; then

$$\frac{x}{\beta} = \frac{y}{\alpha + q},$$

$$\frac{\beta}{\upsilon} = \frac{\alpha}{\gamma}.$$

Multiplying these two equations, we get:

$$\frac{x}{\delta} = \frac{y \cdot \alpha}{\gamma \cdot (\alpha + q)} = \frac{y \cdot (\alpha + p)}{p(\alpha + q)}.$$

As a rule, the lens B is adjusted with its focal point at the centre of the pupil of the illuminated eye A, in which case p and q are nearly the same; and hence the magnification is

$$\frac{\delta}{\ } = \frac{p}{y}.$$

Employing the value of y in LISTING's schematic eye, the image δ is found to be magnified 2, 3 or 4 times according as the focal length of the convex lens is 30, 45 or 60 mm, respectively. This is the actual magnification of the real image of the retina. The magnification for the observer is

$$\frac{p}{yc}\times 25,$$

where c denotes the distance Cd in centimeters; the conventional distance of distinct vision being taken as 25 cm.

Provided the convex lens is placed very close to the patient's eye, the field of view is limited in this method by the pupil of this eye. But the farther away the lens is, the more magnified the pupil becomes, until at length when it is in the vicinity of the focal plane of the lens, the edge of the pupil disappears entirely from the field, and the extent of the field then is determined merely by the aperture of the lens. Here also, as in the preceding method, the extent of the field can be found by treating the lines of sight of the observer's eye as if they were rays of light. The lens B forms an image of the centre of the pupil of the observer's eye somewhere near its focal point, that is, approximately in the pupillary plane of the patient's eye; and thence the lines of sight diverge to the fundus of this eye. These lines intersect at or near the anterior nodal point of this eye, depending on the adjustment of the lens B; and therefore they proceed through the patient's eye practically without being refracted, as indicated by the dotted lines in Fig. 103. If u and v denote the aperture of the lens B and the diameter of the portion of the retina that is in the field, respectively, then

$$\frac{v}{y}=\frac{u}{p}.$$

With such small lenses the aperture may very well be made equal to half the focal length, that is, $u=p/2$; and hence, in this case:

$$v=\tfrac{1}{2}y=7\tfrac{1}{2}\text{ mm}.$$

Accordingly, under such circumstances a larger field is commanded than is practicable by the first method, without artificial dilatation of the pupil by atropin.

VII

Illumination system of the ophthalmoscope

The eye may be illuminated in any of the three ways mentioned above, that is, directly by a light or by an opaque mirror with a hole in it or by a transparent unsilvered plate of glass used as a mirror.

If the eye is illuminated without any mirror at all, the image of the retina has to be inverted to be seen distinctly, and it takes considerable skill. Indeed, this method is not to be recommended unless there is no other instrument at hand except a simple convex lens of short focus. The procedure is as follows. The observer, protected by a screen from the direct light of the lamp, as shown in Fig. 97, looks right past this light in the other person's eye, and holds a convex lens of from 2 to 4 inches focus in front of this eye, as in Fig. 103. In order to find the correct adjustment, the lens is placed at first close in front of the eye, and then gradually brought back from it until the pupil is so magnified that its edges disappear behind the edges of the lens. A real inverted image of the retina is seen then at d (Fig. 103). In order to determine the brightness of illumination of this image, the first thing to do is to trace the bundle of rays that proceeds from the point a of the retina, according to the directions given under proposition V above. These rays are converged by the ocular media towards the point b, and then by the lens to the point d. Beyond d the bundle of rays is divergent and wide enough by the time it gets to qq for the rays to fill the pupil of the observer's eye completely; which means that the place on the retina of the illuminated eye can be seen in its entire actual degree of brightness. This "actual" brightness and the "normal" (greatest possible) brightness are in the same ratio to each other as the part of the bundle of rays qq that returns to the source of light is to the entire bundle of rays (see Prop. V). Now if the source of light is large enough and properly adjusted, it takes comparatively few rays to return past the light and fill the pupil of the observer's eye completely; and in this case the "actual brightness," of the place a on the retina is not much less than the "normal brightness," and the apparent brightness for the observer is equal to the "actual brightness."

The illumination of the eye A by an opaque mirror with a hole in it (Fig. 104) is far more convenient for observation. In the diagram, A and B are the eyes of patient and observer, respectively; C is the convex lens and SS is a perforated mirror. The image of the point a on the retina is focused at d and viewed by the observer through the hole in the mirror. Only a small portion of all the rays coming from a goes through the hole in the mirror. The rest are reflected and may return to the source of light. The mirror SS may be concave (Ruete), plane (Coccius), or convex (Zehender). Moreover, a convex lens L can be interposed between the mirror and the source, as shown in the figure. Evidently, by this method the brightness of the illumination may be nearly normal (see Prop. V).

When the pupil of the eye A is in the focal plane of the lens C, the field of view depends on the size of the lens, as found above. How

much of the retina can be illuminated? Since all the light the observer gets passes through the lens, it is evident that the illuminated part of the retina cannot exceed the blurred image of the lens; and this blurred image itself, as was shown in VI above, corresponds to the observer's field of view. This image, in all its parts, will have its maximum brightness when light goes from every place in the lens to every place in the pupil. This condition will be satisfied, provided the pupil of the illuminated eye is equal to or less than the image of the mirror SS (or of the lens L) that is made by the lens C somewhere near the pupil, and provided light goes from every point of the mirror (except, of course, where the hole is bored) to every part of the lens C. But the latter requires that the lens shall be situated at the image of the source D in the mirror, and that the size of the lens must be equal to that of this image or less than it.

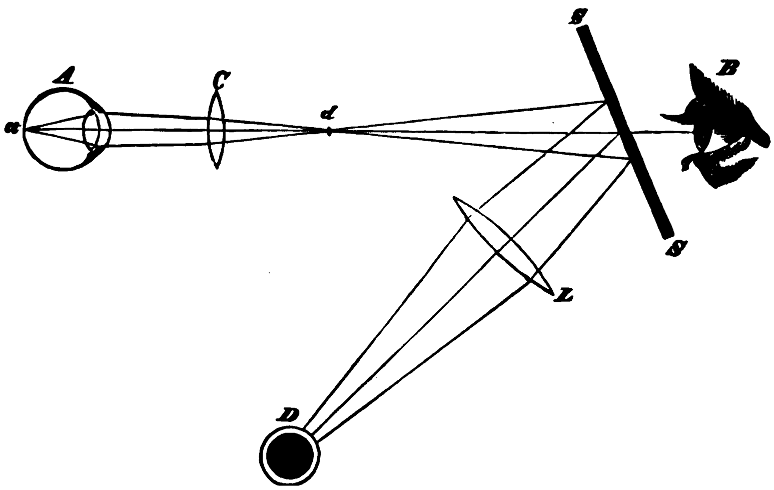

Fig. 104.

To give an example of such constructions, suppose that we wish to get a 4-fold magnification with the ophthalmoscope, and that, consequently, a lens C of 60 mm focus is employed with an aperture of 30 mm. The mirror S, which may be a concave glass with a hole in it, must be placed so far from the image d that the observer can accommodate for it, that is, about 150 mm away. In this case, therefore, the distance of S from C would be 210 mm. By equation (14b), §9, the image of the mirror in the lens will be 60/150 or 2/5 of its own size. As this image is to be of the same size as the pupil of the illuminated eye, which with artificial dilatation may have a diameter of 10 mm, the diameter of the mirror should be 25 mm.

The focal length of the mirror is determined by the condition that it has to produce an image of the light-source that covers the lens C. The diameter of the flame of a large Argand burner is about 15 mm. If the diameters of the lens and flame are taken as 30 mm and 15 mm, respectively, and if these values are substituted for β_1 and β_2 in equation (14b), §9, and if also $f_1 = CS = 210$ mm; the focal length of the mirror comes out to be $F = 70$ mm; and the distance of the source from the mirror must be 105 mm.

In case one p efers to use a plane mirror and a convex lens, as in Fig. 104, instead of a concave mirror, the sum of the distances of the two lenses L and C from the centre of the mirror must be used in the above calculation, instead of the distance of the mirror from the lens C.

When the mirror and lens are in two separate parts which the operator holds in position, of course, it is not possible to keep them adjusted at exactly the distance calculated by the formula; and in fact good images can be had with fairly large variations from the correct distance. However, it is well for the observer to know, at any rate, the best conditions for holding his instrument.

The conditions are more unfavourable when the observation has to be made with a perforated mirror and a concave lens; as represented in Fig. 105, where, as before, A and B designate the two eyes and S the mirror. Suppose that α denotes the fraction of the cone of rays that comes into the observer's eye from the point a on the retina of the other eye; and that $(1-\alpha)$, therefore, is the other part of this light that is reflected by the mirror to the source of illumination. Under these circumstances, the actual brightness of this part of the retina, according to proposition V above, will be $H(1-\alpha)$, where H denotes the "normal" brightness there. As before, let J denote the area of the apparent pupil of the patient's eye (A), and let R denote the corresponding magnitude in the observer's eye; also, let g denote the distance between the two apparent pupils, and h the distance for which the eye A is accommodated; then the cross section of the part of the bundle of rays that enters the observer's eye is

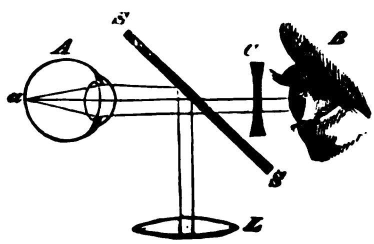

Fig. 105.

$$\alpha J \cdot \frac{(h-g)^2}{h^2}.$$

Generally, this cross section will be smaller than R. The brightness as it appears to the observer in this case will be

$$H \cdot \alpha(1-\alpha)\frac{J \cdot (h-g)^2}{Rh^2}.$$

For $\alpha = 1/2$, the expression $\alpha(1-\alpha)$ has its greatest value; and hence the best arrangement, so far as brightness is concerned, is when half of the rays go to the observer's eye and the other half are reflected back. In this case the brightness attains the value

$$\tfrac{1}{4}H \cdot \frac{J \cdot (h-g)^2}{Rh^2}.$$

A large source of light placed near the eye is used to get as great a field as possible in the illuminated eye; or a convex lens placed at L can be used for this purpose. If the image of the source in this lens covers the pupil completely, the entire blurred image of the lens L in the eye A will be illuminated.

The illumination by means of a transparent glass plate in the convex lens method gives only one quarter of the brightness that can be gotten with a silvered mirror with a hole in it. On the other hand, this mode of illumination in the concave lens method is sometimes superior. For example, suppose that the mirror SS in Fig. 105 is not silvered and has no hole in it, but is just a plate of glass or several plates put together. Along every ray incident on the mirror, a certain portion of light α will be transmitted through it, and another portion $(1-\alpha)$ will be reflected from it. The brightness of the light reflected from the mirror is equal to $H(1-\alpha)$, where H denotes the "normal" brightness of the retina at a, for the case of light incident directly on it. The cross section of the bundle of rays emitted from a at the place where they enter the eye B is

$$J\frac{(h-g)^2}{h^2}.$$

Since only the portion α of the light goes through the plate (or plates), the brightness as seen by the observer is

$$H\cdot\alpha(1-\alpha)\frac{J\cdot(h-g)^2}{R\cdot h^2}.$$

which also has its maximum value for $\alpha=1/2$, in which case it becomes

$$\tfrac{1}{4}H\cdot\frac{J\cdot(h-g)^2}{R\cdot h^2},$$

provided

$$R<\frac{J(h-g)^2}{h^2}.$$

This condition will usually be satisfied for normal eyes, because, as a rule, the pupil J of the illuminated eye will be smaller than the pupil R of the observer's eye. When the pupil J is artificially dilated with atropin, this will not be the case, and then the apparent brightness is simply $H/4$. The method of observation with a mirror with a hole in it is better in this last case, because then the brightness is given by the above expression, provided

$$R<\alpha\cdot\frac{J(h-g)^2}{h^2}$$

and

$$\alpha=1/2$$

When normal eyes are examined without the use of atropin, the same brightness might be obtained by means of both methods of illumination, provided the pupils of the two eyes did not alter their sizes. But the silvered mirror reflects on the whole more light into the illuminated eye and causes the pupil to contract more; so that under these circumstances the transparent mirror may give a larger field of view and a greater brightness. Besides, it illuminates the visible surface of the retina uniformly, whereas the blurred image of the hole in the other type of mirror has the effect of a shadow, so that the illumination is greater at some places than at others. And, lastly, in case of the unsilvered mirror the corneal reflex is less troublesome, because the light reflected from the mirror is more or less polarised, and on being reflected from the cornea without change of polarisation, most of it fails to get through the transparent glass plate or plates.

In order that the unsilvered mirror shall reflect as much as half of the incident light, it may consist of a single glass plate or of several such plates put together, only the mirror must be inclined at a suitable angle depending on the number of plates. For one plate the proper angle of incidence is 70°, for three plates it is 60°, and for four plates it is 56°.

Various types of ophthalmoscopes

1. Helmholtz's ophthalmoscope with reflecting glass plates and concave lenses; shown in section and actual size in Fig. 106; and as seen from in front, half-size, in Fig. 107. The illustrations include an improvement in the original construction, which was added by the instrument-maker Rekoss and which consists in two rotatable discs containing the requisite concave lenses. The three glass plates constituting the mirror are designated by *aa*. These form the sloping face of a rectangular prism box, the bottom of which is a right triangle, as seen in section in Fig. 106. The two perpendicular sides of this hollow prism are made of metal plates, covered on the inside with black velvet to absorb the light as much as possible. The smaller one of the metal plates is fastened to the frame of the instrument in such fashion that it can be rotated around the optical axis; and there is an opening in it corresponding to this axis. The glass plates are held against the prismatic box by a rectangular frame; and the frame itself is fastened to the triangular base of the prism by two screws *ee*. The glass plates are inclined to the optica axis of the instrument at an angle of 56°.

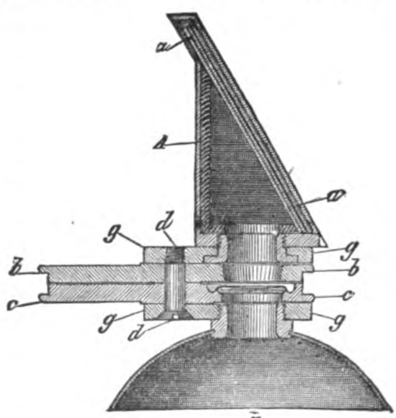

Fig. 106.

Moreover, two discs *bb* and *cc* turn around an axis *dd* inserted in the metal frame *gg*; and each of these discs has five circular openings, four of which

contain concave lenses of from 6 to 13 inches focus; the fifth opening being left empty. These openings, one after the other, can be turned into the optical axis of the instrument; and thus the observer with his eye in the hollow eyepiece B may look through each of them in turn and through the glass plates aa. As shown in Fig. 106, the empty opening of the disc bb, and one of the openings with a lens in it belonging to the disc cc, are in position for the observer. The latter can therefore use any one of the eight lenses by itself or any combination of a pair of them. To keep the discs in place, each of them has little notches around the circumference which catch in the ends of two springs h.

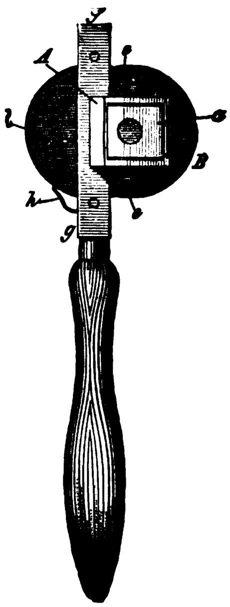

Fig. 107.

Of all the forms of ophthalmoscope with movable mirror, this original type still seems to the writer the best for examination of the eye with a concave lens, that is, with high magnification, without artificial dilatation of the pupil, and particularly in cases where the eye is very sensitive to light. The reasons for this opinion are the same as those given above in the theory of illumination by transparent glass plates. When a healthy eye is inspected through this instrument, it can stand the illumination for hours at a time without being blinded. The author himself has often showed his retina to twenty students in succession and not felt any discomfort; whereas with a silvered mirror the patient cannot bear the illumination for five minutes without being blinded by it. And therefore for most physiological experiments the writer prefers this type of instrument to other forms. On the other hand, for clinical purposes a larger field and more brightness with less magnification are generally desirable; and so for observations of that kind the silvered mirror with a hole in it, and combinations of convex lenses, are usually employed.

In using the ophthalmoscope above described, the observer stands or sits right in front of the patient, with a bright lamp at his side. A screen is adjusted to shade the patient's face. First, the observer takes the ophthalmoscope, and, before looking through it, focuses it about at the right place in front of the patient's face, turning it until the bright reflex from the glass plates falls on the eye. Then he looks through the instrument and sees the redly illuminated retina. Supposing he cannot at once accommodate his eye for the finer details of the retina, with the forefinger of the hand that is holding the instrument he turns one of the discs containing the lenses until he finds the proper concave lens for his purpose. In case the illumination of the retina vanishes, all that is necessary is to watch the reflex of the mirror in the patient's face and bring it back on to the eye.

2. RUETE's ophthalmoscope, with perforated concave mirror, mounted on a stand, as shown in Fig. 108. On a round wooden base a hollow upright a supports a round wooden rod b that fits in a, and that can be raised or lowered, and fastened at any height by a spring on its lower end. A semi-circular brass holder c rests on top of this rod, and can be raised or lowered with it, and turned to one side or the other. In this contrivance is inserted a concave mirror d with a hole through it in the centre. The diameter of the mirror is about 8 cm, and its focal length about 27 cm. By means of screws which can be tightened or loosened the mirror can be turned around its horizontal diameter at any inclination. In the middle of the vertical rod a there are two wooden collars e and f which may be turned around a, and each of which carries

a horizontal arm. On the arm *g* a black screen is supported whose duty is partly to cut off the light of the lamp from the observer, and partly also, if necessary, to cut down the light reflected from the mirror into the illuminated eye; which can be done by shading a portion of the mirror by the screen. The arm *h*, which is a foot long and graduated in inches, carries two uprights *i* and *k* which can be shoved either way on the horizontal arm. These uprights are supports of lenses, convex or concave, according to circumstances, which can be adjusted at the right height; the business of the lenses being to produce a distinct image of the fundus of the patient's eye (*A*) for the observer (*B*) to see. The illustration shows how it is done.

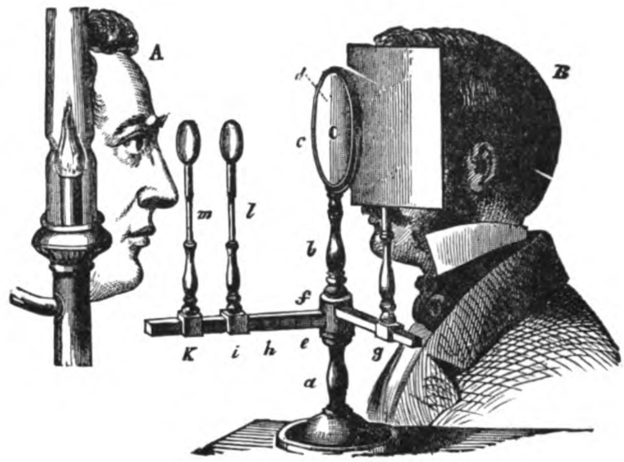

Fig. 108.

The instrument is not well adapted for observations with concave lenses, which in clinical practice at any rate are perhaps used less frequently; because the two eyes cannot be brought close enough together to see more than a very small field. On the other hand, the instrument seems to be very convenient for clinical observations with convex lenses; especially if an assistant is present to adjust the patient's head so that his pupil is at the focus of the light. By using a second convex (ocular) lens (which, however, should perhaps be inserted behind the mirror), a sort of little telescope can be constructed which will give a higher magnification. The brightness is very great with this instrument; but it does not permit of observing the image formed on the retina.

3. Epkens' ophthalmoscope, with plane perforated mirror, mounted on a stand; as modified by Donders and van Trigt. The complete instrument is shown in plan in Fig. 109, and in elevation in Fig. 110. The mirror *D*, shown by itself in Fig. 111, is a silvered plate of glass with the silvering removed at the centre, making a hole of about the diameter of the pupil. Subsequently, Donders had a hole made in the mirror, as in Coccius' method, so as to prevent the light coming to the observer from being diminished by reflection. The mirror is mounted inside of a cubical box *EE* and can be turned by the screw *F*. The eye to be examined is adjusted at *N*, and the observer's eye opposite to it at *O*. Here there is a disc with a set of lenses similar to that added to the Helmholtz instrument by Rekoss. The lenses used by Donders were three convex lenses of focal lengths 20, 8 and 4 cm, and three concave lenses of focal lengths 16, 10 and 6 cm.

In Epkens' original instrument the cubical box was connected with a conical tube, and a lamp placed at the end of it where the micrometer *M* is now. If necessary, a convex lens may be inserted in the end of the tube with its focal point not far from the lamp, so that for any one looking in the instru-

ment the entire surface of the lens appears luminous, and thus a larger area of the retina is illuminated. The entire apparatus attached to the upright rod A can be adjusted at the proper level. A circular disc, painted black to cut off superfluous light from the lamp, is inserted at K; and a piece of oilcoth LL is suspended on the lower side of the instrument from the bar Z, to hide the observer's face from that of the patient.

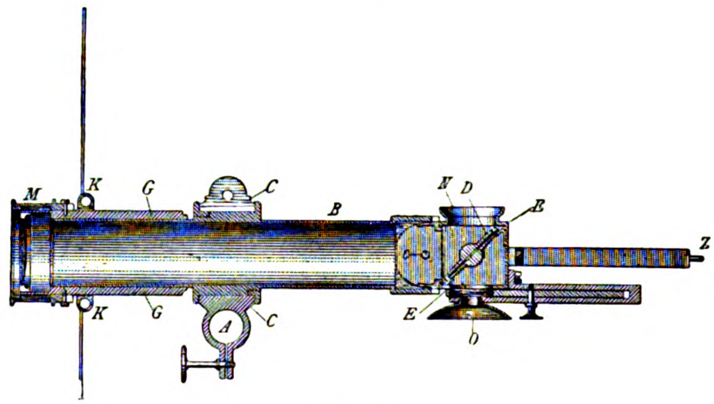

Fig. 109.

However, as it was difficult to examine patients for the correct movements of their eyes, the apparatus was made more flexible by DONDERS and VAN TRIGT. The tube was contrived so that it could be turned in a ring C; and the box EE could be turned around the axis determined by the screws b and c. The lamp was separated from the instrument. At the end of the tube G a micrometer gauge was inserted, the opposite points of which were imaged in

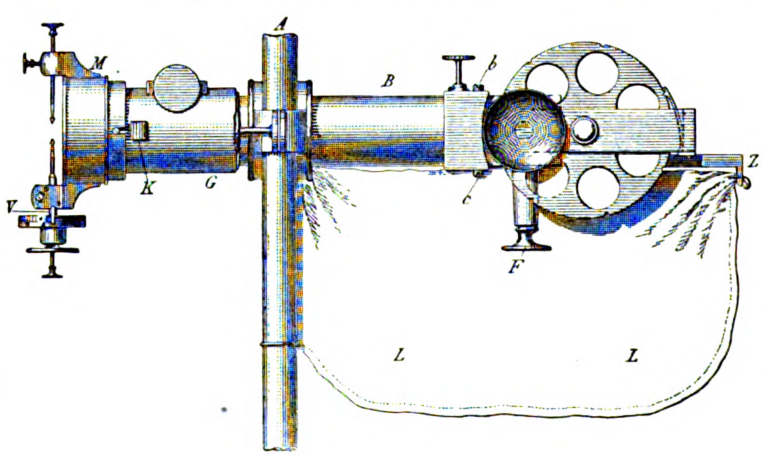

Fig. 110.

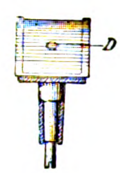

Fig. 111.

the patient's eye when it was accommodated properly. Accordingly, the micrometer could be moved by means of the tube G which slides over the tube B. By turning the micrometer screw V, the distance between the points can be altered and measured. If n denotes their distance apart and x their distance from the anterior nodal point of the patient's eye, then the distance

between the images of these points on the retina is

$$y = \frac{n}{x} \times 15 \text{ mm},$$

where the distance of the posterior nodal point from the retina is put equal to 15 mm.

By inserting a drawing device in the opening at O such as is used in microscopes, and marking on it the intervals between the two micrometer points and the positions of the blood vessels, etc., on the retina, the real dimensions of these, and of similar details on the retina, may be ascertained.

Subsequently, DONDERS inserted another micrometer, this time in the tube B, to be used with near-sighted eyes. In the mouth of this tube he also fitted a conical attachment for the reception of a convex lens of larger diameter; enabling the observer to obtain a more widely illuminated field in the eye in examinations with pupils dilated by belladonna.

The instrument is intended especially for examination of the retina with a concave lens. Besides being convenient and easy to use, it is very accurate and reliable for investigating and measuring retinal images and the more minute details in the fundus of the eye. SAEMANN's portable ophthalmoscope is similar in construction. Imagine that the tube of EPKENS' instrument has shrunk until it is just a mere offset in the side of the box, and that the solid stand is removed, and, finally, that, instead of the disc with the lenses, a device is used for holding one lens at a time; the result will be SAEMANN's ophthalmoscope.

4. COCCIUS' portable ophthalmoscope (Fig. 112), in which a plane silvered mirror with a hole in it is employed, and a lens for illumination. The side of the small rectangular mirror is about 3 cm long; and the diameter of the central hole is less than half a centimetre. The edge of the hole on the side of the illuminated eye is beveled a little. The mirror is fastened to a thin brass plate which is attached beneath to the upright rod b. The lens ordinarily used is about 5 inches in focus, but it can be removed and another lens substituted if desirable. Its position with regard to the mirror can be firmly secured by screwing up the handle e until it clamps the horizontal arm d that carries the upright support for the lens. The instrument can be taken apart and put away in a little box. Like RUETE, COCCIUS used a concave lens as well as a convex lens between the mirror and the lamp. But the concave lens is disadvantageous on account of the reflex, and subsequently several concave lenses were provided which could be shoved in on the back side of the mirror as needed. This ophthalmoscope is very useful for oculists on account of its compactness and transportability. The examination can be made conveniently either with convex lens as in RUETE's instrument or with concave lens as in EPKENS'.

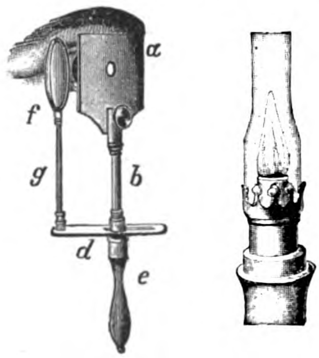

Fig. 112.

5. ZEHENDER's portable ophthalmoscope, with convex perforated metallic mirror and lens for illumination, with holder similar to that of COCCIUS. The only essential difference between this instrument and the one just described is that a convex metallic mirror of about 6 inches radius is used here instead of a plane glass mirror. By adjusting the convex lens nearer to the convex mirror or farther from it, the focal length of the reflecting system can be varied according to circumstances. Another real advantage, in the author's opinion, is in having a metallic mirror, so that the rim of the peep-hole is thin and black

and smooth. In using a perforated mirror and a concave lens to obtain the maximum brightness only half of the bundle of rays coming from a point on the retina can enter the observer's eye, unless the pupil of the illuminated eye is more than twice as large in area as that of the observer's eye; as the writer has proved. Hence, as a rule, part of the pupil of the observer's eye is necessarily covered by the edge of the hole in the mirror, so that a part of this edge is right in front of his eye. Consequently, it is a good thing to avoid having anything in this edge that might reflect light, and this is accomplished with ZEHENDER's metallic mirror much better than with COCCIUS' glass mirror.

6. MEYERSTEIN's prism ophthalmoscope. In place of a metallic mirror a rectangular prism is used in this instrument, the light being reflected from the hypotenuse face. The observer looks through a hole in the prism.

Subsequently, MEYERSTEIN used an illumination lens in conjunction with the perforated prism, a small telescope being inserted between the prism and the observer's eye. Eventually, to reduce the cost, the prism was replaced by a perforated mirror. The writer believes also that the use of the prism proved to have more disadvantages than advantages. There is an attachment by which the whole apparatus can be fastened to the border of the patient's eye; together with an arm in two links that carries a wax candle for illumination. Since the patient's eye is completely screened from external light, the instrument can be used in a brightly lighted room. By inserting or removing the ocular lens of the little telescope, the optical system can be adapted to emmetropic or ametropic eyes.

7. ULRICH's ophthalmoscope. In this instrument the essential features of RUETE's apparatus are combined on a portable tube which has a lateral attachment for holding the light used for illumination.

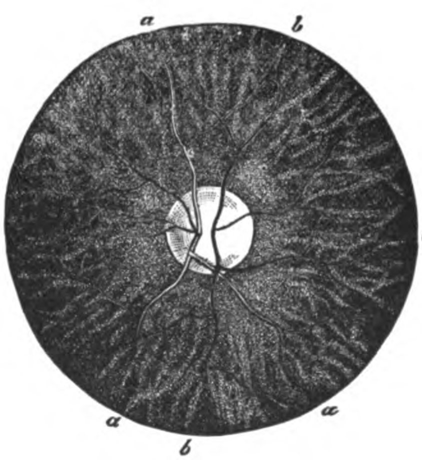

Fig. 113.

The following results of observation of normal eyes with the ophthalmoscope may be mentioned. With strong illumination (with silvered mirror and convex lens) the fundus of the eye appears red; except where the optic nerve enters, and there it is bright white. The blood vessels of the retina, originating from the centre of the white optic nerve, are conspicuously prominent on the red background. The arteries can be distinguished by their lighter red colour and by a stronger light reflex from their surface. In between the blood vessels the fundus of the eye appears in some places bright red and elsewhere brown, depending on the amount of pigment. Often the blood vessels of the choroid can be discerned, as illustrated in Fig. 113, especially in the more peripheral parts. The picture shows the appearance of the optic nerve. Branches of the retinal artery are indicated by *aaa*, and of the retinal vein by *bbb*; and between them can be seen the much wider blood vessels of the choroid. The latter are not always equally clear. In most eyes the pigment layer over these vessels is so thin that they stand out in contrast to the more highly pigmented intermediate places.

With increased illumination there is no very striking change in the appearance of the fundus of the eye except at the place where the optic nerve enters. Apparently, at this place a relatively large amount of light gets through the pigment layer of the choroid, and being reflected from the sclerotica, comes back again. The experiment described in §10 by which the image on the retina can be seen in the inner angle of the eye, and also the entoptical appearance of the choroid figure as seen by light that penetrates the sclerotica, show that right much light can get through the coatings of the eye. This portion of the returning light, due to reflection in the choroid and sclerotica, may be fairly uniform all over the fundus of the eye, even when the brightness of the retina itself varies considerably.

On the other hand, with weaker illumination (with glass plates), the parts of the fundus of the eye in the vicinity of the optic nerve are particularly bright, the brightness diminishing uniformly from here out towards the periphery of the retina. But the place of direct vision is especially conspicuous by its lack of brightness and a more yellowish colour than that of the immediate surroundings; which is not the case with high illumination. The reason may be that with feeble illumination the amount of light that goes back and forth through the pigment layer is scarcely noticeable, and so the perceptible reflex is due mainly to the parts of the retina, especially its blood vessels, that are missing in the yellow spot. In both modes of observation this latter place appears as a tiny bright spot transversely oval in shape. Coccius, who discovered it, attributed the appearance to the reflex from the sides of the foveal indentation; and Donders afterwards proved directly that this little reflex of light is at the place of direct vision. For this experiment a plane mirror has to be used with a concave lens behind it (Donders-Epkens or Helmholtz). The fixation mark may be a lamp-flame or the micrometer tips in Donders apparatus. The eye to be examined is directed towards the image of this object in the mirror, pains being taken to see that it can accommodate for it so as to see distinctly some definite point of the object. Under these circumstances the observer sees a perfectly sharply delineated inverted image of the object focused on the retina of the other person's eye and the reflex of the fovea at the place where the image is most clear-cut. In case this reflex is too faint to be seen at first, it can easily be found by making the subject turn his eye first to one part of the fixation-target and then to another; and the tiny reflex will move about on the retina in response to these minute motions of the eye.

For testing the exactness of the retinal image, the micrometer device is useful which Donders added to Epkens' ophthalmoscope. The author uses for this purpose in his instrument a horizontal thread in front of the light as object of fixation; because with this opththalmoscope the several reflecting surfaces produce multiple images of a narrow vertical line. When the illuminated eye is sharply focused on the object, the image on the retina also is very neat. With any variation of accommodation it becomes vague. Indeed, a delicate object is not at all necessary to observe the variation of the image with varying accommodation. Provided the eye in question is not nearsighted, it is sufficient to watch the retinal image of a distant light while the patient accommodates for far or near vision in the same line of sight. With accommodation for far vision, the image of the distant light also appears distinct; but with accommodation for near vision, it gets hazy. Usually at the same time the details of the retina vanish also unless the observer, with the help of his own accommodation, can follow the new position of the image; and then he has to use another concave lens to be sure that a blurred image of the distant light is projected on the distinctly visible retina. The experiment may be varied too by causing the patient to fixate a distant object permanently,

while the light itself is brought near the eye, enabling the observer to verify that the image of the near light is blurred.

From the most ancient times the brilliant appearance in the dark of the eyes of certain animals, as, for example, dogs and cats, had been observed.[1] In many creatures a portion of the fundus of the eyeball, covered with thin fibres or lamellae, has no pigment and presents an iridescent appearance. This is called a tapetum, which under favourable circumstances is easily visible. An old idea that was widely prevalent was that such eyes were indeed self-luminous, the luminosity being developed by the animals themselves, especially when they were excited; and so there was a disposition to attribute this alleged development of light to the action of the nervous system. This ocular luminosity in the case of animals is most striking in a dark room when light coming from behind the observer goes close past his head to the eye of the animal; and for this very reason the actually incident light might often be hidden from the spectator. The eyes of white rabbits that are devoid of pigment and of albinos generally ought to shine in the same way by internal light. PREVOST[2] was the first to show that the so-called shining of animal eyes never occurred in complete darkness and is not produced either arbitrarily or by emotion, but is invariably due to reflection of incident light. GRUITHUISEN[3] found the same result independently and showed that the tapetum, together with a certain "extraordinary refraction" in the crystalline lens, is responsible for the phenomenon. He saw this glow even in the eyes of dead creatures. These facts were verified by RUDOLPHI,[4] J. MÜLLER,[5] ESSER,[6] TIDEMANN[7] and HASSENSTEIN.[8] It was RUDOLPHI who pointed out that it is necessary to look in the eye in a particular direction to get this luminous effect. ESSER gave the right explanation of the origin of the colour as being due to seeing variously coloured parts of the retina through the pupil. Lastly, HASSENSTEIN found that the glow occurred when the eye was compressed in the direction of its axis, and conjectured that it is also produced arbitrarily by living creatures who possess an ability of contracting the axis of the eye. Thus, the luminosity was admitted to be a reflex phenomenon, but the conditions of its shining or not were not made clear.

In the case of the human eye, the luminosity had not been noticed formerly except in rare kinds of disease, particularly in connection with swellings in the fundus of the eye. BEHR[9] observed it even when the iris was gone, and found that the observer had to look in the diseased eye in a direction almost parallel with the incident rays. This is the fundamental condition in BRÜCKE's method of observing the luminosity of the eye. The ocular glow when the iris is absent is striking, because the illumination of the retina is much greater in such cases; and, besides, the power of accommodation of the eye is lost.

Finally, W. CUMMING[10] and BRÜCKE[11], independently of each other, discovered the way of making healthy human eyes appear luminous, the observer looking almost parallel to the incident light. BRÜCKE had already used the

[1] See, for example, PLINY, Book XI, Chap. 55. (J. P. C. S.)

[2] *Biblioth. britannique.* 1810. T. 45.

[3] *Beiträge zur Physiognosie und Eautognosie.* S. 199.

[4] *Lehrbuch der Physiologie.* I. 197.

[5] *Zur vergleichenden Physiologie d. Gesichtssinns.* Leipzig 1826. S. 49. — *Handbuch d. Physiologie.* 4. Aufl. I. 89.

[6] KASTNERS *Archiv für die gesamte Naturlehre.* Bd. VIII. S. 399.

[7] *Lehrbuch der Physiologie.* S. 509.

[8] *De luce ex quorundam animalium oculis prodeunte atque de tapeto lucido.* Jena 1836.

[9] HECKERS *Annalen.* 1839. I. S. 373.

[10] *Medico-chirurgical Transactions.* XXIX. p. 284.

[11] J. MÜLLERS *Archiv für Anat. u. Physiologie.* 1847. S. 225.

same method previously in studying animal eyes that had a tapetum. WHARTON JONES[1] states that about the same time BABBAGE had showed him a silvered glass mirror with a bit of the silver removed, intended for throwing light into the eye and looking through the aperture. This device suggests immediately COCCIUS' ophthalmoscope; but as BABBAGE does not appear to have used any lenses in conjunction with his mirror, the most that can be said about his procedure is that he was able to obtain some notion of the parts of the retina in an unusual way. Probably, this is why he did not publish an account of his contrivance at the time.

The other aspect of the question, that is, why the parts of the retina, even when they are illuminated, as, for example, in animal eyes with a tapetum and in eyes of albinos, cannot be recognized by the observer, has frequently been discussed. As long ago as the beginning of the eighteenth century, MÉRY[2] noticed in the case of a cat immersed in water that the eyes were not only luminous, but the blood vessels of the retina could be discerned. LA HIRE[3] gave the correct explanation of this latter fact. He was aware that the refraction of the rays had to be altered in order to make the eye appear luminous, but he did not know how to give any more complete explanation. The same was true of KUSSMAUL,[4] who shows that the retina will be bright and recognizable, provided both the cornea and lens are removed, or provided some of the vitreous humor is taken out so as to shorten the axis of the eye.

So far as the author is aware, he was the first[5] to make clear the connection between the directions of the entering and outgoing rays, and to discover the real reason of the blackness of the pupil, and hence also the principle of the construction of the ophthalmoscope. For illumination he used plane unsilvered glass plates, and for studying the retina a concave lens. On the other hand, TH. RUETE was the first to use a perforated mirror and the method of observation with a convex lens. The new instrument quickly assumed extraordinary importance in ophthalmology, and consequently a great number of different types of ophthalmoscope were designed, the most characteristic of which have been described. But no essentially new principles for the illumination or investigation of the retina are contained in any of these constructions.

The theory of the illumination of the eye and of the ophthalmoscope as given above by the author has not been modified in any essential respect. The improvements which STELLWAG VON CARION has endeavoured to make in it are not improvements in the author's opinion. This ophthalmologist, sincerely desirous of employing the principles of physics in his own science, was led into errors by the fundamentally incorrect ideas as to the intensity of illumination and brightness that were current in previous works on these subjects.

1704. MÉRY in *Annales de l'Académie des sciences*. 1704.
1709. LA HIRE, ibid. 1709.
1810. PREVOST in *Bibliothèque britannique*. XLV.
GRUITHUISEN, *Beiträge zur Physiognosie und Eautognosie*. S. 199.
RUDOLPHI, *Physiologie*. I. 197.
1826. J. MÜLLER, *Zur vergleichenden Physiologie des Gesichtssinns*. Leipzig S. 49.
ESSER in KASTNERS *Archiv für die gesamte Naturlehre*. VIII. 399.

[1] *Archives générales de Médecine*. 1854. II.

[2] *Annales de l'Acad. d. sc.* 1704.

[3] Ibid. 1709.

[4] *Die Farbenerscheinungen im Grunde des menschlichen Auges*. Heidelberg 1845.

[5] H. HELMHOLTZ, *Beschreibung eines Augenspiegels zur Beobachtung der Netzhaut im lebenden Auge*. Berlin 1851. Also in VIERORDTS *Archiv für Physiol. Heilkunde*. II. 827. (See also English translation by T. H. SHASTID, Chicago, 1916.—J. P. C. S.)

1836. HASSENSTEIN, *De luce ex quorundam animalium oculis prodeunte atque de tapeto lucido.* Jenae.
1839. BEHR in HECKERS *Annalen.* Bd. 1. S. 373.
1844. E. BRÜCKE, Über die physiologische Bedeutung der stabförmigen Körperchen. J. MÜLLERS *Archiv für Anatomie und Physiologie.* 1833. S. 444.*
1845. E. BRÜCKE, Anatomische Untersuchung über die sogenannten leuchtenden Augen bei den Wirbeltieren. Ibid. 1847. S. 387.*
KUSSMAUL, *Die Farbenerscheinungen im Grunde des menschlichen Auges.* Heidelberg.
1846. W. CUMMING in *Medico-chirurgical Transactions.* XXIX. 284.
1847. E. BRÜCKE, Über das Leuchten der menschlichen Augen, in J. MÜLLERS *Archiv.* 1847. S. 225* u. 479.*
1851. H. HELMHOLTZ, *Beschreibung eines Augenspiegels zur Untersuchung der Netzhaut im lebenden Auge.* Berlin.
1852. TH. RUETE, *Der Augenspiegel und das Optometer.* Göttingen.
H. HELMHOLTZ, Über eine neue einfachste Form des Augenspiegels, in VIERORDTS *Archiv für physiologische Heilkunde.* II. 827.
FOLLIN in *Archives générales de Médecine.* 1852. July.
A. COCCIUS, *Über die Ernährungsweise der Hornhaut.* Leipzig.
FROEBELIUS, *Mediz. Zeitung Russlands.* 1852. No. 46.
1853 A COCCIUS, *Über die Anwendung des Augenspiegels nebst Angabe eines neuen Instruments.* Leipzig.*
A. C. VAN TRIGT, *Dissertatio de Speculo oculi.* Utrecht.—*Nederlandsch Lancet.* Ser. 3. Dl. II. 430. In German with notes by SCHAUENBURG.1854.
H. A. O. SAEMANN, *De speculo oculi.* Regiomonti.
R. ULRICH, Beschreibung eines neuen Augenspiegels in HENLE u. PFEUFFERS *Zeitschrift für rationelle Medizin.* Neue Folge. IV. 175.*
MEYERSTEIN, Beschreibung eines neuen Augenspiegels. Ibid. S. 310.
FOLLIN et NACHET, *Mém de la Société de Chirurgie.* 1853. III.
SPENCER WELLS, *Medical Times.* Sept. 1853.
1854. DONDERS, *Verbeteringen van den oogspiegel, in Onderzoekingen gedaan in het Physiologisch Laboratorium der Utrechtsche Hoogeschool.* Jaar VI. bl. 131* u. 153.*
ANAGNOSTAKIS, *Essai sur l'exploration de la ré ine et des milieux de l'oeil sur le vivant au moyen d'un nouvel ophtalmoscope.* Paris 1854. (A perforated concave mirror.) Also in *Annales d'oculistique.* Feb. and March 1854.
STELLWAG VON CARION, *Theorie der Augenspiegel.* Wien.*
G. A. LEONHARD, *De variis oculorum speculis illorumque usu.* Leipzig.
TH. RUETE, *Bildliche Darstellung der Krankheiten des menschlichen Auges.* Leipzig. Numbers 1 and 2; also under the title: *Physikalische Untersuchung des Auges* S. 23-27.*
W. ZEHENDER, Über die Beleuchtung des innern Auges mit spezieller Berücksichtigung eines nach eigener Angabe konstruierten Augenspiegels, in GRAEFES *Archiv für Ophthalmologie.* I. 1. S. 121.*
1855. LIEBREICH, ibid. I. 2. S. 348.
STELLWAG VON CARION, *Zeitschrift der Ärzte zu Wien.* XI. S. 65.*

Supplement

The form of ophthalmoscope that has finally been most generally adopted by ophthalmologists is one that resembles most the COCCIUS or ZEHENDER instrument described above, except that, instead of a plane or convex mirror in conjunction with a convex lens for illumination as used in those constructions, a concave mirror, 1 inch in diameter and of 5 or 6 inches focus, without any convex lens, has been substituted. The mirror is sometimes made of metal, having the advan-

tage of a better aperture with sharp, non-reflecting rim; or else it is silvered on glass with a hole in the centre. In the latter arrangement the surface of the mirror is better protected against injury. But it has a disadvantage, especially in the method of observation with erect image; because the edge between the reflecting surface and the aperture cannot be made as small and sharp as with a metal mirror.

COCCIUS' method of observing the fundus of one's own eye is described in the second part of this work. Any perforated mirror will do for this purpose, but a convex mirror is best. Another autophthalmoscope, by which the left eye examines the illuminated retina of the right eye, has been described by F. HEYMANN. Light enters the right eye through a hole in a plane mirror; and the left eye looking towards the hole in the mirror sees the reflected image of the right eye. In front of the right eye, as in RUETE's ophthalmoscope, a convex lens (of about 2 1/4 inches focus) is placed with its focal point in the pupil of that eye. This lens produces an inverted image of the retina at its focus. Behind this image there is a reflecting rectangular prism which deflects the rays towards the perforated mirror. Two other convex lenses, one between prism and mirror, and the other in front of the left eye, form a kind of little bent telescope; through which the left eye sees the image of the retina of the right eye, and which also makes it impossible for both eyes to be accommodated for the hole in the mirror at the same time. In order to vary the position of the place on the retina that is under observation, HEYMANN inserts also in front of the observing eye a prism spectacle glass of variable power, whose refracting edge can be turned in different directions.

GIRAUD TEULON's binocular ophthalmoscope is described in the third part of this book.

1855. E. JAEGER, *Beiträge zur Pathologie des Auges mit Abbildungen in Farbendruck.* Wien.
— Idem, Ergebnisse der Untersuchung des menschlichen Auges mit dem Augenspiegel. *Wien. Ber.* XV. 319-344.
1856. CASTERANI, Ophthalmoscope. *Cosmos.* VIII. 612.
— W. ZEHENDER, Über die Beleuchtung des innern Auges durch heterozentrische Glasspiegel. *Archiv. für Ophthalm.* II. 2 S. 103-130.
1857. J. PORRO, La lunette panfocale, employée comme ophthalmoscope. *C. R.* XLV. 103 to 104. *Cosmos.* XI. 96-97.
— A. BUROW, Über Konstruktion heterozentrischer Augenspiegel und deren Anwendung. *Archiv für Ophthalm.* III. 2, S. 68-80.
— SCHNELLER, Ein Mikrometer am Augenspiegel. Ibid. III. 2. S. 121-186.
R. LIEBREICH, *De l'examen de l'oeil au moyen de l'ophthalmoscope.* Bruxelles (From MACKENZIE's translation of the *Traité pratique des maladies des yeux.*)
1859. A. ZANDER, *Der Augenspiegel, seine Formen und sein Gebrauch.* Leipzig and Heidelberg.
1861. O. BECKER, Über die Wahrnehmung eines Reflexbildes im eigenem Auge. *Wiener Med. Wochenschrift* 1860. S. 670-672; 684-688. (Image of posterior surface of lens reflected in cornea.)

1863. BUROW Jr., Notiz betreffend die Beobachtung des eigenen Augenhintergrundes. *Archiv für Ophthalmol.* IX. 1 S. 155-160.
1863. F. HEYMANN, *Die Autoskopie des Auges.* Leipzig.
— R. LIEBREICH, *Atlas der Ophthalmoskopie.* Berlin. Hirschwald.
1864. C. SCHWEIGGER, *Vorlesungen über den Gebrauch des Augenspiegels.* Berlin.
— A. COCCIUS, Beschreibung eines Okulars zum Augenspiegel. *Archiv für Ophthalm.* X. 1. S. 123-147.
— R. SCHIRMER, Über das ophthalmoskopische Bild der Macula lutea. Ibid. X. 1. S. 148-151.
— WINTRICH, Über die Benutzung des zweckmässig abgeblendeten zerstreuten Tageslichts zur Oto-, Ophthalmo- und Laryngoskopie. *Erlanger Medic. Neuigkeiten.* 1864. 9 April.

Note by A. Gullstrand

"Die von mir aufgestellte Theorie des Augenleuchtens und der Augenspiel hat keine wesentlichen Veränderungen erfahren." These words of HELMHOLTZ are still true today. Undoubtedly, the construction of the instrument has been essentially improved and its region of application essentially extended. The ophthalmologist who uses the ophthalmoscope every day not only for the most subtle diagnoses of diseased conditions of the fundus of the eye, but also for investigating the ocular media, and, in various ways, for finding the refraction of the eye, knows best how to estimate HELMHOLTZ's immortal service to mankind.

This is not the place to describe these methods and their results; but the solution of one problem may be mentioned briefly which could not have been even proposed at the time when the ophthalmoscope was invented. The *photography of the fundus of the eye*, which, thanks to the sensitiveness of modern dry plates, is no longer impracticable, affords useful results at present which we owe chiefly to the work of DIMMER[1] in this field. The main difficulty to be overcome was to get rid of the corneal reflex of the source of light. Since one part of the pupil is used for the light that illuminates the fundus of the eye and another part for the passage of the light that is diffusely reflected from the fundus, the external incident light that is reflected from the cornea may be suitably intercepted. On similar principles THORNER[2] has constructed a stationary reflex-free ophthalmoscope, and WOLFF[3] has constructed a portable electric ophthalmoscope. Both of them have photographed the fundus of the eye.

[1] FR. DIMMER, *Die Photographie des Augenhintergrundes.* Wiesbaden 1907.
[2] W. THORNER, *Die Theorie des Augenspiegels und die Photographie des Augenhintergrundes.* Berlin 1903.
[3] H. WOLFF, Zur Photographie des menschlichen Augenhintergrundes. *Arch. für Augenheilk.* LIX. S. 115. 1908.

Appendices to Part I

by

A. Gullstrand

I. Optical Imagery

When HELMHOLTZ's original contributions were published on the Dioptrics of the Eye, the characteristic conception of optical imagery that was current at that time assumed that rays of light emanating at a point were reassembled approximately at another point, and that thus a certain point of the image was associated with a definite point of the object. This conception continues to be very widespread at the present time, although it cannot be reconciled with actual facts as now known. It was well understood that in no single instance were rays of light emanating from a point on the axis of a centered optical system completely reunited at one point, but that an aberration occurs or an error, depending partly on the different refrangibility of rays of different wave-lengths, the so-called chromatic aberration, but manifesting itself also in case of monochromatic light, and known then as the spherical aberration because it depends on the form of the refracting surface and because formerly only spherical surfaces were used. Students of optics were aware also that a point of an object which was at a considerable distance from the axis of the instrument was not reproduced as a point, since the corresponding bundle of refracted rays is astigmatic; and they had the formulae for calculating this astigmatism. The constitution of an oblique bundle of refracted rays and its lack of homocentricity continued, however, to be just as unknown as the general laws of optical imagery themselves; since they took account only of those rays proceeding from a point of the object which were in the meridian plane and in a plane perpendicular to this plane, although such rays constitute an exceedingly small portion of the total rays concerned in the optical effect. Thus finding it impossible to investigate the actual imagery in this way, and without knowledge of the general laws, they invented a system, whereby ABBE, for example, omitting all reference to the actual facts in the case of refraction of light, sought to ascertain the purely mathematical conditions of the

imagery of one space in another in such manner that every point and every straight line in one region were reproduced by a corresponding point and a corresponding straight line in the other region.[1] These conditions were in harmony with the known laws of optical imagery in the case of an infinitely small object situated on the axis of a centered system provided with an infinitely narrow stop. The theory of collinear imagery applied to objects of finite extent and stops with finite apertures, which is the basis of the expositions still to be found in modern text-books, constitutes therefore an essentially arbitrary extension of the region of validity of these laws, inasmuch as a system of fictions had to be introduced in place of the real undiscovered laws. Of course, the realities can be represented as aberrations or deviations from the ideal relations of collinear correspondence provided we remember always that the ideal cannot be realized; but in making use of fictions there is the continual and serious danger of overstepping the border between what is true and what is nearly true. This one thing alone might show the advantage of a direct study of the facts themselves as soon as such a study had become possible through the ascertainment of the laws in question.

The reason why the laws of actual optical imagery have been, so to speak, summoned to life by the requirements of physiological optics is due partly to the fact that by means of trigonometrical calculations, tedious to be sure but easy to perform, it has been possible for the optical engineer to get closer to the realities of his problem. Thus, thanks to the labours of such men as ABBE and his school, technical optics has attained its present splendid development; whereas with the scientific means available a comprehensive grasp of the intricate relations in the case of the imagery in the eye has been actually impossible.

The necessary reorganization of the theory of optical imagery could not be interwoven with the most recent investigations because the latter had been worked out principally in two dimensions, whereas the general imagery-problem is three-dimensional. Ever since the general investigations of STURM[2] and HAMILTON,[3] it sufficed to determine the constitution of a bundle of rays in terms of magnitudes of the first and second order, without, however, keeping in mind the

[1] See J. C. MAXWELL, On the General Laws of Optical Instruments. *Quart. Journ. Pure and Appl. Math.*, II, 1858, 233-246; also MAXWELL's Scientific Papers, I., 271-285. (J. P. C. S.)

[2] CH. STURM, Mémoire sur l'optique. *Journ. de Math. pures et appliquées.* 1838. — Mémoire sur la théorie de la vision. *Comptes rendus de l'Acad. des sc.* t. XX. 1845.

[3] W. R. HAMILTON, Theory of systems of rays. *Transactions of the Royal Ir. Acad.*, vol. XV. 1828. *Supplements* in vol. XVI. Part 1. 1830; vol. XVI. Part 2. 1831; vol. XVII. 1837.

conditions of validity of the laws that were ascertained by use of these magnitudes. STURM discovered that all the rays of a bundle go approximately through two mutually perpendicular focal lines, provided the aperture of the bundle is infinitesimal as compared with the distance of the focal lines from the stop and from each other. But in oblique incidence of light in optical instruments the latter condition is not even approximately the case, and in the case of the eye the so-called astigmatic difference is as a rule less than the diameter of the pupil; so that as a matter of fact the notion of these focal lines is fundamentally false. Hence, to begin with, it was necessary to make a thorough study of a bundle of optical rays in general and to discuss the special cases in detail taking account of magnitudes of the third[1] and fourth[2] orders, and not merely of the first and second orders, as was customary heretofore; and in place of the notion of the focal lines to substitute a knowledge of the precise geometrical magnitudes that determine the form of the caustic surface together with the position, dimensions and form of the narrowest section of the bundle of rays. In case of the peculiar complicated structure of a bundle of rays refracted into the eye, a special investigation was necessary of the so-called circular points (umbilics)[3] of the refracting surfaces and the corresponding bundles of normals. However, the aberration in the eye as ascertained by a study carried out in this way was found to be so great, and the blur circles on the retina even with the best focusing turned out to be so large, that the role in the imagery assigned to the narrowest cross section of the bundle cannot possibly be correct. This can easily be shown by taking a double convex lens of wide aperture and using it to focus on a screen the image of the filament of an incandescent lamp. The sharpest image of the filament is obtained when the lens is focused so that the image appears to be surrounded by a veil; whereas the image is decidedly inferior when the lens is focused so that the veil disappears, although in the latter case the image is produced by using the narrowest section of the ray-bundle. This shows that the light-distribution inside the section of the ray-bundle is of primary importance, whereas the dimensions of the section are of secondary importance. In place of the approximate measures of the dimensions of the narrowest cross sections as found by series-development, it was all the more necessary to substitute the precise geometrical magnitudes that determine the form of the caustic surface; because it is the section of this surface that controls the distribution of the light

[1] A. GULLSTRAND, Beitrag zur Theorie des Astigmatismus. *Skand. Arch. f. Physiol.* Bd. II. 1890. S. 269.

[2] Idem, Allgemeine Theorie der monochromatischen Aberrationen und ihre nächsten Ergebnisse für die Ophthalmologie. *Nova Acta Reg. Soc. Sc. Ups. Ser.* III. 1890.

[3] A. GULLSTRAND, Zur Kenntnis der Kreispunkte. *Acta Mathematica.* 29. 1904.

within the section of the ray-bundle. Another point necessarily connected with the matter just mentioned involves a revision of the conception of the function of the stop. The main business of the stop, according to the old view, was to cut out the rays, and hence it had to be considered as infinitely narrow, which is a fiction with no reality at all; because with an infinitely narrow stop diffraction-phenomena would completely mask the image-appearances. Least of all can this fiction be employed in the case of the eye, where it becomes ridiculous in consequence of the actual size of the stop. Inasmuch as the problem requires us to postulate a stop of arbitrary size, the only exact means of investigating the effect of the stop is by the method of optical projection through the centre of the stop.

These were the main considerations that led to the investigation of optical imagery[1] by ascertaining the fundamental equation from which the general laws thereof are derived. Since the medium of the crystalline lens of the eye is not homogeneous (or isotropic), the laws of optical imagery in heterogeneous media[2] had finally to be studied also before the optical imagery in the eye could be made a subject of thorough investigation

At present the writer will endeavour to give here only such portions of the theory of optical imagery as are required to comprehend the imagery in the eye, but in order not to obscure the results by mathematical intricacies the analytical processes will be entirely omitted. Those details can be found in the contributions to which reference has been made and, so far as some of the simpler questions are concerned, partly also in other articles[3] where the writer has taken pains to make the subject as clear as possible for such readers as are not acquainted with infinitesimal or differential geometry.

General Laws. In the mathematical theory of the laws of optical imagery two experimental facts are necessary and sufficient, namely, the rectilinear propagation of light in isotropic media and the general law of refraction. From the latter, by mathematical deduction, it can be proved, that through each point on any ray of an originally homocentric bundle of rays a surface can be constructed which cuts all the

[1] Idem, Die reelle optische Abbildung. *Kungl. Sv. Vet. Akad. Handl.* Bd. XLI. No. 3. 1906.

[2] Idem, Die optische Abbildung in heterogenen Medien und die Dioptrik der Kristallinse des Menschen. Ibid. Bd. XLIII. No. 2. 1908.

[3] Die Konstitution des im Auge gebrochenen Strahlenbündels. *Arch. f. Ophth.* LIII, 2. 1901. S. 105 — Über Astigmatismus, Koma und Aberration. *Ann. d. Physik.* 4. Folge. 18. 1905. S. 941. — Tatsachen und Fiktionen in der Lehre von der optischen Abbildung. *Arch. f. Optik.* I. 1907. S. 2.

rays orthogonally, and also that the optical length along any ray between two such surfaces is the same for all the rays of the bundle. If the lengths of the different portions of the ray-path in each medium are multiplied by the corresponding values of the index of refraction, what is meant by the optical length along the ray is the sum of these products.[1] In this form the general law of refraction is found to be valid also for anisotropic or "heterogeneous" media, where we have curved paths of light, so-called trajectories, instead of rays; and where the optical length is expressed as a definite integral. In either case the surfaces which are drawn at right angles to the directions of propagation of the light are commonly called wave surfaces, whose normals in the case of isotropic media are identical with the rays of light, but in anisotropic media of continuously variable index are tangent to the trajectories along which the light is propagated. In general, therefore, the investigation of the convergence of the rays amounts to an investigation of the constitution of a bundle of normals to the wave surface, which in turn depends on the form of the surface in question. However, except in a few simple and practically unimportant cases these surfaces cannot be represented by useful algebraic equations, and thus it becomes necessary to investigate the surface in the immediate vicinity of some selected point, and the bundle of rays in immediate proximity to a certain special ray. The bundle of rays which emanates from a point of the object contains one ray which in the stop-space goes through the centre of the stop, and which is therefore called the *chief ray* of the bundle. Thus, the totality of these chief rays, each proceeding from one point of the object, constitutes a homocentric (or monocentric) bundle of rays in the medium where the stop is, which behaves exactly as if the light radiated from the centre of the stop. Precisely as we have to investigate a bundle of rays in the immediate vicinity of a selected ray, so also the imagery of an object can be studied only in the immediate vicinity of a selected object-point. The chief ray belonging to this selected object-point is called the *central ray* or *guide-ray*. The laws of optical imagery are found by investigating the central bundle of object-rays and the bundle of chief rays in the immediate vicinity of the guide-ray, and also by investigating the bundle of contiguous object-rays in the immediate vicinity of the chief rays in question; and are called the *laws of the first order* or of a higher order according as they are obtained by one or more derivations (or differentiations) from the general law of refraction. By reason of the complexity of the problem only the laws of the first order are applicable in the general case, so that when

[1] ¶See Volume II, §19. (J. P. C. S.)

the object is an extended one the formulae in question have to be applied to as many arbitrarily selected guide-rays as are necessary in order to obtain a correct idea of the imagery that is involved.

The laws of the first order in regard to convergence of rays are derived from the *general constitution of a bundle of optical rays* by employing STURM's formulae; which enable us to find the magnitudes that determine the wave surface of the bundle of refracted rays, provided the refracting surface and the bundle of incident rays are given. A characteristic property of a bundle of optical rays considered as normals to the wave surface is, that, in general, any given ray of the bundle will be intersected by contiguous rays in one point (which is a singular or special case) or else in two separate points (which is the rule). Thus along any particular ray chosen at random, the bundle of rays is ordinarily *astigmatic* and has therefore two *focal points* (or image points)[1]; whereas in case the two focal points are identical, the bundle of rays is said to be *anastigmatic* along this ray. Designating one of these points as the *primary focal point*, we may speak of the plane containing the contiguous rays that meet the given ray in this point as the *primary principal section* of the bundle of rays with respect to the ray in question; and of the line perpendicular to the primary principal section at the primary focal point as the *primary focal line*. Similarly, contiguous rays lying in the secondary principal section of the bundle with respect to the given ray, that is, in the section perpendicular to the primary principal section, will meet this ray in the *secondary focal point*; and the *secondary focal line* will be a line perpendicular to the *secondary principal section* at the secondary focal point. Thus in a bundle of rays which is astigmatic with respect to a certain ray the contiguous rays that meet this ray are all contained in one or other of the two principal sections; whereas if the astigmatism vanishes along this ray, so that the two focal points are coincident, all the contiguous rays (neglecting magnitudes of order higher than the first) will intersect the ray in question.

[1] ¶The only satisfactory rendering of the German word "*Fokalpunkte*" is "focal point," as meaning simply a point of intersection of a lot of rays of a bundle. Thus, any pair of so-called "conjugate" points is in this sense a pair of "focal" points. But "the focal points of an optical system" (for which the Germans have the special word *Brennpunkten*) are not even a pair of conjugate points. If we call these latter (as is done to some extent in the translation) the "principal focal points" (*Hauptbrennpunkten*), there is again a possibility of confusion with the "principal points" (*Hauptpunkten*) of the system; which, however, could be avoided by calling them the "unit points," as some English writers do. The reader must be on his guard to distinguish between all these terms, as employed in the text; and also between similar expressions, as, for example, "focal distance" (*Fokalabstand*) and "focal length" (*Brennweite*). The term "focal planes" may be used to mean simply a pair of conjugate planes or to refer to the principal focal planes of the system; etc. (J. P. C. S.)

As a rule, both the focal points and the principal sections vary from one ray to another, and the loci of the first and second focal points are two surfaces called the first and second *caustic surfaces*, respectively; sometimes referred to also as the two sheets of the caustic surface. As a result of this structure of the ray-bundle, a convergence of all the rays at one point is to be regarded as a singular event, so that as a rule the idea of *ray-convergence is concerned only with contiguous rays and occurs only on the caustic surfaces.* If a line is drawn on a surface which cuts a bundle of rays, the totality of rays which meet this line constitutes what is called a *ray-surface* (or ruled surface). The way to study such a surface is along some particular ray just as in the study of a bundle of rays; and it follows that the plane containing this ray which is tangent to the ruled surface either revolves continuously through 180° as we proceed along the ray from infinity in one direction to infinity in the other, or else the tangent-plane remains stationary. In the former case the ruled surface is called a *skew surface* or *"scroll"* with respect to the given ray; and the tangent-plane at the first focal point coincides with the second principal section with respect to the ray, and *vice versa;* and therefore the two focal lines of any given ray will ordinarily be tangent to every skew surface that contains this ray. However, when the tangent-plane to a ruled surface remains the same at every point along the ray, this plane coincides with one of the principal sections with respect to this ray; and contiguous rays of the ruled surface (neglecting magnitudes of order higher than the first) will intersect the ray at its focal points. Thus, since the ruled surface has a focal point along the given ray, it is what may be called a *focal ruled surface* (or a *developable surface* or "torse") with respect to this ray. At the focal point itself the tangent-plane is indeterminate, and therefore every plane containing the given ray is tangent to the ruled surface; and hence also the two focal lines of the ray will in general be tangent to the developable surfaces that contain the ray in question. If a bundle of rays is anastigmatic along a certain ray, every ruled surface containing this ray will be developable or focal with respect to it; and any arbitrary pair of mutually rectangular lines perpendicular to the ray at its focal point may be regarded as the focal lines. Thus, it follows that the general constitution of a real bundle of optical rays is defined by the fact that *the focal lines of any ray, at right angles to each other and also to the ray, are tangent to every ruled surface containing that ray.* Inasmuch as the so-called *focal plane* perpendicular to the given ray at one of its focal points will intersect a ruled surface in a line which may have any curvature, it is obvious at once that in order for this line to be considered as the focal line at this place, the stop-opening must be dimin-

ished until the line in question becomes so short that it is indistinguishable from a piece of any other curved line. The general constitution of a bundle of optical rays may be defined also by the fact that the two focal lines with respect to any given ray are intersected by all the contiguous rays (disregarding infinitesimal aberrations of order higher than the first); but this statement requires an explicit definition of what is meant by a contiguous ray. Draw the ray corresponding to a point on the ruled surface at a finite distance from the given ray, and do the same thing for points nearer and nearer the given ray; then ultimately the distance between the point and the given ray will vanish entirely, and the ray corresponding to this point will be contiguous to the given ray. But this definition will be fundamentally misunderstood if the conclusion is deduced that all the rays of an infinitely narrow bundle go approximately through two STURM focal lines.

Since (neglecting infinitesimals of order higher than the first) all the contiguous rays of a bundle of rays which is anastigmatic along a certain ray meet this ray in the focal point, there occurs in this case a *perfect ray-convergence of the first order*; and the focal point is the optical reproduction or image of the point where the light originates. Hence, in case of a stop of finite aperture, the optical imagery cannot be said to be due to the fact that all the rays coming from an object-point go approximately through the image-point, but merely to the fact that contiguous rays intersect at this point, so that the concentration of the light here is infinitely great as compared with that at a point at a finite distance from this spot. The imagery supposed in the first case above is indeed mathematically correct for an infinitely narrow bundle of rays, but since it is physically impossible on account of diffraction effects to produce an optical image by such means, this imagery represents merely an ideal dreamed of in olden times; whereas the latter case mentioned above describes exactly the actual process as it takes place. If the reality here appears to be little different from the ideal, the explanation is to be found in the circumstance that both in the eye and likewise on the photographic plate shades of brightness are of more importance than absolute brightness.

The criterion of actual *optical imagery* is just this perfect ray-convergence of the first order. As is evident from the foregoing, this convergence of the rays is susceptible of mathematical investigation only along definite rays. The rays chosen for this purpose are the chief rays defined above. Assuming that the chief ray corresponding to each point of the surface of the object has been constructed and traced through the optical system by trigonometrical calculation, and that somewhere in the path of the light a screen is

interposed, we shall obtain on it a *punctual correspondence* with the object by means of *optical projection*, since every point on the screen where a chief ray arrives is the optical projection of the point of the object where this ray originates, and the centre of the stop is to be regarded as the centre of projection. Thus, the optical projection is a mathematical conception which, however, may be illustrated physically by means of any optical system with a very brilliant object, provided the screen is adjusted so that the image is not sharply focused on it; since to every bright point of the object corresponds a bright spot of light on the screen, and as the opening of the stop is made smaller and smaller, the image will become less and less hazy. A special case of general optical projection is realized in the pinhole camera; wherein, generally speaking, every point where a chief ray meets the surface of a screen adjusted arbitrarily is the optical projection of the corresponding point of the object; and every section in which the screen cuts a ruled surface composed of chief rays is the optical projection of a corresponding line on the surface of the object; such that the ratio of the lengths of corresponding line-elements, in case they are both perpendicular to the chief ray in question, represents the so-called *coefficient of linear projection.* Similarly, an optical projection can be produced by means of a bundle of rays emanating from a point of an object. This can be shown physically by stretching a wire at any part of an optical system of wide aperture in the path of a beam of light proceeding from a luminous point, and observing the shadow that is formed on a screen set up in one of the media. If the wire is placed in the medium where the point-source itself is situated and revolved around the chief ray until its shadow on the screen is tangent to one of the principal sections of the bundle along the chief ray, the ruled surface indicated by the wire in this position is a focal or developable surface with respect to the chief ray in both the object-space and the screen-space; since every ruled surface in the object-space is to be considered here as having a focal point coinciding with the luminous point itself. In this particular case the coefficient of linear projection can be replaced by the *coefficient of angular projection*; which is the ratio of the infinitely small angles subtended at the focal points of the developable surfaces by an element of the line projected on the screen and the corresponding element of the wire.

Since anastigmatic convergence of the rays occurs generally only along singular (or extraordinary) chief rays, a punctual optical imagery, characterized by the fact that the separate points of the surface of an object are reproduced as points in perfect ray-convergence of the first order, is for a fixed position of the centre of the stop a mathematical impossibility; and if there is really any such thing as a general optical

imagery, it is simply a question as to the imagery of single lines. *The condition of the optical imagery of lines requires that (neglecting infinitesimals of order higher than the first) all rays contiguous to the chief rays proceeding from the various points of the object-line shall meet the optical projection of this line.* Since in this imagery there is no other punctual correspondence except that obtained by optical projection, and since the rays emanating from a point of the object-line meet the corresponding image-line in different points, obviously the ratio of the lengths of corresponding elements of these lines depends simply on the optical projection; whereas the *magnification-ratio* of the optical imagery is the ratio of the lengths of the corresponding elements of the lines which meet the image-line and object-line perpendicularly; that is, the ratio of the orthogonal trajectories of image-line and object-line, assuming that object-surface and image-surface are both perpendicular to the chief ray. Thus, the magnification-ratio is identical with the coefficient of linear projection of the orthogonal trajectories, supposing these latter could in general be projected into each other.

The fact that there is a general imagery of lines as above explained is proved by the writer's fundamental equation of optical imagery; and the laws thereof result as a consequence. These *general laws of optical imagery*, except for the case of grazing incidence of the chief rays, and also except when there are peaks and edges at the points of incidence, apply without reservation to any optical system with isotropic media of either constant or continuously variable index of refraction; and may be formulated as follows:

At every point where the chief ray makes a finite angle with the surface of an object there are two intersecting lines on the surface, inclined to each other at a finite angle, which will be reproduced in the image-space by perfect ray-convergence of the first order, the image-surface being different for each system of lines.

The tangents to the imageable lines lie entirely in the planes normal to those ruled surfaces which in the image-space are in contact with the principal sections of the bundle of refracted rays. The tangents to the image-lines themselves lie in the same principal sections taken in opposite order.

There is no other imagery with perfect ray-convergence of the first order. The only points of the system of imageable lines that are reproduced by points are the singular points. Singular points in these systems occur only when the bundle of rays after refraction in the image-space is anastigmatic along the chief ray; in which case the two image-surfaces have a point of contact.

The magnification can be expressed only by means of the ratio of the distances of the image-lines to the distances of the corresponding lines on

the surface of the object. However, at the singular points the first term of the magnification is given by the ratio of the length of an element of the image-line to that of the corresponding object-line; and in case of a line composed of singular points the magnification is represented by the ratio of the length of the image-line to that of the object-line.

The product obtained by multiplying the relative index of refraction of the optical system either by the coefficient of angular projection or by the magnification-ratio is invariably equal to unity.

The magnification-ratio of each point and the directions of the tangents to the imageable lines are independent of the position of the stop on the chief ray.

As a rule imageries cannot be combined or compounded. If a different medium is chosen for the image-space, the imageable lines will be altered thereby. The same thing happens when the distance between the object-point and the optical system is varied.

Imageries are absolutely reversible. When the procedure of the rays is reversed, the original image-lines represent one system of imageable lines lying on the surface in question, whereas the tangents to the corresponding original imageable lines lie in the respective principal sections of the bundle of refracted rays.

The optical imagery of the character thus described is a mathematical conception, postulating a fixed position of the centre of the stop and monochromatic light. In the application of the theory to actual physical combinations, it should be borne in mind therefore that, strictly speaking, any point in the plane of the stop can be taken at pleasure as centre of the stop, which will be discussed more in detail presently as to its important practical bearings. In case of compound light the *chromatic differences* also have to be taken into account besides. The latter concern not simply the magnitudes obtained by trigonometrical calculation that determine the path of the guide-ray, but also the positions of the image-lines, their orientation, and that of the imageable lines, and, finally, also the magnification-ratios; all these magnitudes, according to the circumstances of the case, being calculated for different kinds of light. STURM's formulae, which have been extended to include certain singular cases hitherto left out of account, are of service here, and in addition certain other formulae derived by the writer.

If along a guide-ray the plane of incidence is a principal normal plane of the refracting surface, and each successive plane of incidence is either identical with the preceding plane of refraction or perpendicular to it, the imageries can be compounded, because the tangents to the imageable lines and to the image-lines lie everywhere in the plane of refraction and in the plane containing the guide-ray which is perpen-

dicular to this plane, respectively. The so-called *singly asymmetrical systems* characterized by the existence of a plane of symmetry represent a special case under this head that is of practical importance. Although in the general case, where there is no such plane, and the optical system is said therefore to be *doubly asymmetrical*, only the imagery-laws of the first order are applicable, complete laws of the second order have been deduced by the writer in case of the systems first named. Moreover, if the guide-ray is the line of intersection of two planes of symmetry of the optical system, the latter is said to be a *symmetrical system*; and under such circumstances we know also the laws of the third order along the guide-ray which were found originally by SEYDEL in the special case of a centered system consisting of spherical surfaces. Moreover, in singly asymmetrical systems, and even to a greater extent in symmetrical systems, the laws of the first order are considerably simplified. Again, if the optical system is comprised of surfaces of revolution having a common axis which passes through the centre of the stop, and if the surface of the object is likewise symmetrical with respect to the axis, or if it is a plane perpendicular to the axis, the case is that of a *system of revolution*. Systems of this description, of which a centered system of spherical surfaces is a particular case, have a practical importance beyond all others; for which also the imagery-laws become much simplified. Thus since the plane determined by the chief ray and the principal axis is always a plane of symmetry, one pair of corresponding lines of object and image must always be in this plane and the other pair perpendicular to it; and, consequently, image-lines and imageable lines are always parallel circles and meridians; and moreover the imageries can be compounded along every chief ray.

The practical bearing of this mode of classifying the systems will be apparent on comparing the imagery in the case of an ordinary magnifying glass and a bicylindrical lens. In the latter case the laws of a symmetrical system are applicable along the axis, and of a singly asymmetrical system along a guide-ray lying in one of the two planes of symmetry; whereas for any other guide-ray only the general laws in case of double asymmetry are valid. But in case of an ordinary magnifying glass, supposing that the pupil of the eye is on the axis of the lens, the special laws for a system of revolution are applicable along the axis, whereas along any other chief ray the laws are those of a singly asymmetrical system with certain special simplifications due to the properties of surfaces of revolution.

Optical Imagery in a System of Revolution. The simpler laws that apply to an optical system which is symmetrical around an axis may be obtained by elementary methods and suffice for an explanation of the

imagery in the case of the eye. The primary principal section of a bundle of rays emanating from a point of the object is the *meridian plane* determined by the chief ray and the axis of revolution; whereas the secondary principal section is the *equatorial plane* perpendicular to the former and also containing the chief ray. Since the ruled surfaces which are tangent to these planes in the object-space have this same relation in each successive medium, and are therefore focal throughout the system, the bundle of refracted rays may be ascertained by investigating these developable surfaces. The fact that also the entire image-process can be discovered by the study of these two ruled surfaces, follows at once from the general laws of optical imagery given above; for without them all that is really known in this case is the effect of the rays which happen to lie in these two ruled surfaces, and which constitute an exceedingly small portion of the total quantity of rays that are concerned in the imagery.

In Fig. 114 the straight line AC is supposed to be the prolongation of the chief ray which is incident at the point A, whereas BC is the prolongation of another ray lying in the meridian plane and belonging to the same bundle; and AO and BO are the corresponding normals to the refracting surface. The angles of incidence CAO and CBO are denoted by i and $i+\omega$ respectively; and therefore

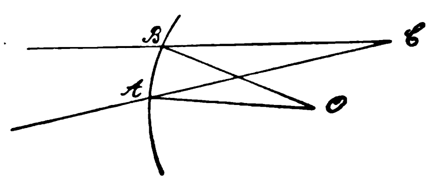

Fig. 114.

$$\angle\, AOB+i=\angle\, ACB+i+\omega,$$

and hence

$$\omega=\angle\, AOB-\angle\, ACB.$$

Now if the point B is taken closer and closer to the point A until BC represents a ray contiguous to the chief ray, AC becomes finally the primary focal distance τ of the bundle of incident rays measured from A, and AO the radius $\rho_{\prime}$ of the primary principal curvature of the refracting surface, and $\angle AOB=\frac{AB}{\rho_{\prime}}$, $\angle ACB=\frac{AB\cos i}{\tau}$. Thus when AB is infinitely small,

$$\frac{\omega}{AB}=\frac{1}{\rho_{\prime}}-\frac{\cos i}{\tau},$$

and, similarly, if i',ω',τ' have analogous meanings for the bundle of refracted rays,

$$\frac{\omega'}{AB}=\frac{1}{\rho_{\prime}}-\frac{\cos i'}{\tau'}.$$

Since

$$\sin(i+\omega)=\sin i\cos\omega+\sin\omega\cos i,$$

and since the angle ω becomes infinitely small along with the arc AB, so that ultimately $\cos\omega = 1$ and $\sin\omega = \omega$, after multiplying both sides by the value n of the index of refraction of the first medium and at the same time dividing by AB, the identity above may be written in the following form:

$$\frac{n \sin (i+\omega) - n \sin i}{AB} = n \cos i \frac{\omega}{AB} = n \cos i \left(\frac{1}{\rho'} - \frac{\cos i}{\tau}\right).$$

Similarly, if n' denotes the index of refraction of the second medium,

$$\frac{n' \sin (i'+\omega') - n' \sin i'}{AB} = n' \cos i' \left(\frac{1}{\rho'} - \frac{\cos i'}{\tau'}\right).$$

Since, according to the law of refraction, the left-hand members of these two equations are equal to each other, we obtain:

$$\frac{n' \cos^2 i'}{\tau'} = \frac{n \cos^2 i}{\tau} + \frac{n' \cos i' - n \cos i}{\rho'} \quad . \quad . \quad . \quad . \quad \mathrm{A}_1$$

If the case is one of reflection instead of refraction, precisely the same method can be used provided we put $n' = -n$, since then also the left-hand members of the two identities above will still be equal to each other. In both cases the magnitudes denoted by τ, τ' and ρ' are always to be reckoned positive in one and the same direction from the point of incidence.

Again, in Fig. 115 suppose that B designates the position of the secondary focal point of the bundle of incident rays, represented here as lying on the prolongation of the chief ray; and let O in this diagram mark the position of the centre of the secondary principal curvature of the refracting surface situated on the normal drawn to the surface through the incidence-point A; and suppose the entire figure is revolved around the line OB as axis.

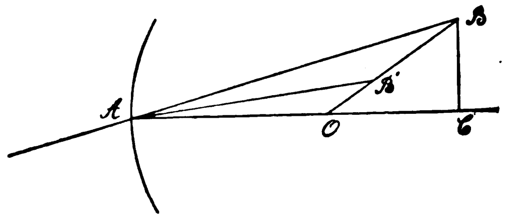

Fig. 115.

As this rotation starts, the point A is carried to an adjacent point of the refracting surface lying in the equatorial plane, and the normal to the surface at this point will be contiguous to the normal AO and will therefore meet it in the point O. Similarly, the ray which falls on the surface here will be contiguous to the chief ray AB and will meet it in the point B. But since a ray in the equatorial plane contiguous to the refracted chief ray must be contained in the same plane as the corresponding incident ray and normal, it must intersect the line OB at a point B', which is therefore the secondary focal point of the bundle of rays situated on the refracted chief ray AB'.

Accordingly, if $\mathfrak{s},\mathfrak{s}'$ denote the distances of the secondary focal points before and after refraction, both measured from the incidence-point A, and if $\rho_{\prime\prime}$ denotes the radius of the secondary principal curvature of the refracting surface with respect to this same point, and finally, if BC is drawn perpendicular to the normal AO, we may write:

$$\frac{BC}{OC}=\frac{\mathfrak{s}\sin i}{\mathfrak{s}\cos i-\rho_{\prime\prime}}=\tan\angle BOC$$

and consequently

$$\frac{n\sin i}{\rho_{\prime\prime}\tan\angle BOC}=\frac{n\cos i}{\rho_{\prime\prime}}-\frac{n}{\mathfrak{s}}.$$

Since a similar equation is obtained for the bundle of refracted rays, and since, according to the law of refraction, the left-hand members of the two equations are equal to each other, we obtain finally

$$\frac{n'}{\mathfrak{s}'}=\frac{n}{\mathfrak{s}}+\frac{n'\cos i'-n\cos i}{\rho_{\prime\prime}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathrm{A}_2$$

The same remarks that were made above with respect to formula A_1 concerning the special case of reflection and the positive directions of the linear magnitudes apply here also. As is obvious from the way they were derived, both formulae are valid not merely for a system of revolution but in any case where one of the principal sections of the bundle of incident rays and a principal normal section of the refracting or reflecting surface coincide with the plane of incidence. Ordinarily as the formulae show, the bundle of refracted rays is astigmatic, and hence in a system of revolution, except under special circumstances, the two image-surfaces are tangent to each other only at the place where the axis meets them, and consequently the axial object-point is the only point that is reproduced by a point.

For this latter point $\cos i=\cos i'=1$ and $\rho_{\prime}=\rho_{\prime\prime}$ and accordingly the two formulæ A reduce to the first of formulae (3), § 9, in the HELMHOLTZ text; where the focal distances for the incident and refracted rays are measured in opposite directions conformably to the old custom, the radius of curvature of the refracting surface being reckoned always as positive.

However, for any point of the object not on the axis *two different imageries* are to be taken into account; and therefore we may designate the imagery of the parallel of latitude determined by the primary focal point of the bundle of refracted rays as the *primary imagery*.

The magnification-ratios corresponding to the two imageries may be obtained as follows: Let p,q, p',q' denote the distances from the incidence-point of the primary and secondary focal points of the bundle of incident chief rays and the corresponding bundle of refracted

rays, respectively. In Fig. 116 the straight line ACD is supposed to represent the prolongation of one of the incident chief rays; whereon the points C and D designate the primary focal points of the bundle of chief rays and the bundle of rays belonging to the given point of the object, respectively. Draw DE perpendicular to the chief ray AD, and mark on it a point E, and suppose that BE is the chief ray which goes through E. Draw AF perpendicular to ACD and meeting BCE in the point designated by F. Then

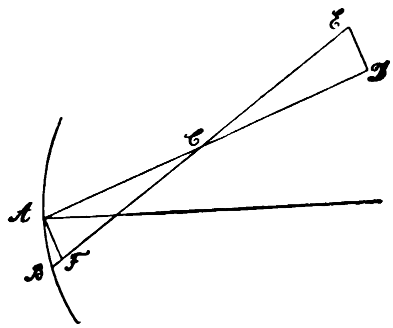

Fig. 116.

$$\frac{AF}{DE}=\frac{AC}{CD}=\frac{p}{\tau-p}.$$

Now if the point E is taken infinitely near the focal point D, so that BE and AD are two contiguous chief rays, the point B may be considered as lying on the tangent to the refracting surface, and hence the angle BAF will be equal to the angle of incidence. If the length DE is denoted by β, then

$$AB=\frac{\beta p}{(\tau-p)\cos i},$$

and hence, multiplying both sides by $n\cos^2 i\left(\frac{1}{\tau}-\frac{1}{p}\right)$, we may write:

$$AB\cdot n\cos^2 i\left(\frac{1}{\tau}-\frac{1}{p}\right)=-\frac{n\cos i\,\beta}{\tau}.$$

An analogous relation connecting the bundle of refracted chief rays and the bundle of refracted rays corresponding to the given point of the object may be obtained in precisely the same way; and by applying formula A_1 to each of the two bundles of rays, it will be seen that the left-hand sides of the two equations are equal; whence it follows that

$$K_1=\frac{n\cos i\,\tau'}{n'\cos i'\,\tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad B_1$$

where the symbol K_1 denotes the *primary magnification-ratio.*

Similarly, for the *secondary magnification-ratio,* we shall find:

$$K_2=\frac{ns'}{n's} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad B_2$$

since, in case Fig. 116 represents the equatorial section, the chief ray of the bundle of incident rays would have to coincide with the normal to the section of the refracting surface made by the equatorial plane,

and instead of multiplying by the expression given above, the factor here would be

$$n\left(\frac{1}{\mathfrak{s}}-\frac{1}{q}\right).$$

In the particular case when $\tau=\tau'=0$, in Fig. 116 $AF=\beta$ and therefore $K_1=\dfrac{\cos i'}{\cos i}$. For $\mathfrak{s}=\mathfrak{s}'=0$, it is evident at once that $K_2=1$.

Since formulae A are addition-formulae containing the products of the reciprocals of the focal distances and the corresponding values of the index of refraction, it is convenient to reckon directly in terms of these functions. Thus, the quotient of the focal distance from any point by the index of refraction is called the *reduced focal distance* with respect to that point. The focal distance is reckoned positive when we proceed from the origin of the measurement to the focal point in the direction that is called the positive direction; and the reciprocal of this distance, which is therefore equal to one of the principal curvatures of the wave surface, is a measure of the convergence of the bundle of rays in the given principal section at the point taken as origin. Hence, the value obtained by multiplying the reciprocal of the focal distance by the index of refraction is called the *reduced convergence*. Employing this idea, we may write equations A and B for both imageries as follows:

$$\kappa^2 B = A + \kappa D, \qquad \kappa K B = A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{C}$$

Here the magnitudes denoted by A and B are the reduced convergences of the bundles of incident and refracted rays, respectively; i.e., $A=\dfrac{n}{\tau}$, $B=\dfrac{n'}{\tau'}$, or $A=\dfrac{n}{\mathfrak{s}}$, $B=\dfrac{n'}{\mathfrak{s}'}$, according as the formulae relate to the primary or secondary imagery, respectively. The magnitude denoted by κ is the value of the magnification-ratio at the point of incidence and is equal to $\dfrac{\cos i'}{\cos i}$ or to 1; whereas K denotes the primary or secondary magnification-ratio at the given focal point. The magnitude denoted by D is the *refracting power* of the surface at the point of incidence and is given by one of the following expressions, namely:

$$\frac{n'\cos i' - n\cos i}{\rho'\cos i'\cos i} \quad \text{or} \quad \frac{n'\cos i' - n\cos i}{\rho''},$$

according as it refers to the case of the primary or secondary imagery, respectively.

It is easy to prove that *formulae* C *are likewise true when the reduced convergences are measured from any pair of points which are conjugate to each other with respect to the imagery in question.* If κ_1 denotes the value of the magnification-ratio for this pair of conjugate points, and if the

reduced convergences at these points measured from the point of incidence are denoted by A_0, B_0, then, besides equations C, we have also the pair of similar equations:

$$\kappa^2 B_0 = A_0 + \kappa D, \qquad \kappa\kappa_1 B_0 = A_0$$

and by subtraction we obtain:

$$\kappa^2 (B_0 - B) = A_0 - A.$$

Moreover, if A_1, B_1 denote the reduced convergences at this same pair of points for the bundle of rays determined by A, B, then

$$\frac{1}{B_1} = \frac{1}{B} - \frac{1}{B_0}, \qquad \frac{1}{A_1} = \frac{1}{A} - \frac{1}{A_0},$$

and therefore also

$$B_1 = \frac{BB_0}{B_0 - B}, \qquad A_1 = \frac{AA_0}{A_0 - A} = \frac{\kappa KB . \kappa\kappa_1 B_0}{\kappa^2 (B_0 - B)}$$

consequently,

$$\kappa_1 K B_1 = A_1.$$

On the other hand, eliminating A or A_0, we obtain

$$\kappa - K = \frac{D}{B}, \qquad \kappa - \kappa_1 = \frac{D}{B_0},$$

respectively; and by subtraction

$$\kappa_1 - K = D\left(\frac{1}{B} - \frac{1}{B_0}\right) = \frac{D}{B_1},$$

wherein, by substituting the value $\dfrac{A_1}{\kappa_1 B_1}$ for K, the following result is obtained:

$$\kappa_1^2 B_1 = A_1 + \kappa_1 D.$$

In order to find the resultant imagery of one of the two imageable systems of lines that is produced by two successive refractions, suppose, to begin with, that the positions have been determined of each pair of focal points in the second and third media corresponding to a point of the object in the first medium; and let κ_1, κ_2 denote the magnification-ratios for the imagery of the given object-lines in the second medium and for the imagery of these image-lines in the third medium, respectively; and, finally, let D_1, D_2 denote the refracting powers of the two surfaces with respect to the given imagery. Then if A, B denote the reduced convergences of any bundle of rays measured at the given object-point in the first medium and at the corresponding focal point in the last medium, respectively, the following pair of values will be found for the reduced convergence at the conjugate point in the second medium:

$$\frac{A}{\kappa_1^2}+\frac{D_1}{\kappa_1}=\kappa_2^2\,B-\kappa_2 D_2,$$

whence, by writing

$$\kappa=\kappa_1\kappa_2,\qquad D=\frac{D_1}{\kappa_2}+\kappa_1 D_2$$

the general equation of the focal distances for the combination of the imageries by two refractions assumes the same form as in case of a single refraction, namely

$$\kappa^2 B=A+\kappa D.$$

Moreover, if K_1, K_2 denote the magnification-ratios for the two surfaces at the focal points of the bundle of rays, the reduced convergence measured in the second medium is given also by the following pair of values:

$$\kappa_2 K_2 B=\frac{A}{\kappa_1 K_1}$$

and hence putting $K=K_1\,K_2$, we obtain:

$$\kappa K B=A.$$

This procedure can be repeated indefinitely. In general, therefore, if for the pair of conjugate points at which the reduced convergences are measured the magnification-ratio with respect to the imagery due to the first n surfaces is denoted by k_n, and if the refracting power of the optical system composed of the first n surfaces is denoted by $\mathfrak{D}_n$, then

$$k_n=\kappa_1\kappa_2\kappa_3\;.\;.\;.\;\kappa_n=\Pi\kappa,$$

and the formula found above for the refracting power of two surfaces may be written in the following form:

$$k_2\mathfrak{D}_2-k_1\mathfrak{D}_1=\frac{k_2^2 D_2}{\kappa_2}.$$

In the case of n surfaces, therefore, we obtain by summation:

$$k_n\mathfrak{D}_n=\frac{k_1^2 D_1}{\kappa_1}+\frac{k_2^2 D_2}{\kappa_2}+\;.\;.\;.\;+\frac{k_n^2 D_n}{\kappa_n}=\sum\frac{k^2 D}{\kappa}.$$

The range of validity of the general image-equations,

$$\kappa^2 B=A+\kappa D,\qquad \kappa K B=A$$

which are applicable along any chief ray in a system of revolution may therefore be extended to any number of media, provided κ denotes the magnification-ratio for a pair of points conjugate to each other with respect to the given imagery, A and B the reduced convergences of a bundle of rays with respect to these points, and K the magnifica-

tion-ratio with respect to the focal points determined by A and B, and provided also that for the compound system consisting of n components

$$D=\frac{1}{\kappa}\sum\frac{k^2D}{\kappa}$$

in which this sum contains one term for each component system.

The angular magnification may be also found by means of these formulae. This term denotes the ratio of the angles subtended at two conjugate points by the distance between a pair of image-lines and the distance between the corresponding pair of imageable lines. Consequently, if the indices of refraction are denoted by n, n', and the elementary arcs of the orthogonal trajectories of the lines which are imaged in each other are denoted by β,β', the angular magnification-ratio with respect to the pair of points, for which the lateral magnification-ratio (as it is called in order to distinguish it from the other) has the value κ, is given by the ratio of the two angles, $\frac{\beta' B}{n'}$ and $\frac{\beta A}{n}$ and hence is equal to $\frac{nKB}{n'A}$, which is the same as $\frac{n}{n'\kappa}$. The continued product of the angular magnification-ratio, the lateral magnification-ratio and the relative index of refraction is therefore invariably equal to unity. Just as we employ the term reduced distance, it is convenient to introduce here the idea of the *reduced angular magnification-ratio as being equal to* $\frac{KB}{A}$, *i.e., equal to the reciprocal of the lateral magnification-ratio.*

Putting $B=0$ in the general image-equations, we find that the reduced distance of the *first principal focal point* is equal to $-\frac{1}{\kappa D}$; and similarly, putting $A=0$, we get for the reduced distance of the *second principal focal point* $\frac{\kappa}{D}$. Again, putting $K=1$, we find for the reduced distances of the *first and second principal points*[1] $\frac{\kappa-1}{\kappa D}$ and $\frac{\kappa-1}{D}$, respectively. Accordingly, by subtracting the latter values from the former, the reduced distances of the first principal focal point from the first principal point and of the second principal focal point from the second principal point are found to be $-\frac{1}{D}$ and $\frac{1}{D}$, respectively. Thus,

[1] ¶It would seem advantageous to call the principal points in this case the "unit points" ($K=1$), as is done by some English writers. (J. P. C. S.)

the *refracting power* may be defined, perfectly generally, as *the reciprocal of the quotient of the second principal focal length by the corresponding index of refraction.*

When *the image-equations are referred to the principal points*, they take the form

$$B=A+D, \qquad KB=A,$$

which expressed in words are equivalent to the following statements: *When rays of light traverse an optical system, the value of the reduced convergence increases by an amount equal to the refracting power of the system*; and *the lateral magnification-ratio is equal to the quotient of the reduced distances of the focal point from the principal point for the corresponding bundles of refracted and incident rays*; whereas the reduced angular magnification-ratio at the principal points themselves is equal to unity. These latter formulae are just as general as the others, and have the advantage of exhibiting clearly the essence of optical imagery; whereas, on the other hand, they have the disadvantage of necessitating a troublesome transformation in applying them to a so-called telescopic or afocal system (which is a better term), characterized by the condition $D=0$; while the general image-equations are applicable in the form in which they are given.

If in the image-equations referred to the principal points the condition is imposed of equal (unreduced) convergences of the corresponding bundles of incident and refracted rays, the result is that

$$\frac{B}{n'}=\frac{A}{n}=\frac{D}{n'-n'}, \qquad K=\frac{n}{n'}$$

and the focal points of the bundle are in this case identical with LISTING's nodal points; the (unreduced) angular magnification at these points being equal to unity. But since the properties ascribed to the nodal points have no reality for rays of finite inclinations except for a system consisting of a single spherical surface, as is too obvious to need explanation, the image-equations referred to these points are of little service, because by means of the reduced angular magnification-ratio the image-equations referred to the principal points secure precisely the advantages that the nodal-point equations are supposed to possess.

On the other hand, in numerous instances the *image-equations referred to the principal focal points* are of real utility. These equations may be obtained either directly from the expressions found above for the distances of the principal focal points from any pair of conjugate points or as follows: If, first B, and then A, is eliminated from the general image-equations, the latter may be expressed in a form which is very useful for certain problems, namely:

$$D=A\left(\frac{1}{K}-\frac{1}{\kappa}\right)=B(\kappa-K),$$

These formulae are applicable, for example, when one of the pair of conjugate points at which the magnification-ratio is equal to κ is situated at infinity, that is, when either $\kappa=0$ or $\frac{1}{\kappa}=0$. Thus, if the reduced convergences of a bundle of rays measured at the first and second principal focal points of an optical system are denoted by L and L', respectively, the following equations are found:

$$KD=L, \qquad KL'=-D.$$

Consequently,

$$LL'=-D^2, \qquad \frac{L}{L'}=-K^2,$$

where K, as usual, denotes the magnification-ratio at the focal points of the bundles of rays whose reduced convergences are L, L'.

If in case of the *combination of two systems* it is desired to find the refracting power $\mathfrak{D}$ of the compound system directly and the positions of the principal points without pursuing the method given above, the necessary formulae for this purpose can be obtained as follows: Suppose the first system is defined by its refracting power D_1 and a pair of conjugate points for which the magnification-ratio is κ_1; and also the second system is defined similarly by its refracting power D_2 and a pair of conjugate points for which the magnification-ratio is κ_2; and let the reduced distance (that is, the distance divided by the corresponding index of refraction) of the first one of the last mentioned pair of points from the second one of the pair named first be denoted by δ. Moreover, let A_1,B_1,K_1 and A_2,B_2,K_2 denote the reduced convergences of a bundle of rays and the magnification-ratios for the first and second members of the system, respectively. Then by the definition given above the refracting power of the compound system is equal to $\frac{D_1}{K_2}+K_1D_2$. By eliminating A_1 and B_2 from the general image-equations we get

$$K_1=\kappa_1-\frac{D_1}{B_1}, \qquad \frac{1}{K_2}=\frac{1}{\kappa_2}+\frac{D_2}{A_2}$$

and since

$$\frac{1}{B_1}-\frac{1}{A_2}=\delta,$$

we find:

$$\mathfrak{D}=\frac{D_1}{K_2}+K_1D_2=\frac{D_1}{\kappa_2}+\kappa_1D_2-\delta D_1D_2.$$

This expression enables us to compute the refracting power of the compound system in terms of the magnitudes that define the two

component systems, and, for $\delta = 0$, it reduces to the expression which was obtained previously for this condition. Let the reduced distances of the first and second principal points (or "unit points") of the compound system as measured from the first one of the pair of conjugate points of the first system and from the second one of the pair of conjugate points of the second system be denoted by H, H', that is, put $H = \frac{1}{A_1}$, $H' = \frac{1}{B_2}$. We have here the condition that $K_1K_2 = 1$ and hence

$$\kappa_1 - \frac{D_1}{B_1} = \frac{1}{\kappa_2} + \frac{D_2}{A_2}.$$

If we substitute on the left-hand side of this equation the expression for A_2 in terms of B_1, and on the right-hand side the expression for B_1 in terms of A_2, the equation may be transformed as follows:

$$B_1\left(\delta D_2 + \kappa_1 - \frac{1}{\kappa_2}\right) = D_1 + D_2 = -A_2\left(\delta D_1 + \frac{1}{\kappa_2} - \kappa_1\right).$$

Then by substituting the expression for B_1 in terms of A_1 and that for A_2 in terms of B_2, the following result may be found:

$$\frac{A_1}{\kappa_1}\left(\delta D_2 + \kappa_1 - \frac{1}{\kappa_2}\right) = \mathfrak{D} = -\kappa_2 B_2\left(\delta D_1 + \frac{1}{\kappa_2} - \kappa_1\right).$$

Hence, *the general formulae for the combination of two systems are*:

$$\begin{aligned} \mathfrak{D} &= \frac{D_1}{\kappa_2} + \kappa_1 D_2 - \delta D_1 D_2, \\ H &= \frac{1}{\kappa_1 \mathfrak{D}}\left(\delta D_2 + \kappa_1 - \frac{1}{\kappa_2}\right), \\ H' &= -\frac{\kappa_2}{\mathfrak{D}}\left(\delta D_1 + \frac{1}{\kappa_2} - \kappa_1\right). \end{aligned}$$

In case the *two component systems are referred to their* principal points (or "unit points"), by putting $\kappa_1 = \kappa_2 = 1$, we obtain:

$$\mathfrak{D} = D_1 + D_2 - \delta D_1 D_2 \qquad H = \frac{\delta D_2}{\mathfrak{D}} \qquad H' = -\frac{\delta D_1}{\mathfrak{D}}.$$

If *one of the component systems is afocal*, then $D_1 = 0$ or $D_2 = 0$, according as it is the first or second system, respectively; and if the other system is referred to its principal points (or "unit points"), that is, if $\kappa_2 = 1$ or $\kappa_1 = 1$, then either

$$\mathfrak{D} = \kappa_1 D_2, \qquad H = \frac{\delta}{\kappa_1^2} + \frac{\kappa_1 - 1}{\kappa_1^2 D_2}, \qquad H' = \frac{\kappa_1 - 1}{\kappa_1 D_2}$$

or

$$\mathfrak{D} = \frac{D_1}{\kappa_2}, \qquad H = \frac{\kappa_2 - 1}{D_1}, \qquad H' = -\kappa_2^2\delta + \frac{\kappa_2(\kappa_2 - 1)}{D_1}.$$

And, finally, if *both component systems are afocal*, by substituting in the expression for δ the value of B_1 in terms of A_1 and that of A_2 in terms of B_2, we find:

$$\frac{1}{\kappa_1\kappa_2 B_2}=\frac{\kappa_1\kappa_2}{A_1}-\frac{\kappa_2\delta}{\kappa_1}$$

whereas the magnification-ratio has everywhere the constant value

$$K=\kappa_1\kappa_2.$$

For example, in the case of a plane parallel plate or of a prism traversed by the chief ray along the path of minimum deviation, for each of the two imageries $\kappa_1\kappa_2=1$, and consequently

$$\frac{1}{B_2}=\frac{1}{A_1}-\frac{\delta}{\kappa_1^2}.$$

If the plate or prism is surrounded by air and if the angles of incidence and refraction at the first surface are denoted by i,i', then for the bundle of emergent rays:

$$\frac{1}{B_2}=\frac{1}{A_1}-\frac{\delta\cos^2 i}{\cos^2 i'} \quad \text{or} \quad \frac{1}{B_2}=\frac{1}{A_1}-\delta.$$

Thus, if the plane of refraction is assumed to be the primary principal section, and if the light were originally homocentric, the distance of the first focal point from the second will be

$$\delta\cdot\frac{\cos^2 i'-\cos^2 i}{\cos^2 i'}.$$

This focal segment, being therefore independent of the position of the radiant point-source, is a measure of the actual astigmatism of the system. Evidently, it is negligible only in case the path of light inside the plate or prism is vanishingly small as compared with the distance of the source.

In physiological optics it is convenient to have available formulae for the *combination of three optical systems*; which, by using the above general methods, may be obtained as follows: Let D_1, D_2, D_3 denote the refracting powers of the component systems and $\mathfrak{D}$ that of the compound system; and let δ_1,δ_2 denote the reduced distances of the first principal ("unit") point of one system from the second principal ("unit") point of the preceding system. In the first place, let us determine the positions of the points conjugate to the principal ("unit") points of the second system in the first and last media. We have:

$$A_1+D_1=\frac{1}{\delta_1}, \qquad B_3=D_3-\frac{1}{\delta_2},$$

$$\frac{\kappa_1}{\delta^1}=A_1, \qquad \kappa_3 B_3=-\frac{1}{\delta_2}$$

and moreover

$$\kappa_1 = 1-\delta_1 D_1, \qquad \kappa_2 = 1, \qquad \kappa_3 = \frac{1}{1-\delta_2 D_3};$$

whence we obtain the following general formula for the refracting power:

$$\mathfrak{D} = D_1(1-\delta_2 D_3) + D_2(1-\delta_1 D_1)\,(1-\delta_2 D_3) + D_3(1-\delta_1 D)_1.$$

The reduced distances of the principal ("unit") points of the compound system measured from the two conjugate points found in the first and last media are, as was proved above,

$$\frac{\kappa-1}{\kappa\mathfrak{D}} \quad \text{or} \quad \frac{\kappa-1}{\mathfrak{D}},$$

where κ is written in place of $\kappa_1\kappa_3$; and hence for the reduced distances H, H' of the principal ("unit") points of the compound system from the first principal ("unit") point of the first system and from the second principal ("unit") point of the second system, respectively, we obtain finally:

$$H = \frac{1}{A_1} + \frac{\kappa-1}{\kappa\mathfrak{D}} = \frac{\delta_1}{1-\delta_1 D_1} + \frac{\delta_2 D_3 - \delta_1 D_1}{\mathfrak{D}(1-\delta_1 D_1)}$$

$$H' = \frac{1}{B_3} + \frac{\kappa-1}{\mathfrak{D}} = -\frac{\delta_2}{1-\delta_2 D_3} + \frac{\delta_2 D_3 - \delta_1 D_1}{\mathfrak{D}(1-\delta_2 D_3)}.$$

If the compound system is symmetrical with respect to the middle component, then, since $\delta_2 = \delta_1$ and $D_3 = D_1$, we have:

$$\mathfrak{D} = 2D_1(1-\delta_1 D_1) + D_2(1-\delta_1 D_1)^2,$$

$$H = -H' = \frac{\delta_1}{1-\delta_1 D_1}.$$

So far as *choice of signs* is concerned, the image-equations for the case of a single surface were derived on the general assumption that the distances of the focal points and centre of curvature of the refracting or reflecting surface from the point of incidence were counted positive in one and the same direction, and that a positive sign for the magnification-ratio means "erect" (or "sympathetic") imagery. Moreover, since in case of reflection, the sign of the index of refraction has to be reversed, and in compound systems the positive direction must be consistent throughout, it is best to define this direction with reference to the way the light travels in the object-space. If we choose here the direction which corresponds to the motion of the light, then since at every reflection both the positive direction with respect to the motion of the light and the sign of the index of refraction are reversed, a positive sign of the reduced convergence implies that the rays in that case are convergent. A positive sign of the refracting power means that the

system acts as a convergent system. The reduced distance from one point to another will always be positive provided in passing from the former to the latter the direction of the light is followed. Hence, we have the *following general rules.* That direction anywhere along the chief ray is always the positive direction which corresponds to the way the light goes in the object-space. The indices of refraction of the media which are traversed by the light after an odd number of reflections are to be reckoned as negative. The distance from one point to another is positive if we proceed from the first point to the second by going in the positive direction; and the radius of curvature is the distance measured from a point of the surface to the centre of curvature. Finally, the magnification-ratio is positive when, looking along the given chief ray in the positive direction, we see the lines imaged in each other lying on one and the same side of this ray.

The reduced convergence and refracting power may be measured in terms of any suitable unit. But in ophthalmology the *dioptry*[1] has become established as the unit of the refracting power of a lens; being defined as the refracting power of a lens surrounded by air whose focal length is one metre. Accordingly, it is advisable to adopt this unit generally, and therefore to define it as follows: *The dioptry is the unit of the reciprocal value of a length* (*the principal focal length or conjugate focal distance*) *divided by the corresponding index of refraction; the distance in question being expressed in terms of the metre as the unit of length.*[2]

The simple and convenient mode of exhibiting all the laws of imagery in the case of a system of revolution in a form that is applicable along any chief ray, which as a special case may coincide with the axis of revolution itself, was made possible by introducing the ideas of reduced convergence and refracting power. However, it can readily be seen that the general image-equations, in spite of their different form, are identical with equations (7) and (7d) as given in § 9 in the original text of this book; while equations (7b) and (7c) are easily recognizable also in the form used above. On the other hand, in the above discussion no special importance was attached to the equations referred to the nodal points. The reason of this was because, as already stated, so far as rays of finite slope are concerned, the essential characteristic of the nodal points has no reality except for the case of a single spherical surface. In accordance with the view that

[1] ¶Spelled also *dioptre* (French; also an English usage) and *diopter* (American). The origin of the word is the same as that of *dioptrics*. Essentially, the *dioptry* is simply a unit of curvature, that is, it is merely a geometrical magnitude. (J. P. C. S.)

[2] A. GULLSTRAND, Über die Bedeutung der Dioptrie. *Arch. f. Ophth.* Bd. XLIX, 1. S. 46. 1899. It should be noted that in this paper the coefficient k used for afocal systems denotes the reduced angular magnification-ratio.

the imagery of an object of finite dimensions might be considered as approximately similar to that of the central portion, the geometrical constructions may enable us to visualize the relations of the conjugate points to each other and to the cardinal points. But if we insist on sticking to what is actually the case, this method is no longer justifiable, because the laws on which the construction depends are not valid except for the fiction of rays which are inclined at infinitesimal angles, and therefore such constructions often create false notions. Neither the nodal points nor the principal ("unit") planes or principal focal planes, whose characteristic properties are likewise due to the fiction of rays inclined at infinitesimal angles, afford any advantage in showing what actually takes place, and so these ideas also have been cast aside here as so much useless ballast. The same is true in regard to the geometrical constructions alluded to, and even the diagrams (Figs. 114-116) employed above in obtaining the image-equations for the case of a single surface were intended merely to spare the reader the otherwise unavoidable differential calculus.

There is no fiction anywhere at all in the preceding treatment. It is always a question of precise geometrical magnitudes and, so far as the magnification is concerned, of coefficients and ratios expressly defined as limiting values. This is the distinction that must be made between the idea of reduced convergence and the ideas of the optical divergence of rays and the optical inclination of a ray,[1] which have no validity except for the fiction of rays inclined at infinitesimal angles.

The opposite way was pursued by ABBE and his school[2] in their treatment of the laws of optical imagery, for they proceeded to derive the theory of collinear imagery from simple geometrical assumptions without reference to the phenomena of refraction and reflection of light. But since we do not have collinear imagery except for the fiction of infinitely narrow bundles of rays and for an exceedingly small element of the surface of the object on the axis of a system of revolution, whereas new fictions have to be introduced along other chief rays, collinear imagery is found to be simply an unattainable ideal when once the general laws of imagery have been ascertained. If it is alleged that in many modern optical instruments the reality does not fall very far short of this ideal, the explanation is due to an accumulation of mathematical singularities in the construction. We must be on our guard against supposing that it is possible to attain this ideal exactly, and certainly we have no right at all to assume that it represents the kind of imagery that occurs in the case of the eye.

[1] H. v. HELMHOLTZ, *Handbuch der physiologischen Optik*, 2. Auflage, Hamburg and Leipzig 1896. S. 66, 71.

[2] S. CZAPSKI, *Grundzüge der Theorie der optischen Instrumente nach* ABBE. 2. Auflage, Leipzig 1904.

In *applying the imagery-laws* it must be kept steadily in view that all that they tell us is the positions of the focal points and the magnification-ratios along the chief ray under consideration. Therefore, in *investigating the image formed on a screen* the effects due to an arbitrary choice of the chief ray need to be specially considered. As part of the system of revolution, the surface of the screen itself must either be a transversal plane perpendicular to the axis, as is generally the case, or a surface of revolution around the same axis as that of the instrument; and, moreover, only such rays coming from points of the object as cross the axis can act as chief rays. In case the image-point corresponding to the axial object-point lies on the screen, the two image-surfaces will be tangent to the screen at that point. Along a chief ray going through the centre of the stop, which proceeds from a point of the object at a finite distance but not far from the axis, the bundle of rays will be astigmatic. But if the size of the stop is finite, and provided the ratio between the distance of the object-point from the axis and the diameter of the stop does not exceed a certain magnitude, the same bundle of rays will be anastigmatic along another ray going through the stop. The anastigmatic image-points arising in this way and corresponding to object-points which are not far from the axis lie on a surface which is tangent to the screen at its axial point. Practically, we never have to deal with the image of a mathematical point; and hence within a certain finite region surrounding the axial image-point, whose extent depends on the size of the stop, the image on the screen is practically indistinguishable from that which would be due to an actual point-to-point reproduction. Evidently, the extent of this region will be greater and greater in proportion as the curvatures of the two image-surfaces are more and more nearly alike and coincide more nearly with the curvature of the screen. Now when we investigate a bundle of rays proceeding from a point which is immediately adjacent to the part of the surface of the object that corresponds to this region, it appears that it cannot have an image-point on the screen unless it lies along a ray which meets the axis but passes eccentrically through the stop; and therefore only one of the two systems of imageable lines on the surface of the object will be reproduced in the corresponding zone of the screen. In case of a plane screen, generally it is the system of meridian lines that is reproduced in this zone by positive optical systems, because in an ordinary optical system of spherical surfaces the concave sides of the image-surfaces are generally turned towards the instrument, the primary image-surface being nearer the system than the secondary. But in the part of the screen beyond this zone there is no image-point along any ray at all, so that here the image is formed simply by optical projection.

In accordance with the process just described, it is easy to show, *e.g.* by using a simple photographic landscape lens, that in the central region of the ground-glass screen focused to get the sharpest image of the axial point of the object, all the points and lines of the object will be reproduced with equal sharpness; whereas in a surrounding zone lines of the object in a meridian plane show up better than others; and beyond this region the definition is about like that which can be obtained with a pinhole camera.

When we investigate in this way the image of a figure consisting of radiating (meridian) lines and parallel (concentric) circles, centered with respect to the axis and placed at right angles thereto, it appears that for a small stop the position of a circle in the image will vary with the position of the stop on the axis, and that if the object-circles are at equal distances apart, they will not be equally spaced in the image. This experiment proves that the magnification-ratio along the axis, which is independent of the position of the stop, does not generally give the magnification of the image, but simply the limiting value which it approaches more and more nearly as the size of the object is steadily diminished. Hence the image of the system of parallel circles is not similar to the object; whereas, by virtue of the characteristic properties of a system of revolution, the meridian lines do give an image that is similar to the object, even when the screen is moved out of focus, so that the entire image is formed by optical projection. As the size of the object is diminished, the lack of similarity in the image of the parallel circles becomes less and less, until it disappears entirely at the instant when the object collapses into a point on the axis. Owing to the degree of accuracy demanded by the investigation, it is always necessary to calculate trigonometrically a greater or less number of chief rays in order to ascertain where they cross the screen; and then the image-equations will enable us to find the focal points where the chief rays intersect the two image-surfaces. The values of the secondary magnification-ratios can usually be found directly from the position of the secondary focal point, because, owing to the similarity between object and image in case of the meridian lines, this ratio is equal to the ratio of the distance of the focal point from the axis to that of the object-point. But the values of the primary magnification-ratios have to be obtained from the image-equations. However, since the focal points generally will not lie on the screen, linear projection coefficients have to be employed to investigate the image. Now in a system of revolution not only the imageable lines and the corresponding image-lines but their orthogonal trajectories coincide with the meridian lines and parallel circles, so that these trajectories can be projected into each other also. Let us therefore distinguish here

between the *primary linear projection coefficient* for projection of the meridian lines and the *secondary linear projection coefficient* for projection of the parallel circles. Just as in the case of the secondary magnification-ratio, the secondary linear projection coefficient may be defined as the ratio between the distances from the axis of the points where the given chief ray meets the screen and the surface of the object. But the primary linear projection coefficient is found as follows: Let δ' denote the reduced distance of the screen from the first focal point as measured along the chief ray; and let $\mathfrak{B}$ denote the reduced convergence of the bundle of chief rays at this point. Then the distance of an adjacent chief ray from the focal point is to its distance from the point where the first ray crosses the screen in the same ratio as the distances of the two points from the point of intersection of the two rays are to each other, *i.e.*, as $\frac{1}{\mathfrak{B}}\left(\frac{1}{\mathfrak{B}}-\delta'\right)$, and therefore $1-\delta'\mathfrak{B}$ is the linear projection coëfficient at the two points. Consequently, the *primary linear projection coefficient* for the corresponding object-point is

$$C_1=K_1(1-\delta'\mathfrak{B});$$

and a similar formula defines the second of these coefficients. The application of this formula implies that the bundle of chief rays has been traced into the image-space, and that the ordinary image-equations have been employed for this purpose. The actual linear magnification in the projection depends on the inclinations of the surfaces of the object and screen to the chief ray. If w, w' denote the angles between the chief ray and the normals to these surfaces, the limiting value of this magnification is equal to $C_1\frac{\cos w}{\cos w'}$. For the projection of parallel circles this limiting value is equal to the secondary linear projection coefficient.

The imagery-laws of the first order here treated apply also in general to singly asymmetrical systems. However, in such systems the magnification-ratios and the linear projection coefficients must be calculated in the same way as in case of the primary magnification-ratio and primary linear projection coefficient in a system of revolution. The plane of symmetry, which in a singly asymmetrical system is not a meridian plane at all, is called the *tangential plane*; and the plane perpendicular to it is called the *sagittal plane*. The formulae for the imagery along the axis in *symmetrical systems* may be found from the formulae for singly asymmetrical systems by putting the cosine of the

angles of incidence and refraction equal to unity throughout. In order to distinguish the imageries in the two planes of symmetry, one of these planes is called arbitrarily the primary plane and the other the secondary plane. In a system of revolution the equations obtained in this way for the imagery along the axis are independent of the orientation of the plane of symmetry.

Imagery-Laws of Higher Order. The imagery laws of *second order* completely developed for singly asymmetrical systems lead to formulae which, for example, in the case of systems of revolution, enable us to ascertain the inclinations of the image-surfaces along a chief ray and the asymmetry-value of the first magnification-ratio. The latter magnitude tells how much this ratio varies from one chief ray to another adjacent chief ray. Other formulae are expressions for the *asymmetry-values* of the bundle of rays and afford information as to degree of convergence of the rays. The general, doubly asymmetrical bundle of rays is defined by four magnitudes of this kind; whereas the singly asymmetrical bundle of rays which occurs in systems of revolution is characterized by two asymmetry values.

Proceeding from one ray to the next in the plane of symmetry of a singly asymmetrical bundle of rays, we find that the primary focal points (or image points) all lie on the section of the primary caustic surface that envelops the rays. This section which may be called the τ-curve is the line ${}_{\tau}F_{\prime}{}_{\tau}$ in Fig. 117 where $F_{\prime}, F_{\prime\prime}$ designate the two focal points lying on the ray $OF_{\prime}F_{\prime\prime}$. The rays which proceed in the

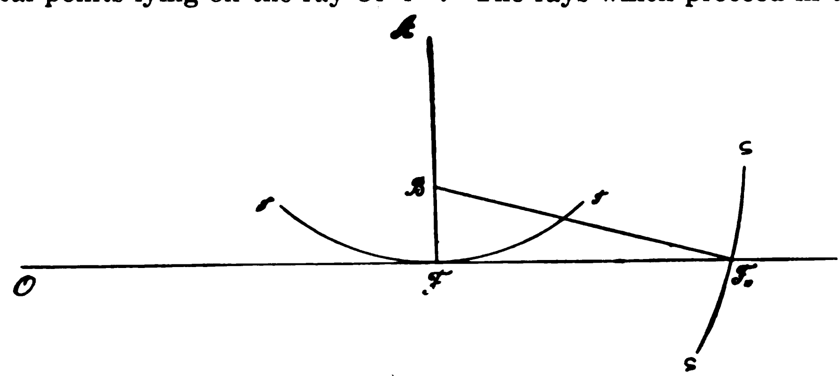

Fig. 117.

plane of the diagram are to be constructed therefore as tangents to this curve, and hence the curved line which passes through O and intersects all the rays orthogonally will be a section of the wave-surface with respect to this point. If the secondary focal point is marked on

each ray, the locus of these points will be a curved line, the so-called $\mathfrak{s}$-curve, indicated in the diagram as $\mathfrak{s}F_{\prime\prime}\mathfrak{s}$, which for a finite magnitude of the astigmatic difference $F_{\prime}F_{\prime\prime}$ always cuts the ray at a finite angle. The radius of curvature $AF_{\prime}$ of the τ-curve, denoted by R, is called the *direct asymmetry-value* of the bundle along the ray $OF_{\prime}F_{\prime\prime}$. On the other hand, if the normal is drawn to the $\mathfrak{s}$-curve at $F_{\prime\prime}$ meeting the primary focal plane at B, the distance $BF_{\prime}$ of this point from the primary focal point, denoted by S, is called the *transverse asymmetry-value.* According to their definitions, the signs of the magnitudes R, S depend on the choice of the positive direction. Therefore, in the case represented in the figure, they are both positive provided the upward direction is defined as positive. In a system of revolution it is a good plan to consider the distance of an object-point from the axis always as positive, and throughout the entire system to treat as positive the direction of any line drawn in this same direction perpendicular to the chief ray. The signs of the astigmatic difference (or distance between the two focal points) and the angle θ which the $\mathfrak{s}$-curve makes with the secondary focal plane are defined by the equations

$$E = \mathfrak{s} - \tau, \qquad S = -E \tan \theta.$$

When the asymmetry-values have the same sign, the form of the primary caustic surface at the focal point is saddle-shaped, because the centre of curvature of the section made by the primary focal plane is at the point where the line $AF_{\prime}$ meets the tangent drawn to the $\mathfrak{s}$-curve at $F_{\prime\prime}$. Now since the curvature of this section is equal to $-\dfrac{S}{E_2}$, it increases with increasing values of S so that the saddle-shape becomes more and more tubular or gutter-shape.

If the angle u between the ray $OF_{\prime}F_{\prime\prime}$ and an adjacent ray is considered as positive when we reach this ray by starting at O and proceeding in the positive direction, we can write the following identities:

$$R = -\frac{d\tau}{du}, \qquad S = -\frac{d\mathfrak{s}}{du},$$

and therefore the asymmetry-values are the limits of the rates of change of the focal distances measured from the wave-surface in passing from one ray to an adjacent ray.

For a first approximation, the degree of ray-convergence is determined by the magnitudes of the asymmetry-values. When it is necessary to have more exact knowledge about this, these values can be calculated along several rays; for along each ray there is a point on the τ-curve where the two curvatures of the primary caustic surface can be found, besides another point on the $\mathfrak{s}$-curve where the inclination of this curve can be ascertained. The latter curve throughout is a

cuspidal edge, as is evident from the fact that the meridian plane is a plane of symmetry. Hence, the curve in which the secondary caustic surface is cut by the secondary principal plane has a cusp at the secondary focal point, where the two branches of the curve are tangent to each other and to the ray $OF_{\prime}F_{\prime\prime}$.

Picturing these relations as deviations or aberrations of the separate rays, we can say that in an astigmatic bundle of rays the distance of the point of intersection of a ray with the primary focal plane, belonging to the chief ray, from the corresponding primary focal line represents the primary lateral deviation of the ray. The secondary lateral deviation is defined similarly with respect to the secondary focal line. Thus, the *primary and secondary lateral deviations* are equal to

$$-\frac{u^2}{2}R-\frac{v^2}{2}S \quad \text{and} \quad -uvS,$$

respectively, where v denotes the angle which the ray makes with the meridian plane. However, these formulae are approximately correct only for infinitely small angles and give merely the deviations that depend on the second power of the angle of inclination. Consequently, they are inserted here not to be used, but simply in order to show the connection between the asymmetry-values that represent exact magnitudes and the ordinary conception of deviations.

Let us employ here the symbol s to denote the length of an arc of the curve made by the intersection of the meridian plane with the wave-surface, and at the same time also introduce the following other symbols, namely,

$$D_{\prime}=\frac{1}{r}, \qquad D_{\prime\prime}=\frac{1}{\mathfrak{s}}, \qquad \frac{dD_{\prime}}{ds}=U, \qquad \frac{dD_{\prime\prime}}{ds}=W;$$

then generally:

$$U=\frac{R}{r^3}, \qquad W=\frac{S}{r\mathfrak{s}^2}\cdot$$

The magnitudes U, W, called the *direct and transverse curvature-asymmetries*, respectively, which accordingly are the rates of change of the principal curvatures of the surface from one point to a contiguous point, are not to be employed for the wave-surface but for the various refracting or reflecting surfaces of the system. If in Fig. 117 the straight line $OF_{\prime}F_{\prime\prime}$ is supposed to represent the normal to a refracting surface of a system of revolution, the $\mathfrak{s}$-curve would have to be a straight line coinciding with the axis, since in a surface of revolution the centre of curvature in the secondary principal section lies on the axis of revolution itself. Therefore for a surface of revolution, and for the bundle of chief rays of a system of revolution whose wave-

surface is a surface of revolution, the magnitudes denoted by W and S are given by the formulae $W = -\frac{(\rho_{''} - \rho_{'}) \tan\theta}{\rho_{'}\rho_{''}^2}$ and $S = -(q-p)\tan\theta$, where θ denotes the angle between the normal to the surface or the chief ray of the bundle and a transversal plane perpendicular to the axis.

If the bundle of rays is anastigmatic along a certain ray, the two caustic surfaces are in contact with each other at the focal point. The radius of curvature of the τ-curve is the same as in the anastigmatic bundle of rays, and the curvature of the $\mathfrak{s}$-line is found from the transverse asymmetry-value by means of the expression

$$-\frac{R-2S}{S^2}.$$

In the case shown in Fig. 118 $(R-2S)$ has therefore the same sign as R. When the two asymmetry-values have the same sign, the focal points of a consecutive ray lie both on the same side of the focal plane at F, and the intersection of the two caustic surfaces with this focal plane is a curve which has a cusp at the focal point common to both surfaces. The tangents to the two branches of the curve which come together in the cusp make with each other the angle whose trigonometrical tangent is

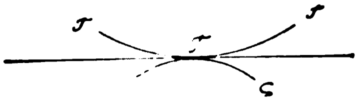

Fig. 118.

$$\frac{2\sqrt{RS}}{R-S}.$$

Since in the cases that ordinarily occur the direct asymmetry-value exceeds the transverse value, this angle whose bisector lies in the plane of symmetry is an acute angle, and consequently the cross-section of the bundle of rays is a characteristic figure similar to an arrowhead.

If the calculation is carried a step further by another differentiation, the *imagery-laws of third order* are derived; which give, for example, formulae that enable us to find in a system of revolution the curvatures of the image-surfaces at the point of intersection with the axis, together with the distortion-value of the primary magnification-ratio and the coefficient of variation of the asymmetry-value. The latter is the differential coefficient of the asymmetry-value for a consecutive chief ray with respect to the distance of the object-point from the axis, supposing this distance to become gradually less and less; and accordingly the direct asymmetry-value is always three times the limit of the transverse asymmetry-value. Again, the

distortion-value is the variation of the primary magnification-ratio depending on the second power of the angle of inclination, which is obtained by passing to a consecutive chief ray. Moreover, the *value of the aberration* along the axis is obtained.

The wave-surface corresponding to the axial object-point is evidently a surface of revolution; which must be true likewise concerning the primary caustic surface; whereas the secondary caustic surface is represented by a piece of the axis of revolution itself. Moreover, since the axis is a line of symmetry, the section of the primary caustic surface made by a meridian plane must have a cusp at the primary focal point where the axis is tangent to it, and where the radius of curvature must be zero, since the asymmetry-values vanish here. If the axis of revolution with which the $\mathfrak{s}$-line coincides here is represented by the straight line OF in Fig. 119, and if the curve AFB is the locus of the centres of curvature of points lying on the τ-curve (*i.e.*, if AFB is the evolute of the τ-curve), the so-called *aberration-value* denoted by A is equal to the radius of curvature of the curve AFB at the focal point F, and the lateral deviation depending on the third power of the angle of inclination of the ray is equal to $-\dfrac{u^3A}{6}$. According to the convention as to the sign of the radius of curvature, a positive value of A, commonly spoken of as positive "spherical aberration," indicates that

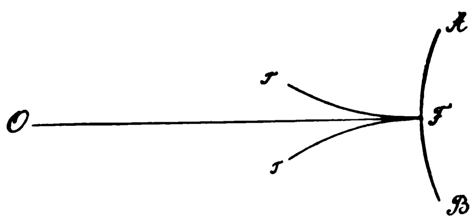

Fig. 119.

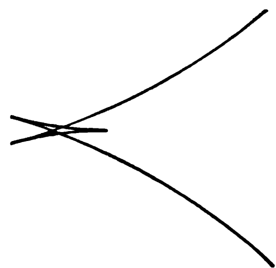
Fig. 120.

the cusp of the τ-curve points along the positive direction. When the "spherical aberration is corrected," the τ-curve has at least three cusps, and under certain circumstances even more; but, though it might be mathematically possible to make them all collapse into a single point, this cannot be practically achieved. A case of this sort is exhibited in Fig. 120, for which the aberration-value at the axial focal point is positive. Along the adjacent rays both the direct and transverse asymmetry-values are found to be positive, as is evident in the first case from the curvature of the τ-curve; and in the second case because, the primary caustic surface being a surface of revolution, the curvature of the section made by a plane perpendicular to the axis is negative.

Along the rays which are tangent to the τ-curve at the two symmetrical cusps the direct asymmetry-value is equal to zero. For rays of greater slope this function is negative, whereas the transverse asymmetry-value continues positive until it vanishes for the ray which is tangent to the τ-curve at the point on the axis where the two branches intersect. The lateral deviation continues negative so long as the slope of the ray has not attained the slope of the tangent to the τ-curve that passes through the axial focal point; thereafter for rays of greater slope it is positive. Accordingly, if aberration means the same thing as deviation of rays, we can speak of "corrected spherical aberration" only for a definite ray-slope. Since the rays with slope-angles smaller than this are in the same relation to each other as in case of "positive spherical aberration," the diagram Fig. 120 represents the case which in the phraseology of optical engineering is that of "corrected spherical aberration with positive zones" for the aperture corresponding to the critical slope-angle above mentioned.

These notions are not sufficient for the exactitude required in physiological optics. If the rays are drawn which are tangent to the τ-curve at the two cusps which point away from the focal point, a conical surface will be generated by revolving these lines around the axis, and a similar conical surface corresponds to it in the stop-space. The section of this surface made by a plane perpendicular to the axis will be a curve $U=0$ or $R=0$, according as we have the wave-surface in mind or the corresponding bundle of refracted rays, respectively; while along every ray intersecting this curve the direct asymmetry-value in the image-space is equal to zero. The diameter of this curve, together with the diameter and position of the cuspidal edge of the caustic surface generated by the two cusps during the revolution may be readily calculated for a given system, and may be found experimentally for a system whose elements are not given. For example, all that is necessary in this case is to observe the curves of the caustic surface projected on a screen perpendicular to the axis while the screen is adjusted until the curves in question are no longer visible. In case the curve corresponding to the last cross section of the caustic surface does not coincide with the contour of the cross section of the bundle or rays, it represents the cuspidal edge mentioned above, and if the stop-opening is diminished until the contour and edge coincide, the diameter of the stop then will be the diameter of the curve $R=0$. However, the degree of ray-convergence inside this curve depends on the distance of the edge from the focal point and varies inversely as this distance. Again, with respect to rays that intersect the plane of the stop outside this region, the curve $W=0$ or $S=0$ constructed in similar way controls this case. Along those rays crossing this curve

which meet the axis at the place in the image-space where the two branches of the τ-curve come together, the refraction is anastigmatic; and hence at this point there is complete ray-convergence of the first order along an infinite number of rays. Finally, the lateral deviation of the edge-ray affords a measure of what the writer calls the *total peripheral aberration*. The necessity of keeping these ideas separate and distinct can be readily seen from the fact that, in the typical case of the "corrected spherical aberration" represented in Fig. 120, the aberration-value is positive along the axis, whereas it is negative for rays corresponding to the curve $R=0$; while the total peripheral aberration will be positive or negative according to the size of the stop. The total peripheral aberration is considered positive when the edge-rays behave as in ordinary positive aberration, *i.e.*, when the lateral deviation is negative.

If an astigmatic bundle of rays has two planes of symmetry, it is said to be a *symmetrical astigmatic bundle* along the ray corresponding to the line of intersection of these planes; and is defined by four aberration-values A_1, G_1, G_2, A_2; the *direct* and *transverse aberrations* in the primary principal section being measured by A_1 and G_2, respectively; while A_2 and G_1 have similar meanings with respect to the secondary principal section. If the intercepts on the line of symmetry are denoted by s_1, s_2, and if the angles between this line and the projections of a ray on the two principal sections are denoted by w_1, w_2, then

$$A_1=-\frac{d^2 s_1}{d\,w_1^2},\quad G_1=-\frac{d^2 s_1}{d\,w_2^2},\quad G_2=-\frac{d^2 s_2}{d\,w_1^2},\quad A_2=-\frac{d^2 s_2}{d\,w_2^2},$$

and there exists the general relation

$$G_1-G_2=s_2-s_1\,.$$

The primary and secondary surfaces have cuspidal edges which cut orthogonally the primary and secondary principal sections, respectively. The direct aberration-values have the same geometrical meaning with respect to the curved sections as in the case of a system of revolution. The curvatures of the edges of the first and second caustic surfaces are

$$-\frac{G_2}{(s_2-s_1)^2}\quad\text{and}\quad-\frac{G_1}{(s_2-s_1)^2}\,,$$

respectively; and hence it follows that both edges cannot be straight at the same time. Although it is mathematically possible for the caustic surfaces of a bundle of rays to collapse into actual focal lines, it appears therefore that a bundle of rays constructed on the type of

STURM's conoid is a mathematical impossibility. If the astigmatism is sufficiently minute compared with the diameter of the stop and the amount of aberration, there will be two rays along which the astigmatism is abolished. The relation between the τ-curve and the $\mathfrak{s}$-curve in the principal section containing these rays is shown in Fig. 121. In the other principal section the τ-curve has a cusp at F'', and the curvature of the $\mathfrak{s}$-curve is finite at the point F'. If the astigmatism along the line of symmetry is continually diminished, the two anastigmatic focal points on the τ-curve approach nearer and nearer to the cusp, and the curvature of the $\mathfrak{s}$-curve gets greater and greater, and finally becomes infinite at the moment when these two points coincide with the two focal points into a single point. In the *symmetrical anastigmatic bundle of rays* thus obtained the $\mathfrak{s}$-curve therefore also has a cusp, and the radius of curvature of its evolute at the focal point is

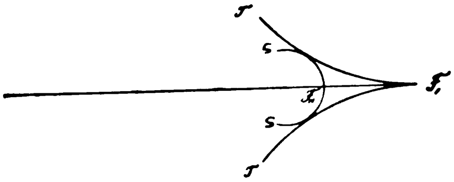

Fig. 121.

$$\frac{4G^3}{(A-3G)^2}.$$

Of the various categories of these bundles of rays, that one for which all the aberration-values have the same sign, and in which the transverse aberration is numerically less than the direct aberration in both principal sections, is of special importance in physiological optics. Under these circumstances both caustic surfaces lie on the same side of the focal plane. The primary surface whose sections made by the principal sections constitute the τ-curve has no edges, and its section made by a plane parallel to the focal plane is a closed curve at finite distance from the focal point which has finite curvature everywhere; whereas the secondary surface always has two edges corresponding to the $\mathfrak{s}$-curve which meet in the focal point, and may have also two other edges besides. The latter is what happens in case the differences A_1-3G and A_2-3G have the same sign. (In the anastigmatic bundle of rays, as results from the relation given above, the two transverse aberration-values coincide.)

The greater the difference A_1-A_2, called the *astigmatism of the aberration*, the more pronounced becomes the phenomenon of astigmatism, varying in degree with the size of the stop, since the position of the cross section of the bundle that is most suitable for the imagery depends on the size of the stop and the direct aberration. But in proportion as the astigmatism of the aberration is less prominent,

the better is the determination of the kind of ray-convergence by means of the difference A_1+A_2-6G, which is a measure of the *diagonal astigmatism of the aberration*. The four edges that we have here on the secondary caustic surface, and that go through the cusps, necessitate eight cusps in a section of this surface parallel to the focal plane, arranged in one of the two typical forms shown in Fig. 122; which, particularly in the second type, involve similar indentations in the section of the bundle of rays. When $A_1=A_2=3G$, the wave-surface has perfect contact of fourth order with a surface of revolution or may be itself a surface of this description. In the latter case the secondary caustic surface collapses on the axis of revolu-
a surface of this description. In the latter case the secondary caustic surface collapses on the axis of revolu-
tion; whereas in the other case it involves a greater number of edges depending on the aberration-values of higher order arranged according to the same scheme and a greater number of similar indentations of the cross section of the bundle of rays.

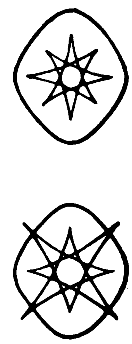

Fig 122.

Corresponding to the aberration-values are the four *coefficients of flattening* ("Abflachungswerte") on the wave-surface denoted by Φ_1, Ω_1, Ω_2, Φ_2, of which Φ_1, Ω_2 are the measures of the *direct and transverse flattening* in the primary and secondary principal sections, respectively, while Φ_2, Ω_1 have the same meanings for the secondary principal section. The connections between these magnitudes and the principal curvatures D_1, D_2 are exhibited in the following equations:

$$A_1=\frac{\Phi_1}{D_1^4},\quad G_1=\frac{\Omega_1}{D_1^2\,D_2^2},\quad G_2=\frac{\Omega_2}{D_1^2\,D_2^2},\quad A_2=\frac{\Phi_2}{D_2^4},\quad \Omega_1-\Omega_2=D_1D_2(D_1-D_2)\,;$$

and moreover

$$\Phi_1=\frac{d^2D_1}{ds_1^2},\quad \Omega_1=\frac{d^2D_1}{ds_2^2},\quad \Omega_2=\frac{d^2D_2}{ds_1^2},\quad \Phi_2=\frac{d^2D_2}{ds_2^2}\,;$$

where s_1, s_2 denote the lengths of arcs of the primary and secondary principal sections of the surface, respectively.

However, the coefficients of flattening have to be used merely for characterizing the refracting surfaces of a system, whereas the bundles of rays are defined by means of the aberration-values. Hence, in a system of revolution the coefficient of flattening Φ at the vertex of each surface must be known in order to calculate the aberration along the axis.

In cases where a caustic surface has multiple curves of intersection with a screen, a multiple imagery of the corresponding system of lines

is produced, as follows from the general imagery-laws. Particularly easy to produce is a double imagery of lines in the symmetrically astigmatic bundle of rays with large aberration-values; because for certain positions of the screen there are two parallel sections of the caustic surface in the plane of the screen, which are not curved too much. But even in systems of revolutions where the caustic surface of the axial bundle makes a circular section with a plane screen perpendicular to the axis, a double imagery of a short line intersecting the axis may be obtained, since the section corresponding to an adjacent chief ray has approximately the same form; and on the supposition of circles corresponding to the different points of the object-line, an impression is produced of two parallel lines, with an intervening space, however, which is brighter than the surroundings. In the same way a multiple imagery may be the result of the edges occurring in the two caustic surfaces of a symmetrically anastigmatic bundle of rays.

Generally, therefore, different imageries correspond to different adjustments of the screen; and what position of the screen is best will depend on the accidental nature of the object to be reproduced and the requirements as to definition or distinctness of the image. For the axial point of the image in a system of revolution, the narrowest cross section of the bundle of rays, as derived from the aberration-value and the size of the stop, at all events does not possess the importance formerly ascribed to it. In general it can be said that the greater the requirement as to reproduction of minute detail, the nearer the screen must be adjusted to the cusp of the caustic surface. The mistiness due to blur-circles which increases with the distance of the screen from the place of the narrowest cross section determines therefore the limit of efficiency of the optical system; that is, according to the degree of ray-convergence.

In the above description[1] of the most important phenomena of *the monochromatic aberrations* entirely in terms of mathematically exact magnitudes there was no room for the proofs; for these the reader must be referred to the writer's works already mentioned. The bridge between this theory and the current method of representing the aberration in a system of revolution has been indicated above. The deviations depending on the asymmetry-value are found usually in books on geometrical optics under the name *coma*, but these phenomena for chief rays of finite slope have hitherto not been correctly treated in such works.

[1] See a somewhat more detailed description as follows: Die Konstitution des im Auge gebrochenen Strahlenbundels, *Arch. f. Ophthalmologie*. Bd. LIII, 2. 1901. S. 185.

II. Procedure of the Rays in the Eye

Imagery-Laws of First Order

1. The Cornea

Anterior Surface of Cornea. During the time that has passed since the first ophthalmometer was made by HELMHOLTZ, there have been great advances and essential improvements in the ophthalmometry of the anterior surface of the cornea. This physiological method of research has been a blessing to practical ophthalmology and to mankind. Every busy oculist uses it daily at the present time. However, convenient as this method is nowadays, the results obtained by it are not due to the improvements. The credit for the method was, and still is, due to HELMHOLTZ; although to others, and mainly to JAVAL and SCHJÖTZ, belongs the credit of bringing the method into ordinary ophthalmological practice. As might be supposed, the necessary adaptations of the method to such uses involved sacrifices in some directions, and even today the scientific investigator finds it advantageous to return to HELMHOLTZ's original construction for the more precise measurements.

The principle of the ophthalmometer is essentially the same as that of the astronomical instrument known as a heliometer. Its plan is to measure a mobile object by shifting the reading or collimation to the object itself. This is effected by bringing in contact two optical images of the object, which involves a "doubling" or duplex mechanism and a collimation device. These mechanisms are both united in the adjustable plane-parallel glass plates in HELMHOLTZ's ophthalmometer. The disadvantage of this construction is noticeable only in two directions. The necessary readjustments and repeated readings, together with the calculations or interpolations in case a numerical table is employed, take a troublesome amount of time; and the arrangements for investigating other normal sections besides the horizontal are too clumsy or inconvenient for the instrument to come into general extensive use. Suggestions of changes in the construction were soon forthcoming. COCCIUS[1] made the calculations and repeated readings unnecessary by using a constant "doubling" device, consisting partly of the glass plates and partly

[1] A. COCCIUS, *Über den Mechanismus der Akkommodation des menschlichen Auges.* Leipzig 1867. *Ophthalmometrie und Spannungsmessung am kranken Auge.* Leipzig 1872.

of a double-refracting calcspar prism, and produced the collimation by varying the size of the object. When HELMHOLTZ's plates are used, it is equivalent to dividing the objective in two parts, and hence also the exit-pupil of the telescope is divided in the same way; but with the double-refracting prism the exit-pupil of the instrument is undivided, and the two double images can be viewed through every point of the exit-pupil. (The exit-pupil is the image of the object-glass in the ocular, which can be seen as a bright disc by pointing the telescope to the sky, and looking along the axis from a point in front of the ocular at the distance of distinct vision.)

Of these two methods of "doubling," namely, with exit-pupil divided and undivided, the former has this disadvantage, that, in case the focusing is not perfectly sharp, there is an apparent displacement of the double images with respect to each other in a direction perpendicular to the line of separation. If, as in HELMHOLTZ's instrument, the line of separation is in the plane of "doubling," although it is true the accuracy of the measurement will not be affected, still a difference of height or level may be erroneously inferred; which will be mentioned again farther on. On the other hand, all the existing constructions with undivided exit-pupil have the disadvantage of chromatic dispersion, which is absent in the case of the plane parallel plates.

For measuring the curvature of the cornea in different normal sections, MIDDLEBURG[1] used a large ring on which the lights could be oriented in various meridians; whereas WOINOW[2] and HELMHOLTZ[3] employed a mirror-arrangement which made it possible to have a stationary source of light. None of these devices was well adapted for obtaining collimation by variation of the size of the object. But the numerous readings that had to be made with the ophthalmometer and the laborious calculations were avoided by LANDOLT[4] in an instrument called at first a diplometer, in which the glass plates were replaced by prisms, and the collination was produced by displacing the prisms along the axis of the instrument.

Originally, flames were used for the object in ophthalmometric measurements. But just as soon as these were replaced by the diffused light reflected from white surfaces, the very modifications could be introduced that made possible the beneficient use of ophthalmometry

[1] Der Sitz des Astigmatismus (nach MIDDELBURG). A letter from F. C. DONDERS to A. v. GRAEFE. *Arch. f. Ophth.* X, 2. 1864. S. 83.

[2] M. WOINOW, *Ophthalmometrie.* Wien 1871.

[3] H. v. HELMHOLTZ, *Handbuch der Physiologischen Optik.* 2. Aufl. Hamburg and Leipzig 1896.

[4] E. LANDOLT, L'ophthalmomètre, *Compte rendu et mémoires du congrès international de Genève.* 1878.

in clinical practice. JAVAL and SCHJÖTZ[1] used for the object a pair of diffusely illuminated white areas, which could be displaced on a circular arc with its centre in the eye of the patient. The arc could be rotated around the axis of the instrument. With this apparatus the investigation of any normal section was just as easy as that of the horizontal meridian. The collimation was easily made by adjusting the relative positions of the white areas. The advantages of constant "doubling" without dividing the exit-pupil were completely realized by using a WOLLASTON prism. And various details of practical construction were contrived to facilitate the manipulation of the instrument.

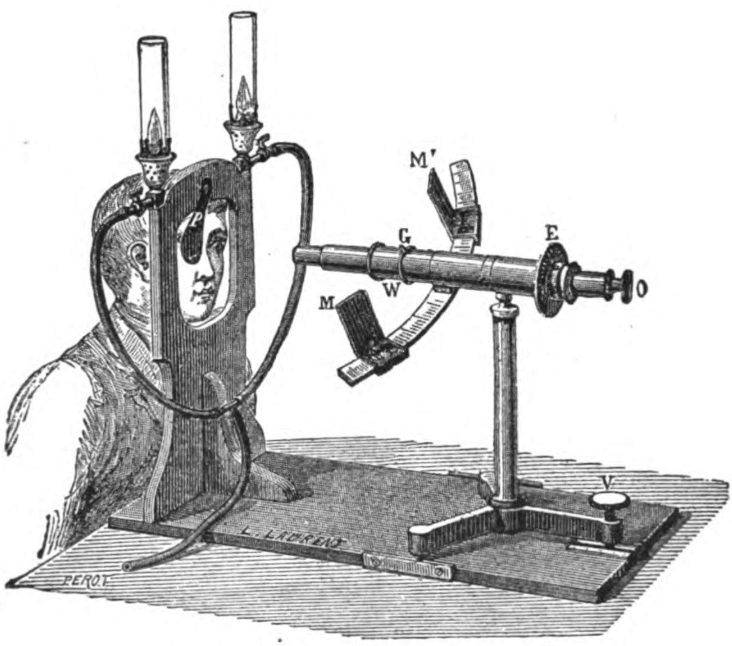

Fig. 123.

The original model of the ophthalmometer of JAVAL and SCHJÖTZ had the external appearance shown in Fig. 123. A notch at E and a pin at G serve as sights for adjusting the instrument. The WOLLASTON prism is inserted at W between two convex lenses, one of which makes the light parallel that comes from the reflex image in the cornea, while the other acts as object-glass of an astronomical telescope. The two white areas or "*mires*" are shown at M and M'. Their actual appearance is illustrated in Fig. 124 in which the distance indicated

[1] JAVAL et SCHJÖTZ, Un ophthalmomètre pratique. *Transactions of the international medical Congress.* VIII. Session. London 1881. III. p. 30. *Annales d'oculistique.* LXXXVI. 1881. p. 5.

by D represents the size of the object, the collimation being made with reference to the lines *ab* and *cd*. The steps or graduations on one of the white areas are calculated so that when the images overlap, each step corresponds to one dioptry of cornea-refraction, supposing there were no difference between the index of refraction of the cornea and that of the aqueous humor. The result of the measurement of a section of the cornea will be given in the form of a certain number of dioptries D, determined by the equation

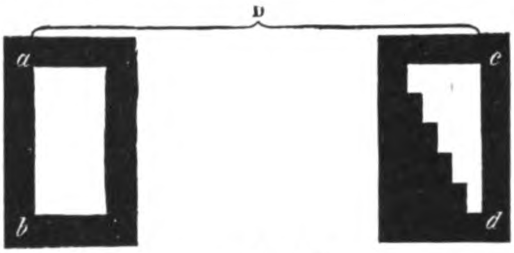

Fig. 124.

$$D=\frac{1000}{\rho}(n-1)$$

where ρ denotes the radius of curvature in millimetres and $n=1.3375$.

Since the "doubling" takes place without division of the exit-pupil, and hence apparent displacements of the images in case of such division are not to be expected here, a difference of level of the images indicates that the longitudinal dimension of the reflex image does not lie in the plane of "doubling," as is the case with the object, and that therefore in the reflection from the cornea a rotation has occurred. The distance to be measured is the distance between the middles of the two lines *ab* and *cd* in Fig. 124. If the cornea is a surface of revolution around the axis of the instrument, this line will be a meridian line, and the same thing is true with respect to its image as reflected in the cornea, and hence there cannot be a difference of level. But if the cornea is astigmatic, with two planes of symmetry, the object-line will not be an imageable line unless it is oriented in one of the principal sections of the cornea; so that the image-line lies also in the same principal section, and a difference of level is again excluded. But if the object-line is between the two principal sections, the reflected image is formed by optical projection. The best way to understand what takes place here is to draw from the two ends of the object-line two lines perpendicular to the principal sections of the cornea, which will make a rectangle whose diagonal is the object-line itself, and whose sides are imageable lines. The two dimensions of this rectangle in the image reflected in the cornea are determined by the two magnification-ratios that are approximately inversely proportional to the refracting powers of the principal sections of the cornea; so that the reflected image is a rectangle whose sides are indeed oriented correctly but are

of different lengths relative to each other; which means that the diagonal of this rectangle is not in the same plane as that of the object. Consequently, its extremities which are to be collimated in the measurement cannot both be brought at the same time in the plane of "doubling," and hence there is a difference of level. The mathematical meaning of this proof is, that in optical projection the inclination to a principal section of a line which does not lie therein is varied when the coefficients of projection for lines which are in the two principal sections have different values; and that when the distance apart of image-lines corresponding to the two imageries is neglected, these coefficients are the same as the magnification-ratios. The proof is strictly valid only for a very small object-line. The exact proof for the size that is employed cannot be given without taking account of the laws of optical projection of higher order. When there is considerable asymmetry in the structure of the cornea, the relations that exist are quite a complicated problem; but if there is a plane of symmetry, it is certain that the difference of level will not vanish except for this plane and the one perpendicular to it, provided the axis of the instrument is in the plane of symmetry.

Easy location of the principal sections, as shown in the regular cases by absence of difference of level, convenient collimation mechanism, and direct reading of the result,—these are the principal advantages of the ophthalmometer of JAVAL and SCHJÖTZ, which, along with the handy form of the entire instrument, have resulted in the employment of the methods of ophthalmometry in ophthalmological practice.

However, these modifications of HELMHOLTZ's original construction are not without disadvantages. The amount of doubling is nearly 3 mm. On the supposition that the investigation is made in a plane of symmetry, this is a measure that depends on the angle between a pair of normals in this plane drawn to the surface of the cornea at points which are about 3 mm apart, a measure that does not give the radius of curvature exactly. Moreover, this measure is not perfectly precise except for a definite value, namely, 45 dioptries, because this is the value for which the instrument is calibrated; but the arc itself is graduated by calculations based on the laws of imagery of the first order that are not entirely permissible for the size of object that is used. Thus errors are introduced depending on the construction, and they are very difficult to eliminate. It is true these errors are not sufficiently great to affect the value of the instrument as it is used in the practice of ophthalmology, but for certain finer physiological researches they limit always the applicability of the results obtained by measurement. Besides, there are other possible errors due to focusing and collimating. With respect to the former, the

distance for which the graduations on the scale are calculated can be obtained only by sharp focusing on the cross-hairs; nor does the "doubling," which is the basis of the calculation, correspond to the real "doubling" unless the focusing is sharp. These errors due to faulty focusing are cumulative. For instance, if the image formed by the object-glass is between it and the cross-hairs, so that the instrument is therefore too far away from the patient's eye, supposing that the "doubling" remains invariable, the object will have to be made larger to give a reflected image of unchanged size. The collimation occurs at a point on the axis. Now if the path of the light is traced backwards from this point, the chief ray lies along the axis at first until it comes to the cemented surface of the WOLLASTON prism where it is split in two rays. The two chief rays which meet the image reflected in the cornea in the two points that are to be collimated diverge, therefore, from the apparent place of this point, and the "doubling" increases with the distance of the instrument from the patient's eye; so that the object must be made greater still than if the "doubling" did not depend on the distance. In order to avoid these errors as much as possible, a very sharp focusing on the cross-hairs is necessary, and if the measurements are to be very precise, it will be well always not to forget HELMHOLTZ's advice[1], to regulate the focusing by parallax displacements of the eye of the observer—which means however that the aperture of the ocular must be sufficiently large.

With regard to the collimation error, the effect of the chromatic dispersion of the prism tends to make it worse, because the edges to be collimated have coloured borders whose appearance is affected by the intensity and composition of the illumination that is used. Modern instruments have transparent *mires* and incandescent electric lights, whereby bright reflected images are obtained without very disturbing coloured borders; as this light is relatively poor in rays of short wave-length, so that the spectrum appears shorter also. This disadvantage can be still further reduced by using coloured glasses. Another thing that influences the amount of the collimation errors is the form of the figure to be collimated, which has undergone numerous modifications since the first model of the instrument. The first improvement was with respect to the collimation for adjusting the level of the images, and consisted in providing the white *mires* with a black line lying in the plane of the "doubling." This serves the purpose in a very satisfactory way, since the line appears continuous without any break in it when

[1] As given by JAVAL, Contribution à l'ophthalmométrie. *Annales d'oculistique.* LXXXVII. 1882. p. 213.

the two images are exactly in level, and since the eye has a remarkable power of detecting a very small displacement of the two parts of this line with respect to each other. On the whole, a similar contrivance is best also for the collimation in making the measurements. The purpose of the steps on the *mire* is to have a convenient way of reading the amount of astigmatism. The best way to do this is to make the adjustment for the first principal section by using a certain one of the *mires* always for this purpose, and then in the other principal section to collimate by displacing the other *mire*. Therefore, at the beginning of the measurement the latter one must always be brought to the zero-point of the scale corresponding to symmetrical adjustment of the *mires* for correct collimation and for the average value of the curvature of the cornea. When the investigation is finished, the curvature in the first principal section is read off by the position of the first *mire*, and the degree and sign of the corneal astigmatism by the position of the second *mire*. On a scale where an interval of 1° corresponds to a dioptry, the zero-point, for example, can be taken on the left-hand part of the arc at 22° from the axis, and a scale can be added reading to 10° in both directions. Then the point on the right-hand part of the arc which is symmetrical to the zero-point will be marked by the number 44. With such an arrangement of the scale the steps on the *mires* are superfluous, and therefore the forms of the figures to be collimated can be constructed solely with a view to the sharpest collimation.

It almost goes without saying that the errors of measurement with modern ophthalmometers, even in the case of long practice and highest skill on the part of the observer, are not to be lightly estimated. The collimation error, indeed, with the instruments now in use, ought not to be more than 1/4 dioptry, and may be rather less when the figures to be collimated are advantageously constructed. But the focusing error, which depends in great measure on the skill of the observer and the repose of the patient, will not be certainly below this value except under the most favourable circumstances; so that as a usual thing the possible error may be estimated at between 1/4 and 1/2 dioptry. While these errors are comparatively unimportant for purposes of ordinary practice, they cannot be left out of account in the more precise measurements of corneal astigmatism or in the measurements of the radius at several points in one and the same principal section.

In case of the measurement just mentioned the large "doubling" comes also into consideration. When we try to get rid of this disadvantage by using a prism with half the "doubling," the collimation errors will be doubled.

In addition to the ophthalmometer of JAVAL and SCHJÖTZ, which was modified in some ways in the new model of 1889[1], a few other types will be briefly mentioned here. The instrument-maker KAGENAAR of Utrecht employed a biprism construction instead of the WOLLASTON prism; and consequently his ophthalmometer has a divided exit-pupil, the line of separation being vertical and lying in the "doubling" plane. The result is there are no difficulties about levelling the reflex images, but the apparent displacements that are possible with divided ex t-pupil have an effect on the final result, which therefore is subject here to an additional source of error. On the other hand, while the HELMHOLTZ plates are retained in the ophthalmometer of LEROY and DUBOIS,[2] the "doubling" in this instrument is constant; and thus *ceteris paribus* the focusing error is reduced, although, otherwise, the difficulty as to contact-adjustment of the two images mentioned above militates against the practical use of the instrument. The most perfect construction from a theoretical standpoint is represented by the SUTCLIFFE ophthalmometer.[3] The exit-pupil is divided here too—indeed divided into five parts—but the apparent displacements thus produced are ingeniously utilized as a check on the sharpness of the focusing. Two mutually perpendicular normal sections are measured simultaneously, as the observer sees three images. In investigating the vertical and horizontal meridians the central part of the exit-pupil produces one image; a second image is due to the upper and lower parts; whereas the third image is formed in the same way by the right and left parts. The apparent displacements that occur in case of error in focusing are accompanied by a "doubling" of the two images last mentioned, which will not disappear until the focusing is right. The "doubling" in the two mutually perpendicular meridian planes of the instrument is variable, the object itself being fixed and having a form very favourable for exact collimation. Thus, while errors of both focusing and collimation appear to be reduced to a minimum, perhaps the greatest advantage of all is to be found in the fact that here the correct collimation in both principal sections is checked by a glance. The measurement of corneal astigmatism as thus obtained is far more reliable, inasmuch as the ordinary method of determination may involve a summation of the errors of two successive focusings, and consequently a bigger error may be made in comparison with the degree of the

[1] SULZER, Description de l'ophthalmomètre JAVAL et SCHJÖTZ. Modèle 1889 in *Mémoires d'ophthalmométrie* par E. JAVAL. Paris 1890.

[2] C. J. A. LEROY et R. DUBOIS, Un nouvel ophthalmomètre pratique. *Annales d'oculistique*. XCIX. 1888. p. 123.

[3] J. H. SUTCLIFFE, One-position ophthalmometry. *The optician and photographic trades review*. XXXIII. 1907. Supplement p. 8.

physiological corneal astigmatism; whereas in this new method such is not the case. Practical experience with this instrument in the Upsala clinic has completely demonstrated its superiority for investigating corneal astigmatism.

However, the great advantage of finding the astigmatism by a single measurement may be obtained also with the ordinary ophthalmometer, because, as the writer has pointed out,[1] this can be ascertained from the contact-difference (*Denivellation*) in a plane making an angle of 45° with the principal sections. For this purpose all that is needed is to make one of the white *mires* in the ordinary ophthalmometer adjustable in a plane at right angles to the plane of "doubling" and to add a corresponding scale. The scale interval corresponding to one dioptry must be half as great as that of the scale in the plane of the collimation-figure which is calibrated for measurement of the radii.

These methods enable us to find simultaneously the radii that are to be compared with each other. The only way to do this, when the problem consists in ascertaining the radii at different points in one and the same principal section, is by photographing the reflex image in the cornea. Such measurements, it must be admitted, take much time and require also special apparatus made for the purpose. Consequently, they are not suitable for the general run of practice, but on the other hand they give a resultant accuracy that previously could not be obtained in any other way. In this method the writer[2] used an object which gave the radius at seven points in one and the same principal section at the same time. The corresponding parts of the object were computed so that their reflex images in a spherical surface all had the same size, for example, about 2/3 mm for a radius of 7.8 mm.

After this concise description of the means that are employed nowadays of investigating in numerous ways the form of the anterior surface of the cornea, let us pass on to a summary of the results. Helmholtz was perfectly justified by the knowledge of geometrical optics in his day in regarding the form of the non-astigmatic cornea as elliptical, because the dioptric behaviour of an ellipsoid was known, but the asymmetry values that are characteristic of the dioptric behaviour of any surface whatever were not known. Accordingly, as soon as the asymmetry of the cornea with respect to the visual axis had been established, there was nothing better to do than to

[1] A. Gullstrand, En praktisk metod att bestämma hornhinnans astigmatism genom den s. k. denivelleringen af de oftalmometriska bilderna. *Nordisk Oftalmologisk Tidskrift.* 1889.

[2] Photographisch-ophthalmometrische und klinische Untersuchungen über die Hornhautrefraktion. *Kungl. Sv. Vet. Akad. Handl.* 1896. Bd. 28.

calculate by means of it the constants of the ellipsoid in question. HELMHOLTZ's own statement,[1] that the "representation of the form of the cornea by an ellipsoid is for the time being a fairly close approximation," is still true today, although subsequent investigations have shown that a different view gives a better approximation.

BLIX,[2] by an entirely peculiar ophthalmometric method of his own, whose greatest advantage, however, lies in the determination of the situations of the ocular refracting surfaces, and which therefore will not be described until later, was the first to demonstrate that the form of the cornea is considerably different from that of an ellipsoid. As was pointed out by AUBERT[3] at the time, these differences may be most simply expressed by saying that in a central "optical zone" the variations of curvature are less, whereas in the rest of the cornea they are greater, than they would have to be if the curvature were elliptical. The curvature of the optical zone, which corresponds roughly to the diameter of a pupil of medium size, is approximately spherical, or at any rate the accuracy that is attainable in ophthalmometric measurements is not high enough to enable us to compute the constants of an ellipsoid that might represent its variation from a spherical form. The form of the cornea has been better ascertained from the researches of SULZER[4] and ERIKSEN,[5] who have supplied us with a considerable mass of data to show that the flattening of the cornea out toward the periphery not only is frequently asymmetrical both horizontally and vertically, but is also in most instances more rapid in the latter meridian than in the former. On the other hand, SULZER's conclusions from this last fact concerning the variation of the astigmatism of the eye with the size of the pupil are due to the erroneous idea that there could be any such thing as astigmatism of an annular zone of the cornea, which is mathematically impossible. And ERIKSEN's views about the astigmatism at various places in the cornea are simply mathematical consequences of the flattening out toward the periphery.

Qualitatively, these researches of SULZER and ERIKSEN are convincing, because the results given above follow from a comparison of the flattening in different directions. But, quantitatively, they are

[1] *Handbuch d. phys. Opt.* 2. Auflage. S. 17.

[2] M. BLIX, Oftalmometriska studier. *Upsala Läkareförenings Förhandlingar.* XV. 1880. S. 349.

[3] H. AUBERT, Nähert sich die Hornhautkrümmung am meisten der einer Ellipse? PFLÜGERS *Arch. f. d. ges. Physiologie.* XXXV. 1885. Die Genauigkeit der Ophthalmometermessungen. Ibid. XLIX. 1891.

[4] La forme de la cornée humaine et son influence sur la vision. *Arch. d'Ophth.* XI. 1891. p. 419. XII. 1892. p. 32.

[5] *Hornhindemaalinger.* Aarhus 1893.

not satisfactory to the same degree, because measurements with an ophthalmometer merely give the angle between two elements of the cornea separated by an interval depending on the amount of "doubling" in the instrument, whereas the measurements were made for smaller angular steps. With the ophthalmometer of JAVAL and SCHJÖTZ the angle measured in case of normal corneal curvature, with the "doubling" usually employed, is more than 20°. If the measurements are repeated at intervals of 5° in the direction of fixation, the reflex image of one of the *mires* in the next four measurements is still inside the part of the surface measured first, and the assumption that these five measurements have given the curvature at five different places is therefore not justifiable. This comment, however, is less applicable to ERIKSEN's researches, because in his work the amount of "doubling" was only 1 mm. The mathematical utilization of the form of the cornea as thus found for investigating the refraction of the rays depends, however, on the assumption that in each successive measurement one end of the object is reflected from the same place on the cornea as the opposite end in the preceding measurement. This requirement, together with the use of sufficiently small surface-elements in the measurements, has heretofore not been fulfilled except in the photographic methods of the writer.

The disc shown in Fig. 125 was employed as object, which was made in such manner that the intervals between one circle and the next were proportional to the radii of the corresponding surface-elements. It was photographed for five different lines of fixation, namely, when the eye was directed right toward the objective, and when the line of fixation was turned into the four principal directions, such that the most peripheral surface-element measured in the central adjustment coincided precisely with the most central element measured in the peripheral adjustment. The photograph was measured on a dividing engine with a microscope, the

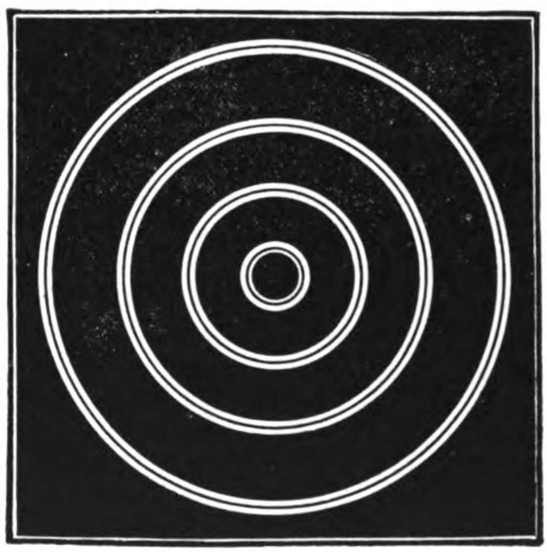

Fig. 125.

readings being corrected by the periodic errors of the screw, as carefully ascertained. The precision of the measurements, checked by numerous trials, was obtained by dividing the rings in two parts as shown in the figure, since the cross-hairs could be sharply set on the bright space in the photographic negative without difficulty. The angles between the normals to the cornea at the different places used for reflection and the normals parallel to the visual axis were known from the construction of the apparatus; and the corresponding radius was found by measurement, since it was considered as belonging to that point where the normal made equal angles with the normals at the two extreme points of the element to be measured. As result of these investigations, the writer obtained the complete measurement and calculation of a typical normal cornea, some data of which are herewith subjoined.

Angle corresponding to the normal at the centre of the measured element	Vertical		Horizontal	
	Upwards	Downwards	Inwards	Outwards
38° 55′ 30″	....	28.5	27.9	32.3
34° 3′ 50″	41.7	36.6	28.4	38.8
29° 14′ 20″	35.2	40.2	37.4	41.2
24° 24′ 50″	37.7	41.2	40.9	43.6
19° 33′ 10″	39.8	42.2	42.5	43.5
14° 37′ 10″	41.7	43.4	42.8	44.0
9° 41′ 10″	42.8	43.8	43.5	43.8
4° 49′ 30″	43.3	43.6	43.8	43.4
0° 0′ 0″	44.5		44.2	

The radii of curvature here are calculated in dioptries by assuming the value 1.3375 for the index of refraction, as is usual in ophthalmometry. A graphical representation of the results of this table is given in Fig. 126 after the method used by Eriksen; which shows very distinctly the comparatively slight variation of the radius of the cornea in the central portions and the rapid flattening in the peripheral regions, together with the asymmetry both vertically and horizontally and the flattening that begins nearer the centre in the vertical section. The irregularities of the curves are due to the unavoidable errors of observation, irrespective of the method, and dependent on the fact that, as a matter of fact, the first refracting surface of the dioptric system of the eye is not the anterior surface of the cornea but the fluid layer resting on it which is responsible for the reflex images employed in the ophthalmometric investigation. In the photographic method, doubtless, it would not be difficult to smooth out these

irregularities by exact mathematical methods, since seven measurements are always taken at once, and by virtue of the construction of the disc the various errors cannot amount to anything more than

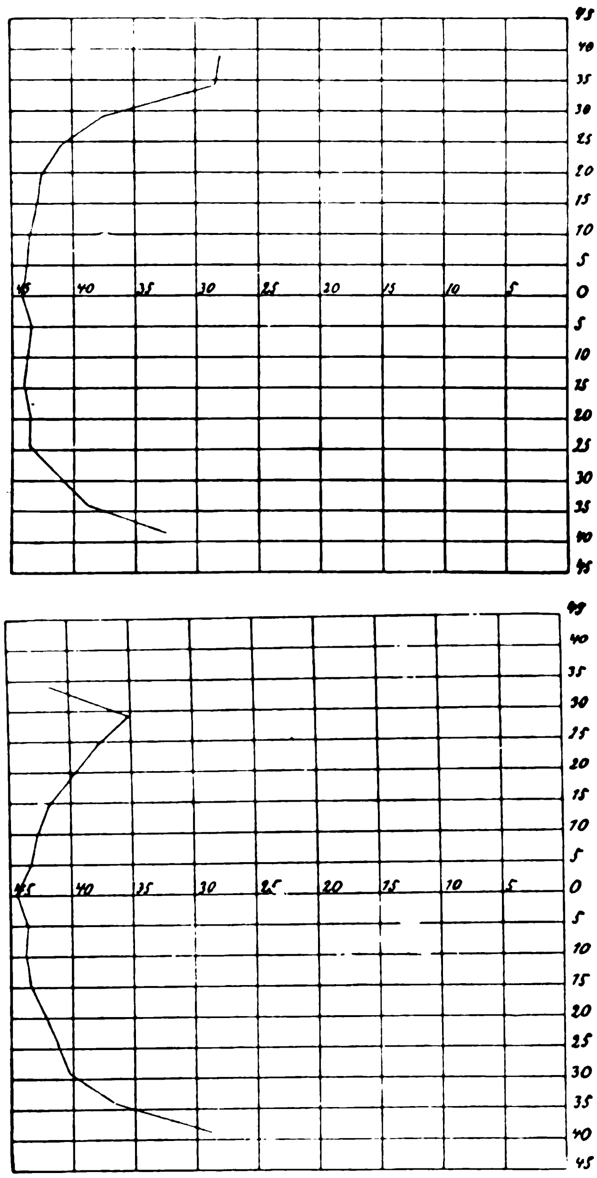

Fig. 126.

that one of the radii of two adjacent elements comes out just as much too large as the other is too small. But this mathematical labour would scarcely be worth while, for, although the curvature asymmetry of the cornea at the point of incidence of the effective chief ray in the

ocular imagery might thus be determined, this value would be of little further use until we knew more than we do at present about the form of the lenticular surfaces.

We must not omit to mention here that MATTHIESSEN[1] endeavoured to compare the curvature of the horizontal section as obtained above with that of an ellipse. The writer had assembled in a table the coördinates of all the points of the cornea employed in the reflection, and from these coördinates MATTHIESSEN computed the tangential ellipses. Now if the curvature were elliptical, the ellipses thus obtained ought to have coincided approximately with one another. But the semi-axes majores of these ellipses vary from 14.46 to 8.62, although the first measurement on the nasal side and the two first on the temporal side were left out of account—and hence the central part of the optical zone was not considered. In the writer's opinion therefore the agreement is not satisfactory, but at the same time he can endorse MATTHIESSEN'S conclusion, that from the rest of the 15 coordinates there is a basis for making an ellipse in the horizontal section of the cornea which agrees fairly well with the average values of the measurements of other workers. All that can be concluded from this is, that, in vision with a pupil of medium size, the assumption of a spherical shape for the utilized part of the cornea is provisionally the best approximation; and in cases where the excentric parts of the cornea are predominantly effective (as in certain investigations of the lens constants) the ellipse may continue to be considered as a better approximation than any similar hypothesis.

Moreover, the curves exhibited in Fig. 126 represent the qualitative results obtained by SULZER and ERIKSEN most faithfully, as might have been expected from the fact that these diagrams are derived from a typically normal cornea, while the researches of those writers relate to a large number of eyes. Another thing in common is that the starting point corresponds to the line of fixation of the patient when he looks right into the objective of the instrument. The point designated by zero in the preceding table is therefore the point of the cornea for which the normal is parallel to the line of sight (*Visierlinie*); and since this is always the origin in modern clinical ophthalmometry, it may be called the *ophthalmometric axial point.*

Starting from this point, the *form of the normal cornea* can be described by saying that there is a central optical zone where the curvature is approximately spherical, and which extends horizontally about 4 mm, and somewhat less than this vertically, and is decentrated

[1] L. MATTHIESSEN, Über aplanatischen Brechung und Spiegelung in Oberflächen zweiter Ordnung und die Hornhautrefraktion. PFLÜGERS *Arch. f. d. ges. Physiologie.* XCI. 1902. S. 295.

outwards and usually also a little downwards; and that the peripheral parts are considerably flattened, decidedly more so on the nasal side than on the temporal, and usually more so upwards than downwards.

Out from the ophthalmometric axial point the pupil also is decentrated outwards and usually a little downwards, corresponding indeed in extent very nearly to the decentration of the optical zone, so that in the most typically regular eyes the latter can be considered as approximately centered with respect to the pupil. Accordingly, in ascertaining the centering of the refracting surfaces of the eye, the best procedure is to take the normal to the cornea which goes through the centre of the pupil as being the *optical axis of the eye*. Strictly speaking, the eye has no optical axis, because the refracting surfaces are not exactly centered; but a line has to be chosen that satisfies the requirements of such an axis approximately. The orientation of this axis is easily found, by placing a perforated round white disc at the objective of the ophthalmometer or of a telescope, whose reflex image is adjusted concentrically to the pupil by motion of the patient's eye; and then all that remains to be done in order to obtain the required data is to adjust the fixation mark.

However, more important for the dioptrics of the cornea than this measurement is the *angle of incidence of the line of sight*, and the orientation of its plane of incidence. The special significance of the line of sight, which with respect to the imagery-laws of the first order is the chief ray of the bundle of effective rays in distinct vision, and which proceeds from the point of fixation to the apparent centre of the pupil, consists partly in the fact that for the actual convergence of rays in the eye it has the same *role* as in the fictitious collinear imagery is ascribed to the visual axis (*Gesichtslinie*) that passes through the anterior nodal point; because its orientation can be exactly ascertained. Practically, indeed, it is quite immaterial whether we speak of the line of sight (*Visierlinie*), line of fixation (*Blicklinie*) or visual axis, so far as inclination to the optical axis is concerned, because the differences between these angles are below the limit of the possible errors. But since the position of the line of sight can be accurately found, whereas that of the visual axis cannot, it is better generally to reckon only with the former, as was first pointed out by Blix.[1] The angle of incidence thereof was measured by Leroy[2], and afterwards by the writer.[3] The most accurate results are obtained with the Helmholtz ophthalmometer, by placing the fixation mark on the

[1] Loc. cit.

[2] De la Kératoscopie ou de la forme de la surface cornéenne, déduite des images apparentes réfléchies par elle. *Arch. d'ophth.* IV. 1884.

[3] Loc. cit. *Skand. Arch. f. Phys.* II. 1890.

prolongation of the axis of the instrument and adjusting a small source of light so that the reflex image is seen in the centre of the pupil. By rotating the plate-carriage on the ophthalmometer until the source of light lies in the plane of "doubling," and by turning the plates until the double images of the source each coincide with an edge of the doubled pupil, the proper adjustment of the source of light can be made. In this delicate experiment it often appears that the angle of incidence depends on the size of the pupil, and hence this angle should be ascertained always for a medium size of pupil of 4 mm. The quotient of the distance of the light-source from the axis of the ophthalmometer by the distance between the vertex of the cornea and the plane passed through the source perpendicular to the axis is equal to twice the tangent of the required angle of incidence. By similar centering of the reflex image of a round white disc this angle can be found without an ophthalmometer. In normal eyes the writer has found it to vary from 0° to 6°, and sometimes he has obtained negative values; in which case, therefore, the source of light had to be shifted towards the nose from the axis of the instrument. The angle between the plane of incidence and the horizontal plane may amount to as much as 30° in perfectly normal eyes, and it usually extends in the direction from above inwards to below outwards when in the perfectly normal eye there occurs a perceptible deviation from the horizontal. For very small values of the angle of incidence, generally every orientation of the plane of incidence is possible. The angle between the line of sight and the optical axis is always larger than the angle of incidence, and as a rule is rather more than half as much again. Thus, if P, P' in Fig. 127 designate the real and apparent positions, respectively, of the centre of the pupil, and if u denotes the angle between the line of sight and the optical axis and i the angle of incidence, then

$$\sin u : \sin i = \rho : (\rho - d),$$

where d denotes the apparent depth of the anterior chamber of the eye, and ρ denotes the radius of curvature of the optical zone of the cornea. For small angles the value found is sufficiently accurate when the angles themselves are substituted for their sines; and for an apparent depth of the anterior chamber of 3 mm and a radius of 7.8 mm, the ratio is 1.625. It may be remarked that the first of the measurements given by HELMHOLTZ at the end of §3 for the position of the centre of the pupil with respect to the axis of the cornea, as determined by the constants of the ellipse, shows complete agreement between this axis and the optical axis which is assumed here, within the limits of the errors of observation; while the differences that occur in the other two measurements are not greater than can be

explained by the asymmetry of the horizontal section of the cornea and the consequent deviation from the elliptical form.

Accordingly, it would seem to be justifiable to employ the angle of incidence of the line of sight and the inclination of this line to the optical axis, both of which angles can be easily measured at present with the usual apparatus, instead of the angle α between the visual axis and the axis of the corneal ellipse, which HELMHOLTZ uses. The angle which HELMHOLTZ[1] calls β, and which in the literature of physiological optics is sometimes denoted by α and sometimes by γ, is the angle between the line of fixation and the normal to the cornea that goes through the centre of the optical zone of the cornea. This angle is entirely without significance for the dioptrics of the eye.

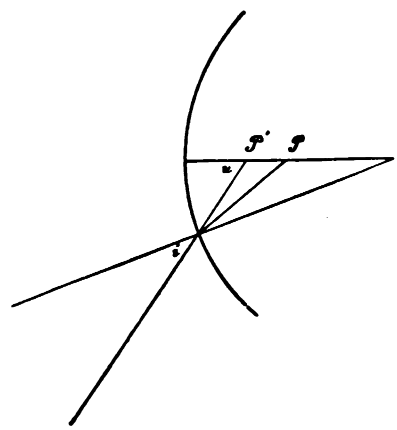

Fig. 127.

With respect to the *curvature of the optical zone of the cornea*, in consequence of what has been stated above concerning the accuracy of measurements with modern ophthalmometers, we may attach comparatively high value to the old investigations, which have been compiled by DONDERS.[2] For 110 adult men the average value was 7.858 mm, the maximum and minimum values being 8.396 and 7.28. For 46 women the corresponding values were 7.799, 8.487 and 7.115. From these numbers HELMHOLTZ derived the schematic value 7.829. Much more extensive numerical data are available for modern ophthalmometry. STEIGER[3] found the average value of the corneal refraction of 1916 eyes to be 43.03 dioptries, corresponding to a radius of 7.843 mm. SULZER[4] takes the average value as 43.7 dioptries, corresponding to a radius of 7.723 mm. The discrepancy here can easily be accounted for by the unavoidable personal equation. In view of the fact that the third decimal figure is uncertain, and since SULZER's material was not as extensive as STEIGER's, perhaps the nearest approach to the truth is to estimate the *ophthalmometric average value of the radius of curvature of the optical zone of the cornea* at 7.8 mm; and, on the basis of DONDERS' numerical results as given above, to put the physiological limits at 7 and 8.5 mm—numerical values which may henceforth be

[1] *Handbuch d. phys. Opt.* 2. Auflage. S. 19.

[2] F. C. DONDERS, *On the anomalies of accommodation and refraction.* London 1864.

[3] ADOLF STEIGER, *Beiträge zur Physiologie und Pathologie der Hornhautrefraktion.* Wiesbaden 1895.

[4] Loc. cit.

generally assumed and which are not inconsistent with exact investigations. So far as the limiting values are concerned, the lower one may be surpassed in microphthalmos and in keratoconus, and the upper one in flattening of the cornea in post-operative pressure or after ulcerations; but under all circumstances these limits are likely to be wide enough for physiological relations. From STEIGER's results the variations are found to lie between 7.5 and 8.1 mm in at least about 80% of the cases, since in 88.5% the refraction was found to be between 41.25 and 45 dioptries. The *sexual difference of corneal curvature* as indicated by the above data of DONDERS was established by the larger material of STEIGER (mean value for boys and girls 42.89 and 43.15 dioptries, respectively). His investigations indicate that the cornea becomes flatter with advancing years; but it is doubtful whether the material was sufficient to decide this question. However, his measurements prove that there is a connection between the radius of the cornea and the distance of the pupil; just as a similar connection between this radius and the size of the body and the circumference of the head is indicated by the investigations of BOURGEOIS and TSCHERNING,[1] since a larger anthropometrical measure implies a larger mean value for the radius of the cornea.

However, although the curvature of the optical zone of the cornea is nearly spherical both vertically and horizontally, this is not the case with the zone as a whole. For the normal condition is an appreciable astigmatism. This *physiological corneal astigmatism* was definitely ascertained by the ophthalmometric measurements carried out first by NORDENSON.[2] Concordant results of various investigators show that its average value amounts to between 0.50 and 0.75 dptr, and that the section of least curvature is not very far from the horizontal direction or the longitudinal extent of the eye-slit. As evidence thereof STEIGER's figures may be used. He found a mean value of 0.78 dptr in 3170 eyes. But when the eyes were left out of account that had an astigmatism of more than 2.0 dptr, which are always to be regarded as pathological cases, he obtained with 3073 eyes an average value of 0.70 dptr, and in two-thirds of these the corneal astigmatism was between 0.50 and 1.0 dptr, and in nearly seven-eighths of them between 0.25 and 1.25 dptr. In 89.4% of the eyes the direction of the section of least curvature was horizontal.

A *change of corneal astigmatism with the age* of the patient has been

[1] Recherches sur les relations qui existent entre la courbure de la cornée, la circonférence de la tête et la taille. *Ann. d'oculistique.* XCVI. 1886.

[2] E. NORDENSON, Recherches ophthalmométriques sur l'astigmatisme de la cornée chez les écoliers de sept à vingt ans. Ibid. XC. 1883.

certainly shown by the researches of SCHÖN,[1] STEIGER and PFALZ.[2] The astigmatism is said to be *with the rule* or *against the rule*, according as the section of least curvature makes an angle not greater than 30° with the horizontal or vertical planes, respectively, or coincides with one or the other of these planes (also referred to sometimes in the literature of the subject as "As. rectus" or "As. perversus" and "direct astigmatism" or "inverse astigmatism"). It is found that, *with advancing years the physiological astigmatism with the rule decreases, whereas the percentage of cases with corneal astigmatism against the rule increases.* Connected with this variation is an increase of the number of cases in which the principal meridians of the astigmatism are not the horizontal and vertical planes. This is also easily explained, since static influences, which with distinct astigmatism were not sufficient to produce an appreciable change in the form of the cornea, are necessarily more and more in evidence with steadily diminishing astigmatism.

The geometrical surface that represents the form of the optical zone of the cornea when its curvature may be regarded as spherical both horizontally and vertically is a toric surface. This is a surface generated by the revolution of the arc of a circle about an axis in its plane, or, according to the geometrical definition, is the enveloping surface of a sphere whose centre moves along a circle.

According to a general law of the theory of curved surfaces, known as DUPIN's theorem, the form of any section of the anterior surface of the cornea made by a plane perpendicular to the axis and infinitely close to the ophthalmometric axial point is elliptical; and the axes of the ellipse are to each other as the square roots of the radii of principal curvature, so that the vertical axis is the axis minor in normal physiological astigmatism. But if we suppose a number of such sections of the cornea to be made in succession, the form of each curve being ascertained by the ratio between its horizontal and vertical diameters, on account of the marked flattening that occurs in the vertical section of the cornea, the contour of these slices will be found to be such that the deeper we go, the less this ratio becomes. It is still problematical as to what is the depth when this ratio is unity. But that it does occur before we come to the base of the cornea, can be seen from the coördinates calculated for the case illustrated by the curves above (which is the only case so far for which such a calculation can be made). From the table in question it is sufficient to give here three points in each direction as follows:

[1] W. SCHÖN, Die Akkommodationsüberanstrengung usw. *Arch. f. Ophth.* XXXIII. 1. 1887.

[2] G. PFALZ, Über Astigmatismus perversus — eine erworbene Refraktionsanomalie. *Zeitschr. f. Augenheilkunde.* III. 1900.

Nasal		Temporal		Upwards		Downwards	
x	y	x	y	x	y	x	y
0.860	3.556	0.831	3.483	0.905	3.680	0.856	3.541
1.231	4.218	1.168	4.084	1.298	4.383	1.201	4.157
1.792	5.048	1.579	4.692	1.680	4.948	1.636	4.801

In this table x denotes the depth of the section from the ophthalmometric axial point, and y denotes the distance of the point on the surface from the ophthalmometric axis. If the 12 points are constructed, it is found that, for a given value of x the values of y are larger upwards than inwards, and larger downwards than outwards; and hence that the vertical diameter of the section is greater than its horizontal diameter. However, this is on the assumption that the outer portion of the eyeball is similar in form. Hence, from the more considerable flattening of the normal cornea vertically, it follows that a section of the outer portion of the eyeball just behind the cornea and perpendicular to the line of sight must have a longer diameter vertically than horizontally. It appears, moreover, that if the cornea were not under the influence of external forces, its natural form would have to show astigmatism against the rule. How the actual form of the cornea comes to be different from this natural form, is exactly what might be expected from the action of external forces due to the pressure of the eyelids. Since this pressure is exerted only upwards and downwards, and by reason of the structure of the lid-slit must be stronger in the former than in the latter direction, it must produce a flattening corresponding to the surface of contact, which affects only the vertical section, and must be more pronounced above than below. The compression from above downwards must also result in a direct astigmatism (astigmatism with the rule) of the optical zone. That this mechanism is sufficient from a qualitative point of view to account for the form of the cornea, is evident because the surface of contact with the eyeball, anyhow so far as the upper lid is concerned, extends over the cornea. This is true too, though not to the same extent, with respect to the lower lid, because most people habitually look downwards below the horizontal plane.

Several facts can be adduced to show that this mechanism is quantitatively sufficient also. First, the difference between the actual and natural forms of the cornea means an exceedingly slight deformation, which might perhaps be produced by the interaction of the forces due to the pressure of the eyelids and the effective processes in the genesis of the form of the cornea. However, in the second place, it is easy to show in the ophthalmometric investigation that any increment of these forces produces a sudden deformation, the astigmatism

being considerably augmented by pinching the lids together, accompanied by a pronounced vertical asymmetry of the cornea with increased flattening. And in abnormal cases like keratoconus, the effect of these external forces comes out in the most striking way, as shown by the fact, which the writer could demonstrate, that it is typical to find a large amount of direct astigmatism (that is, astigmatism with the rule) at the vertex of the cornea, together with a marked vertical asymmetry, involving decentration of the vertex downwards and greater flattening upwards, as has been observed in most cases.[1] In very old age the coatings of the eyeball become more rigid, and the eyelid pressure diminishes, the general tone of the tissue being reduced and the fatty matter in the socket drying up. Hence, the difference between the actual and natural forms of the cornea must get slighter, and accordingly we find STEIGER's statistics show a considerable increase of cases of very old persons with inverse corneal astigmatism or astigmatism against the rule. Lastly, with increase of pressure the action of the eyelid pressure must give way against the pressure from within tending to make the cornea assume its natural form; and the clinical investigations of MARTIN[2] and PFALZ,[3] together with the laboratory researches of EISSEN,[4] have demonstrated that increase of pressure in the normal eye is accompanied by corneal astigmatism against the rule. Thus it appears that where the action of the eyelid pressure is promoted, there will be an increase, and where it is excluded, there will be a decrease, of the deformation, which differentiates the actual form of the cornea from its natural form corresponding to the shape of the outer portion of the eyeball. The writer thinks it is fair to conclude that the pressure of the eyelids, or the resistance that is offered by them to the dilatation of the eyeball, is the explanation of the normal direct astigmatism of the optical zone, and likewise of the excess of the peripheral flattening in the vertical section, and of the normal vertical asymmetry of this flattening.

The calculation of the form of the cornea from the ophthalmometer measurements is made as follows: In Fig. 128 the straight line AEF represents the ophthalmometric axis, and the straight line DGE shows the path of a ray proceeding from a point of the object, which,

[1] Ett fall af keratoconus med tydlig pulsation af hornhinnan. *Nord. Ophth. Tidskr.* IV. 1892. S. 142.

[2] G. MARTIN, Études d'ophthalmométrie clinique. *Ann. d'oculistique.* XCIII. 1885. p. 223.

[3] G. PFALZ, Ophthalmometrische Untersuchungen über Kornealastigmatismus. *Arch. f. Ophth.* XXXI. 1 1885. S. 201.

[4] W. EISSEN, Hornhautkrümmung bei erhöhtem intraokularem Druck. Ibid. XXXIII. 2. 1888. S. 1.

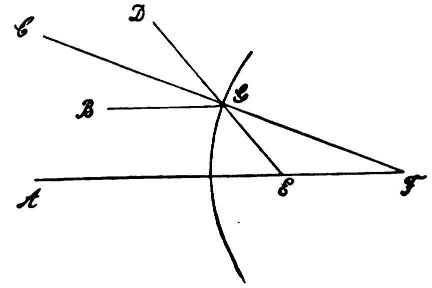

Fig. 128.

after reflection at the cornea, goes to the ophthalmometer in the direction GB. The normal to the cornea at the reflecting point G is CGF, which is inclined to the axis at an angle $\varphi = CFA$. In HELMHOLTZ's ophthalmometer or an instrument with similar "doubling" device the straight lines BG and AF are parallel. Consequently, the angle of incidence is equal to φ and the angle $DEA = 2\varphi$. With other instruments the line BG makes a small angle with the axis, which, however, for the ratio between the distance of the ophthalmometer and the radius of the cornea, and with the limitations in other ways to the precision of the measurements, may be neglected. The same thing is true with respect to the distance of the point E from the cornea. Thus, either the angle 2φ is obtained directly in degrees by measuring the angular distance of the object-point from the axis of the instrument, or tan 2φ is found by dividing the linear distance of this point from the axis by the distance from the cornea of the transversal plane in which the point lies. In this way the angle φ corresponding to a given object-point is ascertained accurately enough. Referring now to Fig. 129, let the ophthalmometric axis again be represented by the straight line AF, and suppose that the light from two different object-points is reflected from the cornea at the two points B, C, at which the normals are BG, CG intersecting in the point G. The smaller the arc BC, the more nearly will the point G coincide with the centre of curvature of this element of the cornea; and ultimately the distance $BG = CG$ will be equal to the radius of curvature ρ. Let D and E designate the feet of the perpendiculars dropped from B and C, respectively, on the straight line drawn through G parallel to the axis; and let the distance of B and C from the axis be denoted by y_1, y_2, respectively; then

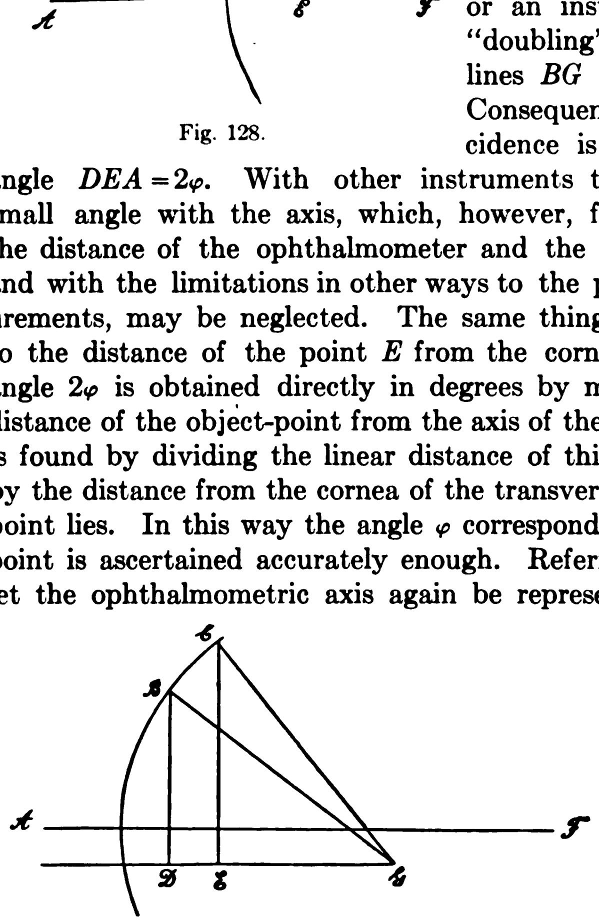

Fig. 129.

$$BD = \rho \sin \varphi_1, \qquad CE = \rho \sin \varphi_2,$$

and, since $CE-BD=y_2-y_1$, the following *general formula for calculating the radius of curvature* is obtained:

$$\rho=\frac{y_2-y_1}{\sin\varphi_2-\sin\varphi_1},$$

where y_2-y_1 is given by the value of the "doubling," and the angles φ_1,φ_2 are known by the positions of the object-points. In the special case, when these points are symmetrically situated with respect to the axis of the ophthalmometer, so that $\varphi_2=-\varphi_1$, the formula may be put in the form given by HELMHOLTZ,[1] namely

$$\rho=\frac{\beta}{2\sin\left(\frac{1}{2}\text{ arc tan }\frac{b}{2a}\right)},$$

where β denotes the value of the "doubling," b the distance apart of the object-points collimated in the reflex image, and a the distance from the cornea of the line joining these points. The smaller the angles are, the more nearly is this formula equivalent to the approximate formula

$$\rho=\frac{2a\beta}{b}\quad\text{or}\quad D=kb,$$

which in the last form, where k denotes the constant of the ophthalmometer, is the fundamental formula in modern ophthalmometry.

In making measurements for different visual directions each new direction must be chosen so that one of the points of the cornea used for reflection in the preceding measurement is utilized in the next determination; as otherwise the measurements are not adapted for calculating the form of the cornea. In this secondary position of fixation a number of radii and corresponding values of the angle φ with respect to the secondary axis are obtained from the measurement. These angles are computed first by adding the amount of rotation of the line of fixation to the angle φ as measured from the ophthalmometric axis of the cornea, and then the values of y with respect to this axis can be found from the general formula. From the relations

$$GD=\rho\cos\varphi_1,\qquad GE=\rho\cos\varphi_2,\qquad GD-GE=x_2-x_1,$$

which are obvious from the diagram (Fig. 129), the values of x may be found by means of the formula

$$x_2-x_1=\rho(\cos\varphi_1-\cos\varphi_2),$$

and thus as the result of the calculation we have obtained the co-

[1] *Handbuch d. phys. Opt.* 2. Aufl. S. 16.

ordinates of the points of the cornea used for reflection, the inclinations of the normals at these points and the radii of curvature of the arcs of the measured normal section that are comprised between them. These data are sufficient, and also necessary, in order to investigate by trigonometrical calculation the influence of the cornea on the aberration.

Heretofore, the exact calculation of the form of a cornea has been performed only by the photographic-ophthalmometric investigation. However, the method is essentially equivalent to a series of corresponding measurements that can be made with the HELMHOLTZ ophthalmometer. JAVAL's ophthalmometer can also be adapted to the same purpose by different methods. A method of this kind which is unobjectionable in principle has been proposed by BRUDZEWSKI[1] and by BASLINI.[2] But the calculation of the results is wrong in the case of both of them; for the former employs a relation between the normal and the radius of curvature that is applicable only to a surface of the second degree, and the latter calculates the values of x by a formula that is true only for a circle. If the form of the cornea is to be calculated by means of relations that are true for an ellipsoid, undoubtedly the best way is to go about it regularly and to calculate the constants of the ellipsoid by the method given by HELMHOLTZ.[3]

The rule stated above for calculating the astigmatism from the amount of "Denivellation" (or difference of level in the double image) in a meridian inclined to the principal section at an angle of 45°, is obtained as follows. One of the principal sections of the cornea having been ascertained, the arc of the ophthalmometer is turned through half a right angle, and then the collimation is made, and the "Denivellation" compensated by adjusting one of the *mires* in the direction perpendicular to the plane of "doubling." The collimated points are then the extremities of a line whose optical projection at the focus of the ophthalmometer lies in the plane of doubling. With the degree of accuracy that can be obtained by the instrument, it is permissible to neglect the interval between this focus and the two focal points of the bundle of reflected rays that do not lie exactly in the same plane. Hence, as follows from the formula on page 290, the projection-coefficients for the two principal sections are equal to the corresponding magnification-ratios; and these latter are to each other inversely as the refracting powers in the two principal sections,

[1] K. v. BRUDZEWSKI, Beitrag zur Dioptrik des Auges. *Arch. f. Augenheilk.* XL. 1900. S. 296.

[2] C. BASLINI, Recherches ophthalmométriques. *Arch. d'ophth.* XXIV. 1904. p. 565.

[3] *Handbuch der phys. Optik*, 2. Aufl. S. 17.

since

$$K_1D_1 = K_2D_2 = L,$$

where K and D denote the magnification-ratio and refracting power, respectively, and L denotes the convergence of the bundle of incident rays with respect to the focus of the reflecting surface of the cornea. The tangent of the angle between the projected line and the first principal section is equal to $\frac{K_2}{K_1}\tan\omega$, where ω denotes the angle between the corresponding object-line and the same principal section. Since the tangent of the former angle is equal to unity, it follows that $\tan\omega = \frac{D_2}{D_1}$, and hence

$$\tan(\omega - 45^0) = \frac{D_2 - D_1}{D_2 + D_1} = \frac{c}{b},$$

where c denotes the adjustment that has to be made to compensate the "Denivellation", and b denotes the length of the projected object line as found by the collimation. By EULER's theorem, employing the constant of the ophthalmometer, we may write: $\frac{1}{2}(D_1 + D_2) = kb$ and thus we obtain finally the following mathematical statement of the rule above mentioned:

$$D_2 - D_1 = 2kc.$$

This formula is thus approximate to the same extent as those generally employed in modern ophthalmometry. The errors involved therein are, however, of no consequence in the measurement of astigmatism and do not need to be taken into account until it is necessary to calculate the absolute value of the refraction of the cornea.

The corneal substance. Disregarding the layer of fluid over the anterior surface of the cornea, which in the dioptrics of the eye can be considered as an infinitely thin film bounded by concentric surfaces, and consequently without influence on the ray-procedure; the corneal tissue constitutes the first ocular refracting medium. After ABBE's modern refractometer method had been introduced, the *index of refraction of the cornea* was measured by AUBERT and MATTHIESSEN[1] and found to be 1.377 for the eye of a man fifty years of age, and 1.3721 for that of a child two days old. LOHNSTEIN[2] obtained by calculation from the indices of refraction of the component parts a value intermediate between the two just mentioned. MATTHIESSEN's[3] final

[1] H. AUBERT, *Grundzüge der physiologischen Optik*. Leipzig 1876.

[2] TH. LOHNSTEIN, Über den Brechungsindex der menschlichen Hornhaut. *Arch. f. d. ges. Physiologie*. LXVI. 1897.

[3] L. MATTHIESSEN, *Die neueren Fortschritte in unserer Kenntnis von dem optischen Baue des Auges der Wirbeltiere*. Hamburg 1891.

comparison of results gave the value 1.3763, and since the fourth decimal is always doubtful, 1.376 may be taken as the best schematic value.

Heretofore, no one except BLIX[1] has measured *the thickness of the cornea* in the living eye by an unobjectionable method. His ophthalmometer consists of two microscopes, T, T_1 arranged according to the plan of Fig. 130 with their objectives at O, O_1. In place of the ocular of the second microscope there is a brightly illuminated diaphragm b whose image is produced at the point where the two axes meet. The observer looks through the other microscope and focuses it on this point by means of the cross-hairs at b; and hence he cannot see a sharp image of the diaphragm unless the apparent place either of the principal point or of the centre of curvature of a reflecting surface is at the point of junction of the axes. The microscopes may be adjusted in two ways, partly by moving them along the bisector of the angle between their axes without disturbing their mutual relation to each other, and partly by moving them simultaneously and equally along their axes without changing the point of junction. The latter mechanism is used for measuring a radius of curvature, and the former for measuring the interval between two reflecting surfaces. Thus, the thickness of the cornea is measured by two successive adjustments in which the reflex image, first, in the anterior, and then in the posterior, surface of the cornea is sharply focused. The displacement of the instrument is equal then to the apparent thickness, and the actual thickness is obtained by exact calculations without using approximate formulae. With ten eyes BLIX found thicknesses varying from 0.482 to 0.668 mm. Leaving out the eyes which had these extreme values, the limits were 0.506 and 0.576. These measurements corresponded partly to the vertex as defined by the minimum radius of the cornea, partly to the ophthalmometric axial point, and partly to points that were 20° inwards and outwards from the points first named.

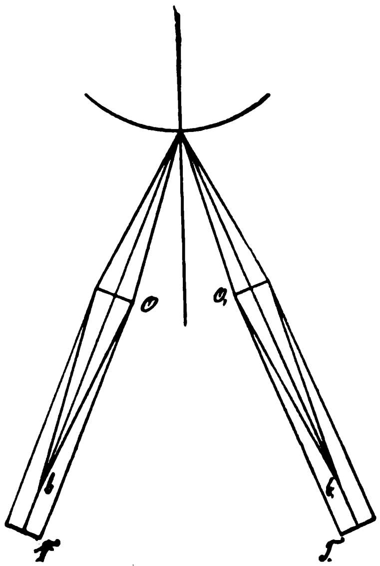

Fig. 130.

[1] Loc. cit.

The fact that Blix was able without difficulty to observe the reflex image in the posterior surface of the cornea and utilize it in measuring the thickness of the cornea, whereas Helmholtz[1] had tried in vain to see this image, was due to the high magnifying power of the microscope, which was necessary in order to see the image separate from the more luminous image in the anterior surface. With the methods of illumination available nowadays there are no difficulties at all about obtaining reflex images from the posterior surface of such clarity that the thickness of the cornea can be determined by the same method as is used for the thickness of the lens. In order to get reflex images as sharp and bright as possible, a source of light must be used with the greatest possible specific intensity. On account of its convenience of adjustment, the incandescent filament of the Nernst lamp (see p. 470) is the only kind of light to be considered, since neither sunlight nor the electric arclight is satisfactory for the purpose, and all other sources of light have much less specific intensity. The lamp should be attached at one end of a closed tube, which has an adjustable slit at the other end, and in the middle of which there is a lens system that can be decentered at pleasure. This lens system is for the purpose of focusing a sharp image of the filament of the lamp on the rear side of the slit. With the slit wide open and the eye protected by a dark glass, the sharp focusing is found by parallax displacements of the eye. By rotating and moving the lamp, and also by decentering the lens system in a direction perpendicular to the length of the slit, it is not difficult to focus the image of the filament right in the middle of the aperture of the slit. The slit mechanism should be the so-called bilateral arrangement with both edges simultaneously adjustable. The tube as a whole should be capable of rotation around its axis and attached to a stand. This source of light gives a luminous line of variable intensity, which in its specific intensity is far superior to all other sources of light that can be conveniently employed for ophthalmometry. For brevity we shall call it hereafter the ophthalmometric Nernst lamp.

For measuring the thickness of the cornea by the method given by Helmholtz as illustrated in Fig. 50, the writer uses two of these lamps with vertical slits placed one exactly above the other in such fashion that the horizontal plane through the vertex of the cornea passes midway between the two slits. With proper intensity of light and a good telescope magnifying twenty times, the reflex images even at the thinnest place are extraordinarily clear when the angle of incidence is about 25°. A small incandescent lamp with a straight vertical filament was used for the source of light whose reflex image in

[1] Über die Akkommodation des Auges. *Arch. f. Ophth.* I, 2. 1855. S. 1.

the anterior surface of the cornea was to be collimated with the reflex image of the ophthalmometric NERNST lamp in the posterior surface. The angle was measured with an instrument similar to a theodolite fastened above the head of the patient, its vertical axis ending downwards in a point. The precise orientation of the vertex of the cornea on the prolongation of this axis was achieved by focusing the telescope, which was capable of rotation around a horizontal axis, first, on this point, and then on the image reflected in the cornea.

The experiment is easy to make when the angular distance vertically between the middle points of the two slits is 12°, in which case the element of the cornea that is used is so small that the imaginary reflex image of the vertical line determined by the two ends of the slit can be regarded as a straight line. The ophthalmometric NERNST lamps and the telescope were adjusted each at an angle of 25° from the zero position of the telescope of the theodolite, and by a preliminary trial the direction of fixation was found for which the angle of incidence was practically the same when the positions of lamp and telescope were interchanged, because with this adjustment the optical axis of the eye does not differ much from the zero position of the telescope of the theodolite. The basis of the calculations was the average results of measurements made on different days in both adjustments. By this arrangement the reflex image in the posterior surface is seen at the place where the normal to the anterior surface crosses the posterior surface, and the calculation is very simple. Since this normal is common to both surfaces, it is the line or axis of centres of the cornea, and since it coincides with the zero position of the theodolite, the measurement gives the angular distance ω of the lamp from this line, and the angle of incidence is $\frac{1}{2}$ $(25° + \omega)$. The procedure of the rays is the same as in Fig. 127 where the posterior surface of the cornea takes the place of the pupil of the eye. Having first calculated the angle of refraction i', we can find the angle u', between the line of centres and the ray refracted at the anterior surface, and hence the thickness d for the given value of the radius ρ of the anterior surface, by means of the following formulae:

$$25^0 - i = u' - i' \qquad \sin u' : \sin i' = \rho : (\rho - d).$$

The eyes of two individuals were measured by the writer with the utmost pains, and the values 0.46 and 0.51 were obtained; corroborating very exactly the results found by BLIX. After BLIX had observed the image reflected in the posterior surface of the cornea, TSCHERNING[1] succeeded in making it visible with a small incandescent

[1] *Optique physiologique.* Paris 1898.

lamp. In one case also he tried to measure the thickness of the cornea, and obtained a result of 1.15 mm. This may have been due to errors in his method, which was not nearly so accurate, and to the small specific intensity of this source of light. From measurements in the living eye, *the schematic value of the thickness of the cornea* in the optical zone may be taken therefore, as BLIX proposed, as being about 0.5 mm.

Measurements with dead eyes have given very discrepant results, the values for the vertex varying between 0.4 and 1.0,[1] and sometimes even exceeding this latter value. This may be due in part to a *post mortem* swelling, and partly also to the method of measurement. In some instances the writer has removed entirely the healthy cornea of a freshly enucleated eye and measured the thinnest place in it with the ordinary micrometer screw used for measuring thicknesses, with the contact surfaces reduced to a diameter of 1/2 mm. The values obtained in this way were between 0.4 and 0.6 mm.

By using the arrangement above described, *the radius of the posterior surface* can be measured in the same way as the radii of the lens surfaces, whereas a direct measurement with the ophthalmometer does not prove satisfactory. With the HELMHOLTZ type of instrument the writer finds that generally he cannot see the reflex images in the posterior surfaces of the cornea, because they are masked by the rays that are diffusely refracted at the edge of the plates. The slits are adjusted horizontally, and the reflex images of the straight horizontal filaments of two incandescent lamps are so focused that each reflex image is the prolongation of one of the reflex images in the posterior surface of the cornea due to the ophthalmometric NERNST lamp. The object represented by the vertical distance between the two incandescent lamps is reproduced then by an image reflected in the anterior surface of the cornea which is equal to the reflex image in the posterior surface corresponding to the vertical distance between the two slits. Just as in measuring the thickness, the fixation mark is so adjusted that the line of centres of the two surfaces of the cornea coincides with the zero position of the telescope of the theodolite, and the angular distances of the observing telescope and NERNST lamps are equal. What has to be measured are the sizes and distances of the two objects, together with their angular distances from the line of centres; these magnitudes for the NERNST lamps and the incandescent lamps being denoted here by b, a, u and b_0, a_0, u_0, respectively. Since for the reflection at the anterior surface of the cornea the angle of

[1] MERKEL in *Handb. d. ges. Augenheilk.* v. GRAEFE u. SÄMISCH. I. Leipzig 1874. S. 44-45.

incidence is equal to u or $\frac{1}{2}(u+u_0)$, the ratio ϵ of the two reflex images β, β_0 is found by the relation

$$\frac{\beta_0}{\beta}=\frac{ab_0 \cos u}{a_0 b \cos \frac{1}{2}(u+u_0)},$$

which follows from the general formula $K=\frac{L}{D}$ for the secondary imagery, provided the distances of the object are reckoned from the principal focus of the corneal mirror. But the ratio of the sizes of the two reflex images of the object represented by the NERNST lamps is equal to the inverse ratio of the reflecting powers of the two reflecting systems in question, since for both reflections the value of L here may be considered the same without sensible error. The reflecting power of the anterior surface of the cornea with respect to the secondary imagery is

$$-\frac{2 \cos u}{\rho_1},$$

where ρ_1 denotes the radius of curvature of this surface in the vertical section; whereas the power of the reflecting system which produces the reflex image in the posterior surface of the cornea, as found by the formula on page 285 is

$$2D_1(1-\delta_1 D_1)+D_2(1-\delta_1 D_1)^2,$$

where D_1 denotes the refracting power of the anterior surface of the cornea with respect to the secondary imagery, and D_2 denotes the reflecting power of the posterior surface for the same imagery. If the radius of curvature of the vertical section of the posterior surface is denoted by ρ_2, and if n denotes the index of refraction of the corneal substance, then

$$D_1=\frac{n \cos i' - \cos i}{\rho_1}, \qquad D_2=-\frac{2n \cos u'}{\rho_2};$$

the angles being denoted just as above in the determination of the thickness of the cornea.

If the point of incidence on the posterior surface of the cornea is designated by P in Fig. 127, then P' will be the first principal point of the reflecting system with respect to the secondary imagery; and δ_1 denotes the reduced distance of P from the point of incidence on the anterior surface, whereas H denotes the distance of P' from this point of incidence. From the formula on page 285 we get

$$1-\delta_1 D_1=\frac{\delta_1}{H},$$

and since in Fig. 127 evidently

$$n\delta_1 : H = \sin u : \sin u',$$

therefore
$$1-\delta_1 D_1=\frac{\sin u}{n\sin u'}\cdot$$

Hence, putting ϵ equal to the ratio of the dimensions of the two reflex images of the NERNST lamps, we obtain

$$\epsilon=-\frac{2\cos u}{\rho_1}:\frac{2\sin u}{n\sin u'}\left\{\frac{n\cos i'-\cos i}{\rho_1}-\frac{\sin u}{\rho_2\tan u'}\right\}\cdot$$

From the final formula

$$\frac{\rho_1}{\rho_2}=\frac{\tan u'\,(n\cos i'-\cos i)}{\sin u}+\frac{n\sin u'\tan u'}{\epsilon\sin u\tan u},$$

where $\epsilon=\dfrac{\beta_0}{\beta}$ as found by measurement, the writer has obtained the values 1.1822 and 1.1811 in the two cases mentioned above. On the assumption that the ratio of the radii of curvature of the horizontal sections of the anterior and posterior surfaces is the same, these values give, for the schematic radius 7.8 mm of the anterior surface, the value 6.6 mm for the radius of the posterior surface.

These investigations with reflex images in the posterior surface were also carried out by the writer for angles of incidence up to 40°, in which case the observation is more easily made. With a little practice, however, the observation is sufficiently successful for the given value $u=25°$. Smaller angles are better because the asymmetry values, on which the lack of similarity between object and image depends, become greater with increasing angles of incidence, tending therefore to vitiate the validity of the imagery-laws of the first order, although the formula is exact for any value of the angle of incidence. After the observer's eye has become sufficiently adapted to the room illuminated only by the ophthalmometric NERNST lamp, the reflex images are seen instantly as soon as the objective of the telescope is fixated; and whenever they become indistinct, it is a good plan each time to make the patient look again in this direction, and then turn his eye gradually towards the fixation mark. The correct adjustment of the incandescent lamp can be regulated best by suddenly shutting off the current and then focusing it at the instant of re-illumination.

A vast amount of work is involved in investigating the line of centres, and the writer did not have time to make the complete measurements and computations except in the two cases above mentioned. In both instances the good agreement of the results not only with each other but also with the very accurate researches of BLIX so far as the thickness is concerned, and with TSCHERNING's[1]

[1] LAGRANGE et VALUDE, *Encyclopédie française d'ophthalmologie.* III. p. 109. Paris 1904.

latest value of 6.5 mm for the radius of the posterior surface, would perhaps seem to indicate their adequacy also. However, owing to the discrepancy that exists between too small a schematic value of the corneal refraction and the results of anatomical measurements of the length of the eyeball—a matter to be considered presently in detail—further investigations were deemed desirable. The writer therefore has measured four other eyes of different individuals by an approximate method, assuming the line of centres to be a line inclined to the line of sight (*Visierlinie*) at an angle of 6°, and using in the calculation the value of the angle as obtained for the case when the thickness of the cornea was found to be 0.46 mm. Now a discussion of the formulae proves that for a given value of ϵ the ratio $\frac{\rho_1}{\rho_2}$ will be greater in proportion as the cornea is thicker; therefore the value of the radius of the posterior surface found by this approximate method cannot be too small. In this way the following values of $\frac{\rho_1}{\rho_2}$ were found: 1.1864, 1.1734, 1.1486, and 1.1427. For the schematic value 7.8 mm for the radius of the anterior surface, these numbers give values of the radius of the posterior surface between 6.57 and 6.83 mm. Hence, *the ophthalmometric mean value of the radius of the posterior surface of the optical zone of the cornea* can hardly be greater than 6.7 mm, which is the value that is hereafter assumed by the writer.

However, the ophthalmometric mean values of the radii of curvature of the optical zone cannot be used without further consideration for *calculating the ray procedure* by means of the imagery laws of the first order. In a calculation of this kind the curvatures that are involved are those at places where the line of sight (*Visierlinie*) is incident on the temporal side of the ophthalmometric axial point. The radius of the anterior surface must needs be somewhat less at such a point than the ophthalmometric mean value for the whole zone—all the more so because the latter value was found by measurements in which the one point of the cornea used for reflection is at the edge of the optical zone and therefore has probably turned out rather too large. On the other hand, on account of the greater flattening of the anterior surface in the vertical section, the ratio $\frac{\rho_1}{\rho_2}$ may be here somewhat larger than in the horizontal section; and, besides, it must prove to be greater in case the measurement was not made exactly along the line of centres. Therefore, in calculating the ray procedure a radius of the anterior surface should be used that is somewhat smaller, and a radius of the posterior surface that is somewhat larger, than the

schematic values of these magnitudes for the optical zone. Since the difference at the point of incidence of the line of sight and at the vertex of the optical zone cannot be calculated, there is no other alternative in calculating the ray procedure except to identify the curvatures at these two points in advance. For the reasons assigned, the writer assumes for *the schematic values of the radii of curvature at the vertices of the optical zone of the cornea in this calculation the data* 7.7 *and* 6.8 mm.

The index of refraction of the aqueous humor must be known in order to ascertain the cornea system. Previously, the value given by HELMHOLTZ (p. 107) has been quite generally accepted and the numerous refractometer measurements which have been made since (which have been collected by FREYTAG[1]) show scarcely greater variations from this value than are shown by the variations of the values in the case of distilled water. Although more recent investigations tend to give rather lower values, there hardly appears to be any sufficient reason as yet for modifying HELMHOLTZ's schematic value beyond merely discarding the figure in the fourth decimal place as not being at all certain. Similarly, while recent measurements of the index of refraction of the vitreous humor have given a value which in the fourth decimal place is one or two units less than that of the aqueous humor, it may be regarded as practically identical with the latter. The writer assumes therefore for both indices the *schematic value* 1.336.

The constants of the cornea system. If the radii and thickness of the cornea are denoted by ρ_1, ρ_2 and d, respectively (all expressed in metres), and if the indices of refraction of the cornea and the aqueous humor are denoted by n_1, n_2, then from the general formulae for the combination of two optical systems, namely,

$$D_c = D_1 + D_2 - \delta D_1 D_2\,, \qquad H_c = \frac{\delta D_2}{D_c}, \qquad H_c' = -\frac{\delta D_1}{D_c},$$

where

$$D_1 = \frac{n_1 - 1}{\rho_1}, \qquad D_2 = \frac{n_2 - n_1}{\rho_2}, \qquad \delta = \frac{d}{n_1},$$

the following numerical results may now be calculated:

Refracting power D_c . . . . = 43.053 dptr
Position of the first principal point $1000\ H_c$ . . . . = -0.0496 mm
Position of the second principal point
$1000\ (d + n_2 H_c')$ . . . . = -0.0506 mm

[1] G. FREYTAG, *Vergleichende Untersuchungen über die Brechungsindizes der Linse und der flüssigen Augenmedien des Menschen und höherer Tiere in verschiedenen Lebensaltern.* Wiesbaden 1907.

where the vertex of the anterior surface of the cornea is taken as the origin or zero point. The values of the focal lengths are 23.227 and 31.031 mm.

2. The Lens

Unless a BLIX ophthalmometer is available, HELMHOLTZ's method of ascertaining *the positions of the surfaces of the crystalline lens* is still today the best method. The lamps and fixation mark may be connected with a rectilinear scale (HELMHOLTZ[1]) or with a graduated arc (TSCHERNING[2]); or the angles can be read off as in the previous investigation of the posterior surface of the cornea. The advantage of the latter arrangement is that the ophthalmometric NERNST lamps can be used, which are not convenient for adjustment along a straight or curved scale. (TSCHERNING calls the instrument he uses for this purpose an "ophthalmophakometer.")

For measuring the depth of the anterior chamber of the eye, DONDERS[3] uses a so-called corneal microscope which was focused first on the anterior surface of the cornea (made visible with calomel so as to get the exact focus) and then on the edge of the pupil; the apparent position of the pupil being thus found by the change of focus. Perhaps more reliable results can be obtained by the method of MANDELSTAM and SCHÖLER[4] worked out under HELMHOLTZ's supervision, which was used by REICH.[5] An unsilvered plate of glass acting as a mirror is interposed between the microscope and the cornea so as to reflect the light along the axis of the microscope into the eye. The image of the lamp reflected in the anterior surface of the cornea can be shifted by optical means until it is sharply in line with the edge of the pupil. The position of the reflex image as found by calculation will be the apparent place of the pupil. In these methods, as also in that of HELMHOLTZ, what is measured is the distance of the edge of the iris from the anterior surface of the cornea.

Doubtless, BLIX's method described above, which to a certain extent is connected with the one just mentioned, is the most accurate that has been used at all. It determines the distance of the anterior surface of the lens. This distance can be obtained also by HELMHOLTZ's method for measuring the distance of the posterior surface of the lens, as this is a general method of finding the apparent position of a re-

[1] *Handbuch d. phys. Opt.* 2. Aufl. S. 103.

[2] Loc. cit.

[3] Instrument pour mesurer la profondeur de la chambre antérieure et la courbure de la cornée. *Congrès de Londres. Compte rendu.* 1872. p. 209.

[4] L. MANDELSTAM und H. SCHÖLER, Eine neue Methode zur Bestimmung der optischen Konstanten des Auges. *Arch. f. Ophth.* XVIII, 1. 1872. S. 155.

[5] M. REICH, Resultate einiger ophthalmometrischer und mikrooptometrischer Messungen. Ibid. XX, 1. 1874. S. 207.

flecting surface. However, it should be noted that, although this plan gives exceedingly accurate results for the measurement of the thickness of the cornea owing to its smallness, the determination of the position of the surface of the lens is quite a different matter, because the distance between the points where the ray enters and leaves the cornea is too considerable to permit the intervening piece of the cornea to be regarded as the portion of a sphere. Perhaps, therefore, for such measurements as these we ought to calculate an osculating ellipse based on the radii as measured at these places and at the point where the axis used in the measurement meets the cornea; and to employ this curve for trigonometrical calculation of the position of the surface in question. On the other hand, the errors due to neglecting the refraction at the posterior surface of the cornea are of quite secondary importance. In these measurements TSCHERNING adjusts two lamps in a plane which contains the axis of the telescope and determines the axis of the system by observing the reflex images of the lamps in the cornea and in the given surface of the lens, which should appear to be all in a straight line. In the calculation the cornea is regarded as spherical, apparently without observing HELMHOLTZ's precaution of repeating the measurement by changing the direction of the incident light without altering the angle of incidence. Concerning this method, therefore, perhaps it should be said that, while the investigation is certainly much simpler, the results are not so reliable.

There are two other methods of investigating the depth of the anterior chamber, mainly intended for clinical use, namely, that of HEGG[1] with a stereoscopic instrument and variable pointer and that of GRÖNHOLM[2] with CZERMAK's orthoscope, which latter is merely for approximate measurements.

The following values of the distance between the pupillary plane and the vertex of the cornea were obtained by the original method of HELMHOLTZ:

HELMHOLTZ [See §3.]	4.024	3.597	3.739	
KNAPP[3]	3.692	3.707	3.477	3.579
ADAMÜK and WOINOW[4]	3.998	3.237	2.900	3.633

[1] E. HEGG, Eine neue Methode zur Messung der Tiefe der vorderen Augenkammer. *Arch. f. Augenheilkunde.* XLIV. Erg.-Heft. 1901. S. 84.

[2] V. GRÖNHOLM, Eine einfache Methode die Tiefe der vorderen Augenkammer zu messen. *Skand. Arch. f. Physiologie.* XIV. 1903. S. 235.

[3] J. H. KNAPP, Über die Lage und Krümmung der Oberflächen der menschlichen Kristallinse und den Einfluss ihrer Veränderungen bei der Akkommodation auf die Dioptrik des Auges. *Arch. f. Ophth.* VI, 2. 1860. S. 1.

[4] E. ADAMÜK und M. WOINOW, Zur Frage über die Akkommodation der Presbyopen. Ibid. XVI, 1. 1870. S. 141.

whereas v. REUSS[1] consistently got smaller values which evidently cannot be taken into account along with the above. The mean of the preceding is 3.598. In two instances values of 3.921 and 3.651 were found by the method of MANDELSTAM and SCHÖLER; and for three different individuals values of 3.639, 3.708 and 3.652 were obtained by REICH; so that this method gives a mean value of 3.714 mm. The emmetropic eyes investigated by BLIX gave a mean value of 3.515, but it should be noted that only one of the five eyes was measured at the vertex of the cornea, all the others being measured at the point where the line of sight (*Visierlinie*) meets the cornea, so that the true mean value must be a little larger, and probably is more nearly equal to that found by HELMHOLTZ's method.

The results of these investigations justify the assumption of *the schematic value of* 3.6 mm *for the distance between the anterior surfaces of the cornea and crystalline lens.* This was the value that HELMHOLTZ[2] took in his schematic eye, and it has been quite generally adopted. The mean values obtained by STADFELDT[3] and AWERBACH[4] by TSCHERNING's approximate method are: STADFELDT from measurements of 10 eyes, 3.81; AWERBACH, for 15 emmetropic eyes, 3.4; for 28 hypermetropic eyes, 3.5; and for 43 myopic eyes, 3.6.

The determination of the position of the posterior surface of the lens gives its *thickness*. The results of measurements made under HELMHOLTZ's supervision are:

HELMHOLTZ[5] [See p. 112.]	3.414	3.801	3.555	
KNAPP	3.920	3.848	3.776	3.662
ADAMÜK and WOINOW	3.202	3.963	3.944	3.567

the mean value being 3.692 mm. The results of MANDELSTAM and SCHÖLER together with those of REICH give the mean value 3.787 mm; the mean of STADFELDT's measurements is 3.63; and AWERBACH obtained 3.89, 3.94 and 3.88 for emmetropia, hypermetropia and myopia, respectively.

The value that HELMHOLTZ took for his schematic eye was 3.6 mm. According to the numbers given above, on the assumption that the methods of measurements were absolutely accurate, this value should be from 0.1 to 0.2 mm greater. But if the sources of error are taken into account, depending on the asymmetrical flattening of the cornea,

[1] A. v. REUSS, Untersuchungen über die optischen Konstanten ametropischer Augen. Ibid. XXIII, 4. 1877. S. 183.

[2] *Handbuch d. phys. Opt.* 2. Aufl.

[3] A. STADFELDT, *Den menneskelige linses optiske konstanter.* Kopenhagen 1898.

[4] M. AWERBACH, Zur Dioptrik der Augen bei verschiedenen Refraktionen. (Russian.) *Inaug.-Diss. Moskau* 1900. Reviewed in *Jahresber. ü. d. Leist. u. Fortschr. i. G. d. Ophthalmologie.* XXXI. S. 652.

on the unknown form of the anterior surface of the lens, and on using a total index for the index of refraction of the lens, there can be no doubt that HELMHOLTZ's number is within the possible limits of error. Moreover, the lens gets larger and larger as years go on, and according to all measurements of the dead lens that have been made hitherto the thickness increases but never decreases. Also the difference between the thickness of the unaccommodated lens in the living eye and that of the dead lens is found to be less with advancing years. For these reasons, and finally, because, owing to various individual variations of the lens substance that begin to be very manifest soon after the age of youth, the schematic value should apply to a youthful eye. The latter should be taken rather less than the mean value. Hence, the writer concludes that with the present available material of investigation there is no good ground for changing *the schematic value of the thickness of the unaccommodated lens* as assumed by HELMHOLTZ, that is, 3.6 mm.

Curvatures of the Surfaces of the Lens. The same conclusion has been reached also concerning the radii of curvature of the two surfaces of the lens. The variations between the mean values of these dimensions and the schematic values as assumed by HELMHOLTZ are comprised within the limits of the errors that are involved in the methods of measurement. It would seem therefore to be unnecessary to give here the actual figures. It is sufficient to say that investigations by TSCHERNING's method have yielded similar results, mean values according to STADFELDT being 10.9 and 6.0 mm, and according to AWERBACH 10.4 and 6.1 mm. The schematic values of the radii of curvature of the surfaces of the lens as used by HELMHOLTZ are 10 and 6 mm; and apparently they are still the best values to take for this purpose.

In the earlier measurements of the curvatures of the surfaces of the lens, sometimes HELMHOLTZ's original method was employed and sometimes also the reflex images, obtained by using sunlight,[1] or the DRUMMOND lime-light,[2] were measured directly with a HELMHOLTZ ophthalmometer. The ophthalmometric NERNST lamps are peculiarly adapted for the more recent methods. For an accurate computation of the radius from the results found by measurements, all that is necessary is to use the method given above for finding the curvature of the posterior surface of the cornea. The formula given by

[1] B. ROSOW, Zur Ophthalmometrie. A. d. physiol. Labor. des Herrn Prof. HELMHOLTZ. *Arch. f. Ophth.* XI, 2. S. 129.

[2] v. REUSS, loc cit.

HELMHOLTZ,[1] which is applicable for infinitesimal angles of incidence, enables us to obtain an approximate value. The writer's own calculations as performed by the first of these methods, like those of other investigators, indicate no reason for changing HELMHOLTZ's schematic values of the distance or curvatures of the surfaces of the lens.

The measurements of SAUNTE[2] made in TSCHERNING's laboratory are peculiar. Diffused light obtained from an electric arc lamp was employed in a quite complicated arrangement of apparatus. Since there was an angle between the incident light and the ophthalmometric axis just as in the method of measuring the posterior surface of the cornea described above, SAUNTE called his method a "decentrated ophthalmometry." But in spite of the oblique incidence, formulae are used in the calculation that are applicable only for perpendicular incidence, and hence the results are not easily assessed.

TSCHERNING's method as used by STADFELDT and AWERBACH amounts to finding the apparent position of the centre of curvature in a manner similar to that for finding the place of the surface. Thus, a line of centres having been determined, the light is allowed to enter the eye in the direction of the ophthalmometric axis, at a certain angle with the line of centres; and all that is necessary is to measure the angle of incidence and calculate by trigonometry the position of the centre of curvature. However, fairly large angles are required, and in the calculation the surfaces are regarded as spherical, and hence only approximate values can be found in this way, with errors that cannot be estimated. TSCHERNING himself has pointed out the uncertainty in the results due to the feeble specific intensity of the light-source used in the experiment, and intimated that the method was not free from objections. Perhaps, therefore, the differences between the values obtained by STADFELDT and AWERBACH and the schematic values are within the limits of possible errors.

The form of the surfaces of the lens in the living eye has been minutely investigated by BESIO[3] by TSCHERNING's method. He discovered such a pronounced flattening towards the periphery that it certainly cannot be accounted for as due to possible error. On the other hand, on account of the approximate method of calculation, the results in a quantitative sense are probably less trustworthy. According to him, the osculating surface of the second degree was hyperbolic for the anterior surface and parabolic for the posterior surface of the lens. By investigations of the dead lens, with proper

[1] *Handbuch*, 2. Aufl. S. 144.

[2] O. H. SAUNTE, *Linsemaalinger (Linsenmessungen)*. Odense 1905.

[3] E. BESIO, La forme du cristallin humain. *Journal de physiologie et de pathologie générale*. III. 1901. pp. 547, 761, 783.

precautions and use of exact methods of calculation, the fact of the flattening towards the periphery was positively confirmed by DALÉN,[1] although with respect to the anterior surface of the lens HOLTH[2] had obtained a contrary result.

In experimenting with the reflex image in the anterior surface of the lens it has been frequently noticed that for slight movements of the patient's eye the image executes slight movements also but with remarkably high speed, as if there were hollows or furrows on the surface. However, as the reflected light is not due entirely to the anterior surface of the lens, as is indicated by the diffused form of the reflex image, but is reflected partly also from the adjacent layers, it is not possible to draw any conclusions from this phenomenon as to the form of the surface itself.

The substance of the lens, as shown by refractometric investigations, consists of a medium of variable index of refraction. Physiological measurements indicate very clearly that during childhood and until puberty the variation of the index is continuous throughout the lens. However, towards the end of the second decade of life, perhaps usually a little later, light reflexes begin to be manifest; implying a discontinuity in the variation of the index of refraction. With movements of the source of light, these granular appearances or "*Kernbildchen*" (as they were called by HESS[3] who described them first in the eyes of human beings) behave as if they had their origin in a continuous surface. Consequently, in order to investigate the dioptrics of the lens, we need to know the laws of optical imagery not simply for media of continuously variable index of refraction, but also when there are such discontinuities in the variation of the index as may be considered as amounting to surfaces separating two different media of variable index of refraction. Now while we do not know enough about these surfaces of discontinuity[4] to apply to them the laws above mentioned, at the same time, so far as the imagery produced by the surfaces of the lens is concerned, these laws are absolutely requisite, because the ordinary formulae are not applicable to this imagery when the light goes obliquely through the lens. The problem was attacked by

[1] ALBIN DALÉN, Ophthalmometrische Messungen an der toten menschlichen Kristallinse. *Mitt. a. d. Augenklinik d. Carol. Med. Chir. Inst. z. Stockholm*, 1905.

[2] S. HOLTH, Études ophthalmométriques sur l'œil humain après la mort. IX. *Congr. intern. d'ophth. d'Utrecht. Compte rendu.* 1899. S. 386.

[3] C. HESS, Über Linsenbildchen, die durch Spiegelung am Kerne der normalen Linse entstehen. *Arch. f. Augenheilk.* LI. 1905. p. 375.

[4] The idea of discontinuity relates here solely to the mathematical function of the variation of the index and not at all to the manner in which the space is occupied by the substance of the lens.

L. HERMANN[1] and L. MATTHIESSEN,[2] who derived differential equations for the ray-convergence from the formulae for homogeneous media by proceeding to the limit. Now this method is not justifiable unless the same mathematical process has not been employed already in developing the original formulae; and consequently, the differential equations thus obtained for the ray-convergence in a meridian section and for the aberration are not correct. Moreover, these investigations did not include the magnification-ratios. Hence all that was known of the required laws was the differential equation for the ray-convergence along the axis of a system of revolution, as obtained first by LIPPICH,[3] and the differential equation for the ray-convergence in the equatorial section, as derived by HERMANN for an optical system of concentric layers, and by MATTHIESSEN for a system of revolution. In addition, an approximate law of the increase of the index of refraction as found by MATTHIESSEN[4] has been used for integration, just as if it were mathematically exact, leading therefore again to erroneous results.

In obtaining the laws in question, it is evident that the general fundamental equation of optical imagery, and the general laws derived from it (p. 270) for any optical system whatsoever, cannot be employed unless the variation of the index of refraction of the media in it is continuous. When there is a plane of symmetry, the differential equations are simplified and are applicable then to the meridian plane of a system of revolution. In this case we obtain by integration the same equations as were found for homogeneous media, namely,

$$\kappa^2 B = A + \kappa D, \qquad \kappa K B = A,$$

where the refracting power is given by a definite integral corresponding to the summation formula. In addition to the complete laws of the first order, the investigation has led also to formulae for calculating the aberration. At the surface of the lens where the light passes between two media one of which has a variable index of refraction, the refracting power has the same value for the second imagery as if the two media were homogeneous, but not for the first imagery; and, moreover, the formula for calculating the imagery along the axis is different.

It appears from a study of the ray-convergence in the eye for the case of a point-source of light that the wave surface of the bundle

[1] L. HERMANN, Über Brechung bei schiefer Inzidenz mit besonderer Berücksichtigung des Auges. *Arch. f. d. ges. Physiol.* XXVII. 1882. S. 291.

[2] L. MATTHIESSEN, Über den schiefen Durchgang unendlich dünner Strahlenbündel durch die Kristallinse des Auges. Ibid. XXXII. 1883. S. 97.

[3] In a review in *Zeitschr. f. Math. u. Phys.* XXIII. 1878. Hist.-Lit. Abt. S. 63.

[4] *Grundriss der Dioptrik geschichteter Linsensysteme.* Leipzig. 1877.

of refracted rays in the eye is not a surface of revolution at all, but may be considered as having complete contact of the fourth order with such a surface. The star figures seen as radiating lines around the luminous point are due to a definite peculiarity of the wave surface characterized by a wave-like procedure of the line in which the wave surface is cut by a co-axial cylinder. Since the genesis of this form of wave surface has been shown (p. 192) to be due to the passage of light through the lens, the necessary mathematical consequence is that either the lens-surfaces themselves or the *iso-indicial surfaces* of the substance of the lens must have such a form. These latter surfaces, each of which is the locus of points where the index of refraction has one and the same value, constitute therefore a system of surfaces having complete contact of the fourth order with a system of revolution; and consequently it is this system which must be the basis of the investigation.

If the lens is supposed to be composed of a medium of continuously variable index of refraction, this is not perfectly true except for the youthful lens which shows no sign of discontinuity; whereas in case of the riper lens it is just about as accurate as the physiological data heretofore available will permit. The mathematical methods of dealing with the dioptrics of surfaces of discontinuity have been worked out.

The substance of the lens constitutes an optical system which for investigating the dioptrics of the lens is combined in the ordinary way with the systems represented by the surfaces of the lens; and which will be referred to as the *core of the lens*, as proposed by MATTHIESSEN. It is to be regarded, therefore, as a system of revolution of continuously variable index of refraction; and, for complete information as to the dioptrics of the lens, all that remains to be done is to find the law of the variation of the index. Since the total variation of the index of refraction of the crystalline lens as well as the thickness of the lens and the diameter of the pupil are relatively small as compared with the focal length, this law can be expressed in the form of a series, provided refractometer investigations show that the series is convergent. The point where the index of refraction has its greatest value may be considered as the centre of the lens, and taken as the origin of a system of coördinates whose x-axis lies along the axis of revolution and is reckoned as positive towards the retina. If μ_0 denotes the value of the index of refraction at the centre of the lens, and μ its value at the point x, y, the general form of the *indicial equation* including powers of x, y as high as the fourth is as follows:

$$\mu_0 - \mu = \frac{1}{2}(m x^2 + n y^2) + \frac{1}{6}(M x^3 + 3N x y^2) + \frac{1}{24}(p_m x^4 + 6p_0 x^2 y^2 + p_n y^4),$$

since in a system of revolution the coefficients of the missing terms in the series vanish. If only the first two terms are retained, the equation reduces to one of the second degree; and the corresponding *indicial curve*, that is found by plotting the distances measured from the centre of the lens along a diameter as abscissae and the indices of refraction as ordinates, turns out to be a parabola. A series of investigations, carried out by MATTHIESSEN and his successors on the crystalline lenses of the eyes of human beings and other animals, indicates that the indicial curve is approximately parabolic in form, although the osculating curve of the second degree may also be an ellipse of great eccentricity. This result in itself is a complete justification of the propriety of expressing the indicial equation in the form of a series, because it converges rapidly; and it shows also that the indicial equation of the second degree is a first approximation that may be employed so long as it does not conflict with mathematical or physical realities. However, further mathematical study reveals that there are some conflicts of this kind. MATTHIESSEN could not formulate a single indicial equation for the nucleus of the lens, but had to assign to each half of it a special function, according to which the iso-indicial surfaces of one system had a single intersection with those of the other, although in the diagrammatic figure of a meridian plane[1] the points of intersection were arbitrarily rounded off. But this is equivalent to a surface of discontinuity passing through the centre of the lens, which would necessarily give a reflex, and whose existence may be positively denied on the basis of the absolute absence of any anatomical evidence for it. It is true that the most recent measurements of the index of refraction[2] might be compatible with a single indicial equation of the second degree for the case of a symmetrical, accommodating lens, and with one of the third degree for an asymmetrical lens. But this would involve such a great increase in the thickness of the lens for accommodation that it may be definitely ruled out. Hence it appears to be unavoidable for the dioptrics of the nucleus of the lens to include in the indicial equation all the terms of the fourth degree. Another indicial equation, due to MAXWELL, and obtained by developing the reciprocal of the index of refraction in a series of powers, is essentially nothing more than an interesting problem of geometrical optics without physiological reality. When terms of higher powers than the second are neglected, it reduces to MATTHIESSEN's formula.

MATTHIESSEN's parabolic indicial curve, which was used also

[1] Loc. cit. *Grundriss usw.* S. 198.
[2] FREYTAG, loc. cit.

by MONOYER[1] in the latest investigation of this subject, is typically analogous to STURM's focal lines of the general bundle of optical rays, both approximations originating by discarding terms higher than the second and being applied beyond the limits where they are valid. The law of the total index given by MATTHIESSEN affords a perfect illustration of the unfortunate influence that the focal lines have exerted in geometrical optics. Mathematical investigation shows that in general the total index varies with every change of the shape of the lens, and hence cannot be calculated at all from the values of the index of refraction at the centre of the lens and in the outside cortical layer. MATTHIESSEN's law, that the total index is just as much greater than the index at the centre of the lens as the latter is greater than that of the outside cortex, might indeed be correct, provided the indicial surfaces were geometrically similar to each other in strict mathematical sense, and provided that the only possible changes in the form of the lens were such as to preserve this characteristic unchanged. Obviously, from what has been stated above, this singulàr condition in the human eye is ruled out.

The next problem, therefore, is to determine the seven constants of the indicial equation as given above. From measurements of the index of refraction, considering at the same time the limits of accuracy of these methods, not more than three equations can be obtained for this purpose, since we may consider that we know accurately enough the values of the index at the centre of the lens, at its poles, and in the equator. Two additional equations are derived from the values of the radii of curvature of the surfaces of the lens, since there is very great probability for the assumption that the curvatures of adjacent iso-indicial surfaces are equal. From a physiological source one other equation may be obtained, by calculating the refracting power of the lens-core from the loss of refracting power of the eye when the lens has been extracted. However, one more equation still is necessary, and for that MATTHIESSEN's parabolic law can be used provided it is restricted to the axis of the lens, especially as from the investigation on which it is based this law may be considered as correct so long as it does not conflict with the actual facts. Hence the writer has obtained the needful seventh equation by assuming a parabolic indicial curve along the axis for which the constant p_m is equal to zero. However, here he has made a special investigation to prove that a variation from this law along the axis even to an extent that we know would

[1] MONOYER, La théorie des systèmes stratifiés. *Société française d'ophthalmologie. Congrès de* 1908. Paris. The method is a variation of that of MATTHIESSEN, and, like it, does not give any unique indicial equation.

be out of the question is essentially without any influence. For each case the results have been put in such form that if future investigations should give a value of p_m different from zero, the constants of the indicial equation may be obtained directly by substituting this value.

On account of the slow variation of the index of refraction in the vicinity of the centre of the lens, the position of the centre of the lens can only be found approximately. The change of form of the anterior surface of the lens is so considerable during accommodation that the lens itself becomes very nearly symmetrical in shape, probably involving a marked increase in the thickness of the two portions of the lens on each side of the centre. Moreover, after the appearance of sclerosis of the core of the lens, the latter is found usually a little nearer the anterior than the posterior surface. From such considerations the writer has estimated the distance of the centre of the lens from the anterior and posterior surfaces as having the values 1.7 and 1.9 mm, respectively. However, so far as the dioptrics of the lens is concerned, these distances may be considered as being equal.

Concerning the indices of refraction, measurements made prior to the introduction of the refractometer are of no value. Recent measurements with this instrument yield numerical results varying from 1.38 to 1.39 for the outside cortex and from 1.40 to 1.42 at the centre of the lens. Undoubtedly, the most reliable results are those of FREYTAG[1] who carried out numerous measurements with careful regard for the necessary precautions. His determinations have therefore been taken by the writer for the schematic values. From table III of his measurements for eyes of human beings up to 30 years of age the following are the mean values of the index of the superficial layer of the lens:

Anterior Vertex	Equator	Posterior Vertex
1.387	1.375	1.385

and for the same class of eyes table IX gives the mean value of the index for the

Centre of the lens: 1.406.

The difference between the values of the index at the vertices and at the equator, as here established has very great significance for the dioptrics of the lens, and, incidentally, it is this very difference that makes it possible to formulate an indicial equation of the second degree like MATTHIESSEN's for the accommodating lens. The difference between the values found for the two vertices is perhaps of less importance, since, in the first place, this difference has very little effect on the dioptric equations, secondly, it is small, and, thirdly,

[1] Loc. cit.

the *post mortem* drying up that takes place in the eye produces a change in its fluid that renders the result in this respect rather less certain. Accordingly, the writer assumes for the index of the two vertices of the lens the mean value of 1.386 as the schematic value. The index at the equator cannot be used in the calculation until the distance of the layer in question from the centre of the lens is known; and since no measurements of this distance seem to have been made, the writer employs the above result by assigning the schematic index 1.376 to the points whose coördinates are $x=0$, $y=\pm 4.2$; which implies that the mean value of 1.375 given above belongs to a point in the equator estimated as being a little towards the periphery.

The refracting power of the core lens and the total index of the lens would follow from the schematic values thus defined, provided MATTHIESSEN's law of the total index were correct. But as this is not the case, the only thing left to do is to ascertain the value in question by direct investigation. Moreover, the variation of the total index with every change in the form of the lens makes its determination illusory, no matter whether this is done by direct measurement on the dead lens according to HELMHOLTZ's method (as illustrated by Fig. 49) or whether BERLIN's[1] method is used by comparing the lens with and without accommodation in the living eye. On the other hand, the striking difference between the values 1.4519, 1.4414 as found by HELMHOLTZ (p. 110) and the values 1.4260 to 1.4434 as given by STADFELDT[2] may possibly be explained by this change of the total index, together with such sources of error as are unavoidable on account of *post mortem* variations of the physical index of refraction; especially as a variable factor is introduced by the mode of fastening the lens in the experiment, which must affect the value of the total index. Indeed, it would seem to be extremely doubtful whether the total index of the unaccommodated lens in the living eye can be found at all with the dead lens, inasmuch as all the passive relations and particularly the distribution of the tension of the zonule between the various groups of fibres require to be exactly reproduced.

It is quite certain, therefore, that there is no other way of ascertaining the refracting power of the lens and the total index except in the living eye; and the only method that is left is to calculate it from the loss of refraction that the eye suffers when the lens is extracted. This matter was actually within the province of ophthalmology during the past twenty years when it was quite the fashion to treat myopia by

[1] E. BERLIN, Über eine Bestimmung des totalindex der Linse am lebenden Auge. *Arch. f. Ophth.* XLIII. 1897. S. 287.

[2] Loc. cit.

an operation in which the crystalline lens was extracted. It developed from a comparison of the correction of the eye before and after the operation, that HELMHOLTZ's schematic eye does not represent the actual relations close enough, the total index of the lens being too great and the axial length of the eyeball too short.

In this calculation of the total index, the refraction of the eye before and after removal of the lens may be determined, as BJERKE[1] did, with respect to the apparent position of the optical centre of the lens; in which case the loss of refraction is directly proportional to the refracting power of the lens, and the approximation which consists in using an optical centre in place of the principal points of the lens has no effect on the accuracy that is obtainable by the calculation. The best way to perform the calculation is as follows. The general image-equations referred to the principal points of the cornea are

$$B = A + D_c, \qquad KB = A.$$

Let δ denote the reduced distance of the optical centre of the lens from the posterior principal point of the cornea; κ the magnification-ratio at the optical centre; and x the apparent position of this centre with respect to the anterior surface of the cornea. Then substituting $\dfrac{1}{\kappa - H_c}$, $\dfrac{1}{\delta}$ and κ for A, B and K, respectively, we obtain:

$$\kappa = 1 - \delta D_c, \qquad x = \frac{\delta}{\kappa} + H_c.$$

Now since the distances of the optical centre from the two surfaces of the lens are proportional to the radii of curvature, the schematic values as above defined give:

$$\delta = \frac{0.05 + 3.6 + 2.25}{1.336} \text{ mm}$$

$$\kappa = 0.80987 \qquad x = 5.4 \text{ mm}.$$

Let D_c, D_l, and D denote the refracting powers of cornea, lens, and eye as a whole; then the general formula for the combination of two systems is

$$D = D_c + \kappa D_l.$$

Let A, B denote the reduced convergences for a distinct image in the eye before the lens is removed, measured at the apparent place of the optical centre of the lens; and let A_0, B_0 denote the reduced convergences for a distinct image after removal of the lens, with respect to the actual position of the optical centre of the lens; so that

$$\kappa^2 B = A + \kappa D, \qquad \kappa^2 B_0 = A_0 + \kappa D_c.$$

[1] K. BJERKE, Über die Veränderung der Refraktion und Sehschärfe nach Entfernung der Linse. II. *Arch. f. Ophth.* LV. 2. 1903. S. 191.

Since the values of the left-hand sides of the two equations are identical, we have:

$$A_0 - A = \kappa^2 D_l.$$

It is pretty well ascertained that most eyes that have been operated on for cataract require a correction glass of between 10 and 11 dptr; and from this fact the value of A_0 can be calculated. Previous attempts to harmonize this fact with data for a schematic eye have not been successful and have resulted in obtaining an augmented optical effect of the correction glasses used for the calculation, by assuming a great distance between the glass and the vertex of the cornea. For example, TREUTLER[1] places the posterior vertex of the correction glass at a distance of 13 mm from the vertex of the cornea, and estimates the distance of the second principal point of the glass as being 1.5 mm from the glass surface. Thus, using the mean value of the correction, he obtains the value 80.735 mm for the distance of the virtual far point of the aphakic eye from the vertex of the cornea, corresponding therefore to a value $A_0 = 13.27$ dptr.

This value is certainly rather too high, because in making an accurate measurement of the refraction of the eye the distance between the glass and the eye is taken as small as possible, and since, too, in most clinics the eyelashes of those who are operated on for cataract are removed. Nevertheless, if this value is used here by the writer, it is because he considers it as being an outside upper limit.

It is not permissible simply to put $A = 0$ in calculating the refracting power of the crystalline lens, as is usually done, because, in the first place, the normal refraction of the eye as ascertained by trial-case glasses is not emmetropic at the age when most cataracts are extracted, and, secondly, owing to the aberration, this method does not give the value along the axis that is to be used in the imagery-laws of the first order. Whereas the aberration in the eye as a whole amounts to a difference of at least 1 dptr between the latter value and the value obtained by the ordinary method of refraction, the same difference may not exist in case of the eye that has been operated on for cataract. Indeed, the writer has proved ophthalmometrically that that part of the cornea, which ordinarily remains uncovered by the lid and has to be considered when the pupil has a fairly large size, is responsible for a slight positive aberration; as has been confirmed also by the ophthalmoscope in a case of spontaneous reabsorption of the lens. But after the cataract operation there occurs a pronounced flattening of the vertical section of the cornea, which has an effect on

[1] Einige Bemerkungen zu den schematischen Augen. *Klin. Monatsbl. f. Augenheilk.* XL. 1902. S. 1.

the aberration in such fashion that the refraction of the operated eye as found by the trial-case method gives a value that is extremely likely to be the refraction along the axis. Consequently, while the value of A_0 is not influenced by the aberration, that of A is equal to the sum of the mean value of the refraction at the average age of the person who undergoes the cataract operation and the amount of the difference due to aberration between the measured refraction and the exact refraction along the axis. By assuming the schematic value $A = 0.75$ dptr, perhaps the writer may have chosen a lower limit for it. Hence, the value of 19.1 dptr for the refracting power of the crystalline lens as obtained by the formula above can certainly not be too small. Since the refracting powers of the two surfaces of the lens are 5 and 8.33 dptr, the approximate schematic value of the refracting power of the core lens comes out to be 6 dptr as a probable upper limit. Now in the calculation of a schematic eye with this small value for the refracting power of the lens, it appears that it is only with this upper limit that the axial length of the eye is found to be in agreement with the results of anatomical measurements; and hence the writer uses it also as the basis of the calculation of the indicial equation. Let x_1, x_2 denote the abscissae of the two vertices of the lens; μ_1, μ_2 and ρ_1, ρ_2 the indices of refraction and the radii of curvature, respectively, at these points; μ_0, μ_3 the indices of refraction at the points whose coördinates are $x = 0$, $y = +4.2$, and $x = 0$, $y = -4.2$; and D the approximate value of the refracting power of the core lens. Taking the millimetre as unit of length, the writer has therefore used the following numerical values in the calculation:

$$x_1 = -1.7 \qquad x_2 = 1.9 \qquad \rho_1 = 10 \qquad \rho_2 = -6 \qquad D = 0.006$$
$$\mu_0 = 1.406 \qquad \mu_1 = \mu_2 = 1.386 \qquad \mu_3 = 1.376.$$

For $p_m = 0$, these numbers give the following values of the constants of the indicial equation:

$$\begin{array}{ll} m = 0.012537 & n = 0.0010475 \\ M = -0.0023004 & N = 0.00011470 \\ p_o = 0.0011150 & p_n = 0.0016012 \end{array}$$

It should be remarked that these constants are not mere numerical values, but are magnitudes with physical dimensions, and hence vary in value with the choice of the unit of length. Since the linear magnitudes are expressed here in millimetres, the unit of refracting power, reduced convergence, etc., is equal to a thousand dioptries (Kilodioptry).

By means of the indicial equation, the writer has constructed the sections of the iso-indicial surfaces made by a meridian plane, by

calculating a sufficient number of coördinates for values of the index of refraction equal to 1.386 and 1.404. The outer curve (Fig. 131), therefore, does not coincide with the surface of the lens, but has a contact of the second order with it at the vertices only. The parabolic law of the variation of the index of refraction along the axis is shown by the fact that the difference between the value of the index at the centre of the lens and that on the outer curve is ten times as great as the same difference for the inner curve.

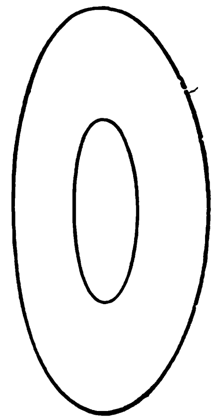

Fig. 131.

The following are the exact values obtained for the refracting power (D_k) of the core lens and the positions of its principal points:

$$D_k = 0.005985 \qquad H = 0.22921 \qquad H' = 0.25752,$$

the reduced distances H, H' being reckoned from the centre of the lens. As compared with previous estimates, the total index has the remarkably low value of 1.4085. As long as the exact image-equations in heterogeneous media were unknown, the fictitious total index was the only method of representing this imagery at all. But now that the core lens has become a definite dioptric conception, perhaps the best way to understand its effect is in terms of the *equivalent core lens;* which is to be understood here as a lens having the same index of refraction as that at the centre of the crystalline lens and suspended in a medium with the same index of refraction as that at the vertices of the crystalline lens; its refracting power and the positions of its principal points being identical with those of the real core lens. Moreover, the refracting power of the equivalent core lens is divided between the two surfaces in the same ratio as it is divided in the real core lens between the two portions of the substance of the lens on opposite sides of the centre. The advantage of using the equivalent core lens is because, so far as the imagery-laws of the first order are concerned, the entire lens system then has precisely the optical characteristics of the actual crystalline lens; whereas, on the contrary, the total index gives a wrong position for the principal points of the lens. The radii and thickness of the equivalent core lens, expressed in millimetres, are found to have the following values:

$$r_1 = 7.9108 \qquad r_2 = -5.7605 \qquad d = 2.4187\,;$$

and if the distance of the anterior surface of the equivalent core lens from the anterior vertex of the lens is 0.5460 mm, the principal points of the equivalent core lens coincide with those of the real core lens.

Employing the formulae on page 285 for the combination of three component systems, we find therefore the following data for the crystalline lens as a whole:

$$D_l = 19.1107 \qquad 1000 n_2 H_l = 2.07792 \text{ mm} \qquad 1000 n_2 H' = -1.39317 \text{ mm},$$

where H, H' denote here, as usual, the reduced distances of the principal points of the lens from the corresponding surfaces.

In the schematic eye it is best to consider the lenticular system as a system of revolution, just as also the surfaces of the cornea are represented as being surfaces of revolution, although the physiological form is not astigmatic. It is extremely probable that the latter is the case with the lens also. For the astigmatism of the entire system, that is, the total physiological astigmatism of the normal eye, does not amount to that of the cornea by itself, which implies that there must be an opposite lenticular astigmatism. As was explained above, ophthalmometric investigations of the cornea show that a section near the base of the cornea made by a plane perpendicular to the ophthalmometric axis is oblong in shape with its longer diameter vertical. And yet it is hardly conceivable that this is the case unless the whole anterior part of the eyeball has a similar shape, so that certain static conditions are thus involved that are calculated to produce an opposite lenticular astigmatism, which would be a natural result both of a corresponding oblong shape of the lens and of a feebler curvature in the vertical section due to greater tension of the zonule. However, due to the sources of error inherent in the methods, previous attempts at measuring lenticular astigmatism in the living eye cannot claim to be very reliable. Generally speaking, the measurements indicated a direct astigmatism of the anterior surface of the lens and an inverse astigmatism of the posterior surface; a result which by itself would seem to indicate pretty clearly its probable dependence on the sources of error that are involved in the unequal flattening of the cornea in different directions and on the decentration of the refracting surfaces. It should be added that an astigmatic refraction in the core lens may be involved in the form of the iso-indicial surfaces, without having to assume that the surfaces of the lens are astigmatic.

3. The Optical System of the Eye

For the compound system composed of the cornea and the lens the expression

$$1000 n_2 \delta = 0.0506 + 3.6 + 1000 n_2 H_l$$

has to be substituted in the general formulae, and thus for the eye as a

whole we obtain:

$$D = 58.636, \qquad 1000\,H = 1.3975, \qquad 100\,n_2H' = -4.2061,$$

which are therefore the fundamental constants of the schematic eye. The complete data, as based on the calculation of the equivalent core lens, are collected in the following table. The numerical results of that calculation were obtained only to three places of decimals, and, consequently, there may be a difference as compared with the numbers given above amounting to as much as one unit in the last decimal place.

SCHEMATIC EYE, ACCOMMODATION RELAXED

Index of refraction of cornea	1.376	
" " " " aqueous and vitreous humors	1.336	
" " " " the lens	1.386	
" " " " the equivalent core lens	1.406	
Position of anterior surface of cornea	0.	
" " posterior " " "	0.5	mm
" " anterior " " lens	3.6	
" " " " " equivalent core lens	4.146	"
" " posterior " " " " "	6.565	"
" " " " " lens	7.2	"
Radius of anterior surface of cornea	7.7	"
" " posterior " " cornea	6.8	"
" " anterior " " lens	10.0	"
" " " " " equivalent core lens	7.911	"
" " posterior " " core lens	−5.76	"
" " " " " lens	−6.0	"
Refracting power of anterior surface of cornea	48.83	dptr
" " " posterior " " "	−5.88	"
" " " anterior " " lens	5.0	"
" " " equivalent core lens	5.985	"
" " " posterior surface of lens	8.33	"
Cornea System		
Refracting power	43.05	dptr
Position of first principal point	−0.0496	mm
" " second principal point	−0.0506	"
First focal length	−23.227	"
Second focal length	31.031	"
Lens System		
Refracting power	19.11	dptr
Position of first principal point	5.678	mm
" " second principal point	5.808	"
Focal length	69.908	"

Complete System of Eye

Refracting power	58.64	dptr
Position of the first principal point	1.348	mm
" " " second principal point	1.602	"
" " " first focal point	−15.707	"
" " " second focal point	24.387	"
First focal length	−17.055	"
Second focal length	22.785	"
Position of the retinal fovea	24.0	"
Hypermetropia along the axis	1.0	dptr

The refraction-state as found by investigating such an eye as this would be that of emmetropia as will be shown presently in discussing the aberration. A section of the schematic eye is represented by Fig. 132 in which the surfaces are regarded as spherical.

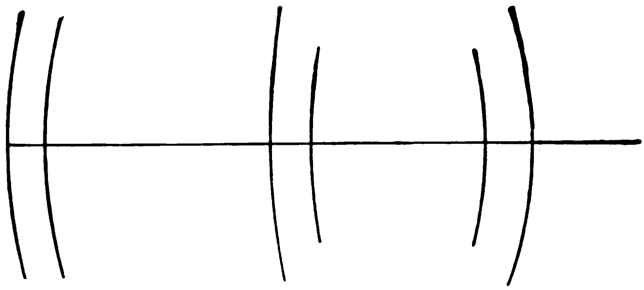

Fig. 132.

From the foregoing explanation it follows that the axial length of the schematic eye is determined by the refraction of the cornea and the mean refraction of the aphakic eye. The greater the refracting power of the cornea and the average hypermetropia of the aphakic eye, the shorter will be the axial length of the schematic eye. By taking account of the difference between the radii of curvature of the cornea at the vertex of the optical zone and the mean radius of curvature of this zone, the value of the refracting power of the cornea given above is the greatest value that is compatible with the preceding metrical results. The same thing is true as to the basis of the calculation by which the hypermetropia of the aphakic eye was found. Obviously, therefore, the length of the axis of the schematic eye cannot be made shorter without doing violence to the facts of the case. It is true that if the lens is removed from the schematic eye, it is 0.1 dioptry more hypermetropic than it should be according to TREUTLER's requirements. Moreover, the axial length of 24 mm is just inside the limits which he thinks are established by measurements of the dead eye. Without attaching undue importance to these measurements, the writer believes

however, that he is almost forced to conclude that they would make it very unlikely that the axial length of the schematic eye is greater than 24 mm. According to MAUTHNER,[1] measurements on dead eyes give more values around 25 mm, which would be a good agreement, allowing 1 mm for the thickness of the sclerotica and choroid. However, on the other hand, lower values have been found in numerous measurements. But in estimating measurements of this kind it should be borne in mind that the length of the eye is diminished by *post mortem* changes; nor is it probable that it could be brought back to its original length by artificial restoration of the intraocular pressure that exists in the living eye, since, by virtue of their anatomical structure, a *post mortem* increase of thickness of the cornea and sclerotica may be accompanied by a contraction of the interior of the eyeball.

The remarkably low refracting power of the lens of the schematic eye is a necessary mathematical consequence of the change of refraction that occurs when the lens is removed. Taking account of the effect of aberration, it cannot be greater in the unaccommodated eye than has been found above. The mathematical connection that has been shown to exist between the total index of the lens and its shape enables us for the first time to calculate a schematic eye that is not contrary to any ascertained fact; whereas so small a value for the refracting power of the lens is, on the assumption of the validity of MATTHIESSEN's law, hopelessly at variance with the results of measurements of the index of refraction, and, besides, would not be sufficient for the low total index in the accommodating eye. If by elongating the axis of the schematic eye to a length of 30.98 mm it were made myopic to such a degree that it would be emmetropic if the lens were removed, the myopia along the axis would be 13.16 dptr at the first principal point, and the far point would be 74.88 mm in front of the vertex of the cornea. A glass of 16.7 dptr would be required to correct this myopia, with its second principal point, according to TREUTLER, 14.5 mm in front of the vertex of the cornea, not taking account of the effect of aberration. However, if the latter is taken into consideration, the glass required would have to be about 18 dptr, which agrees perfectly with the results of clinical investigation.

But if it is thus absolutely necessary to take the aberration into account in the exact schematic eye, on the other hand; it may be desirable to calculate a simplified schematic eye, wherein, neglecting the divergent effect of the posterior surface of the cornea, we may assume that the lens is homogeneous, and the entire system free from aberration. But as the fiction of a homogeneous lens is in itself so different

[1] L. MAUTHNER, *Vorlesungen über die optischen Fehler des Auges.* Wien 1876. S. 422.

from the actual relations that the positions of the principal points of the lens cannot be exactly located, we may as well take advantage of the fiction by assuming the position of the optical centre of the lens. If the calculation is to be performed anyhow with the incorrect positions of the principal points, there is no sense in taking account of the distance of these points from each other. The equivalent surface of the cornea to be used in a simplified schematic eye of this kind has a radius of curvature which agrees very closely with the schematic radius of the optical zone as found by the ophthalmometer, and hence this value has been assumed. By retaining only three decimals in the value of the total index of the lens, the following values have been found for a simplified schematic eye; the assumed values (except in case of the total index of the lens) being the same as in HELMHOLTZ's latest schematic eye,[1] provided the figures in the last two decimal places are discarded in the value of the radius of curvature of the cornea and in the last decimal place in the value of the index of refraction of the aqueous and vitreous humors. (HELMHOLTZ's data therefor are 1.4371, 7.829 and 1.3365, respectively.) The lens in the calculation is an infinitely thin lens with the vertices of its two surfaces at the optical centre of the crystalline lens of the eye.

Simplified Schematic Eye

Data assumed		
Index of refraction of the aqueous and vitreous humors	1.336	
" " " " " lens	1.413	
Radius of curvature of the equivalent surface of cornea	7.8	mm
" " " " " anterior surface of lens	10.0	"
" " " " " posterior surface of lens	−6.0	"
Position of optical centre of lens	5.85	"
Calculated values		
Refracting power of cornea	43.08	dptr
" " " lens	20.53	"
" " " optical system of eye	59.74	"
First focal length of the cornea	−23.214	mm
Second focal length of the cornea	31.014	"
Focal length of lens	65.065	"
First focal length of eye	−16.740	"
Second focal length of eye	22.365	"
Position of first principal point	1.505	"
" " second principal point	1.631	"
" " first focal point	−15.235	"
" " second focal point	23.996	"

[1] *Handbuch d. phys. Opt.* 2. Aufl. S. 140.

A *reduced eye* corresponding to the exact schematic eye and having an index of refraction of 4/3 would have a radius of curvature of 5.7 mm.

According to the accuracy demanded in any given case, one or the other of these models can be employed. With regard to the simplified schematic eye, it should be expressly noted that, due to neglect of the aberration, the total index of the lens is greater than its actual behaviour indicates, as its value in the exact schematic eye is only 1.4085; and, consequently, this latter eye should be used in case it is desired to trace the rays through the lens exactly. In the simplified schematic eye it may be said that we are never concerned with calculating the refraction at the single surfaces of the lens, but the lens is always to be treated as a single unresolvable optical system.

In the schematic eye both the decentration of the refracting surfaces and the obliquity of the line of sight (*Visierlinie*) are left out of account entirely, because the decentration is doubtful in amount, and moreover seems to be variable in extent. As we have already seen that the anterior surface of the cornea does not have any exact vertical plane of symmetry, obviously, the reflex images of infinitely large objects cannot determine any precise centering of the optical system. However, in HELMHOLTZ's experiment illustrated in Fig. 51 the effect of the asymetry of the cornea is eliminated as far as possible by basing it on the osculating ellipse, and there would seem to be hardly any doubt as to the fact that this experiment proves that the lens is decentered with respect to the axis of this ellipse. But since, by virtue of the asymmetrical structure of the cornea, it is not certainly established that the calculated axis of the ellipse is a normal to the cornea, it is not proved by the experiment that the axis of the lens does not in general coincide with the normal to the cornea that passes through the centre of the pupil. Investigations carried out by TSCHERNING's[1] method afford just as little proof of this. If, in observing the reflex images in the surfaces of the lens, the axis of the telescope and two lights, preferably ophthalmometric NERNST lamps, are adjusted all in one vertical plane, then, supposing there was a vertical plane of symmetry, a position of the eye can be found for which all six reflex images would be seen lying in one straight line. But generally this is not the case. When the patient's eye looks right into the telescope, the reflex images in the anterior surface of the lens are on the nasal side of the reflex images in the cornea, and those in the posterior surface of the lens are on the temporal side. If then the look is directed in towards the nose, in most cases, first, the reflex images in the posterior surface of the lens turn into line with the reflex images in the anterior surface

[1] Beiträge zur Dioptrik des Auges. *Zeitschr. f. Physiol. u. Psychol. d. Sinnesorgane.* III. 1892. S. 429 and elsewhere.

of the lens, and, last of all, the latter come in line with the reflex images in the cornea. This experiment shows that as a matter of fact the optical system as a whole has no vertical plane of symmetry (as had been previously ascertained in other ways by ophthalmometric investigations of the cornea); and, provided the surfaces were accurately spherical, or in case the experiment could be performed for very small angles of incidence, it would show that the axis of the lens does not coincide with a normal to the cornea, but passes on the nasal side of the centre of curvature of the cornea. Now in reality very large angles of incidence have to be used to perform this experiment with any success, because indeed it is generally necessary to dilate the pupil artificially, and then we have no right to assume that it is only the optical zone of the cornea that is concerned in the phenomenon. On the contrary, the uneven peripheral flattening in different directions is a factor; and therefore the experiments are not convincing. The same thing applies to the investigation of the vertical decentration where the lights are adjusted in a horizontal plane containing the axis of the telescope. With typically normal eyes the look has to be lifted a little above this plane in order to bring the various reflex images in line with each other.

Accordingly, with respect to the decentration of the refracting surfaces, all that can be said is that the optical system has no axis of symmetry, and that the optical axis of the lens, meaning thereby a common normal to the two surfaces of the lens, meets the cornea in a point that lies outwards and generally a little downwards from the ophthalmometric axial point; as is the case also with the normal to the cornea which passes through the centre of the pupil of medium size. But the fact that this normal, which was defined above as the *optical axis of the eye*, does not coincide with the optical axis of the lens, is so far not proved. Although, obviously, small deviations are extremely likely, they cannot be used in calculations, especially as they have not yet been definitely proved to exist. Investigations of the eye with a luminous point (see below) prove that, if present, they do not affect the imagery in the eye.

Practically, therefore, the only decentration that needs to be considered is that due to the inclination of the line of sight to the optical axis of the eye, which can also be determined by the decentration of the pupil with respect to the ophthalmometric axis of the cornea. In order to find experimentally the position of this axis, we must, as pointed out above, employ a pupil of medium size, preferably 4 mm produced by illumination. For although the apparent centre of the pupil does not seem to vary for small changes of the diameter, very considerable differences occur with greater variations of the diameter.

Thus, in one case where the investigation was made with a Helmholtz ophthalmometer by the method described on page 315, the writer found for the angle between the line of sight and the optical axis 6.5° with maximum pupil and 2.7° with minimum pupil; and hence, on the supposition of a spherical cornea, the displacement of the centre of the pupil towards the nose, occurring during the contraction, would amount to 0.28 mm. However, part of this displacement may be really due to the asymmetrical flattening of the cornea in the horizontal meridian; and, besides, a difference as large as this is perhaps exceptional. By another, but not so sensitive method, Hummelsheim[1] found that the pupil contracted concentrically.

Point-to-point correspondence of object and image occurs in a very narrow region in the vicinity of the axis of the optical system; and the structure of the retina corresponds to this fact in the most perfect manner, since it is adapted for distinct imagery only in a very small part of it. Hence, the excellence of *the peripheral imagery in the eye* is a matter of secondary importance. If, as a first approximation, the optical system is considered as a centered system of revolution, the two image-surfaces are surfaces of revolution concave towards the forward side, the first of which, whereon are focused the image lines of parallel circles, is nearer the optical apparatus than the second on which the radial or meridian lines are reproduced. Since Young's[2] time various investigators have endeavoured to construct these image-surfaces by calculation or to compute the astigmatism for the peripheral imagery, generally obtaining results in which the retina is close to the image-surfaces or between them. However, until the precise form of the posterior surface of the lens is ascertained, not much value, if any, can be attached to such calculations. The effect of the lamination of the lens on this astigmatism was calculated by Hermann;[3] who found that the lamination tended to reduce the astigmatism. But this conclusion is due partly to his assumptions and partly to the fact that the differential equations which he used for the primary imagery are not correct. Computation of the schematic lens with variable index of refraction as given above shows that, for a ray incident at the centre of the anterior surface of the lens at an angle of 25°, and on the assumption that the posterior surface is parabolic, the astigmatism is greater than in a homogeneous lens with the corresponding total index. Of the methods of direct investigation of the peripheral imagery on the retina, the one that gives the least trustworthy results is by observing the image of a source of light through the sclera, as can be

[1] Pupillenstudien, *Arch. f. Augenh.* LVII. S. 33. 1907.
[2] Th. Young, On the mechanism of the eye. *Philos. Transactions*, 1801.
[3] Loc. cit.

done when the light enters sideways in the protruding eye. The measurement of the refraction with the ophthalmoscope by the method of the erect image also cannot compare with the skiascopic method, which has perhaps given the most reliable results so far. By this method DRUAULT[1] found that the retina is probably between the two image surfaces, and that the second surface is beyond it and nearer than the first.

Since the line of sight makes a finite angle with the optical axis, the imagery in the *fovea centralis* of the retina, strictly speaking, belongs to the region of the periphery. However, the resulting astigmatism for an angle of 5° amounts only to about one-tenth of a dioptry, as the writer has shown by calculation with HELMHOLTZ's schematic eye.[2] (TSCHERNING has calculated this astigmatism for different angles and finds much higher values, partly due to using a wrong formula, and apparently due also to his having confused these angles with the angles of incidence. From the formulae on page 277, it can be seen that the ratio of the refracting power of the first imagery to that of the second imagery is equal to $1 : \cos i \cos i'$. But this ratio by itself does not give the astigmatism, because in the primary imagery the principal points do not coincide with the point of incidence. TSCHERNING,[1] however, has not only used $1 : \cos^2 i$ in place of the correct ratio, but has also neglected the distances of the refracting surfaces from each other and from the corresponding principal points; which, together with the confusion of the angles, is responsible for the erroneous results.)

III. Refraction of the Eye

What is meant by the refraction of the eye is its focusing as dependent on the length of the axis and the refracting power of the optical system. This name has become firmly established, but it is not altogether a happy designation, inasmuch as it suggests merely one of the factors of optical focusing, whereas the different states of refraction are generally dependent on variability of the other factor also. It is found to be a great advantage in optical problems to reckon all distances as positive in a certain definite direction—otherwise, the necessary changes of sign are apt to lead to errors. And so on the basis of

[1] Astigmatisme des rayons pénétrant obliquement dans l'œil. Application de la skiaskopie. *Arch. d'ophth.* XX. 1900. p. 21.

[2] Loc. cit., *Skand. Arch. f. Physiol.* II.

[3] Loc. cit., *Encyclopédie française d'ophthalmologie.* III. p. 185.

this principle, in dealing with the subject of the refraction of the eye either distances measured in the direction in which the light goes must be counted as positive, as is usual in other parts of optics, or conversely. The latter method commonly prevailed until HESS[1] first introduced a theoretically more correct mode of treatment. Thus, according to him, the far-point distance and the near-point distance are both negative when these points are real. The disadvantage of this method for the physician who is not trained in physics, and who therefore has to acquire this mode of view, is too slight to be compared with the simplifications that are gained thereby; and so the writer will follow HESS's method, with the conviction that these advantages will be appreciated by oculists and ophthalmologists.

The focusing of the eye is measured by the convergence of the bundle of rays that falls on it when the image is sharply in focus on the retina; and the only other question that has to be decided is at what point this convergence is to be measured. On account of the simpler form of the image-equations as referred to the principal points, the first principal point of the eye is particularly suitable for this purpose. Thus, in the general equation

$$B = A + D,$$

A denotes the refraction of the eye expressed in dioptries, D the refracting power of the optical system, and $\frac{1}{B} = b$ the reduced distance of the retina from the posterior principal point, which may be called the *reduced length of the axis;* whereas $a = \frac{1}{A}$ denotes the distance from the anterior principal point of the point that is exactly in focus. However, for practical reasons, it is desirable to measure the convergence of the bundle of incident rays also at the place where the spectacle lens is worn or, more accurately, at the second principal point of this glass. If g denotes the distance of the first principal point of the eye from the second principal point of the glass, whereby g is always positive, and if A_g denotes the convergence of the bundle of incident rays at the glass, then

$$\frac{1}{A_g} = \frac{1}{A} + g, \qquad A_g = \frac{A}{1 + gA}, \qquad A = \frac{A_g}{1 - gA_g}.$$

In these expressions g is hardly ever more than 15 mm, and A and A_g usually do not exceed 20 dptr. Practically, therefore, the error

[1] C. HESS, Die Refraktion und akkommodation des menschlichen Auges und ihre Anomalien. Leipzig 1902. A separate volume in GRAEFE-SAEMISCH, *Handb. d. Augenheilk.* 2. Aufl. II. T. XII. Kap.

due to neglecting the product g^2A^2 or $g^2A_g^2$ will be of no consequence, and the useful approximate formulae

$$A_g = A(1-gA), \qquad A = A_g(1+gA_g)$$

will be accurate enough. HESS calls A_g the *correction value* of the refraction; and hence the above relations may be expressed by saying, *the difference between the refraction and the correction value is numerically equal to gA% or gA_g%, provided the distance g is given in centimetres; the correction value being always algebraically less than the refraction.* In ophthalmological literature correction value and refraction are sometimes called also "spectacle refraction" and "principal point refraction," respectively. Thus, in myopia the spectacle refraction is numerically greater than the principal point refraction, whereas in hypermetropia it is just the reverse.

Since the act of accommodation consists essentially in a change of the optical focusing of the eye, innumerable states of refraction are comprised within the limits determined by the far point and the near point. If the distances of far point and near point from the first principal point of the eye are denoted by r and p, and if the corresponding convergences are denoted by R and P, respectively, the eye may have every refraction between these values. The difference

$$R-P$$

is the amplitude of accommodation; the refracting power of the optical system being increased, but the refraction itself being diminished by accommodation. However, when we speak simply of the refraction of the eye, it is understood that we refer to its *static* refraction, when it is focused for the far point, and when accommodation, therefore, is relaxed. But if accommodation is exerted, there is a *dynamic* state of refraction, which has its smallest limiting value when the eye is focused for the near point.

Ametropia differs from *emmetropia* in the fact that for the former the refraction has a finite value, this value being positive for *hypermetropia* and negative for *myopia*. The spectacle refraction is the power of the glass that will make an ametropic eye emmetropic; whence is derived also the name correction-value.

As a general rule, the refraction must be determined by *combination of the eye with an optical instrument.* Let D_0, D and D_t denote the refracting powers of the optical instrument, the optical system of the eye, and the compound system, respectively; and let δ denote the distance (or in case the eye is immersed in a fluid, the reduced distance) of the anterior principal point of the eye from the posterior principal point of the optical system which is to be combined with it. Then the data of the compound system can be found by the formulae:

$$D_t = D_0 + D - \delta D_0 D, \qquad H_t = \frac{\delta D}{D_t}, \qquad H_t' = -\frac{\delta D_0}{D_t}.$$

Now if, generally, the convergences of the artificial system (or the reduced convergences, if this system is not surrounded by air) at its principal points are denoted by $A_0 = \frac{1}{a_0}$ and $B_0 = \frac{1}{b_0}$; and if A, B, a, b, and A_t, B_t, a_t, b_t have similar meanings for the eye and the compound system, respectively, the distances being all reduced distances; then B_0 denotes the refraction of the eye measured at the second principal point of the artificial system, and the principal point refraction is obtained from the expression:

$$A = \frac{A_0 + D_0}{1 - \delta(A_0 + D_0)}.$$

Hence, provided the distance of the object is so great that we may put $A_0 = 0$ (as is the case in ordinary spectacle fitting), D_0 denotes the correction value. In case of other combinations we get certain simplifications for special values of δ. If both A_0 and δ vanish, then the principal point refraction is equal to D_0. For $\delta = \frac{1}{D_0}$ we find

$$A = -D_0^2\left(\frac{1}{D_0} + \frac{1}{A_0}\right) = -\frac{D_0^2}{L_0},$$

where L_0 denotes the reduced convergence measured at the anterior focal point of the artificial system; the equation being therefore identical with the general focal point equation. We obtain the same equation for the correction value of the ametropia when $\delta = \frac{1}{D_0} + g$. Hence, for a given value of D_0, the distance of the first focal point of the artificial system from the object is in these cases proportional to the refraction or to the correction value of the ametropia.

The size of the retinal image is given by the two general equations

$$KB = A, \qquad KD = L;$$

the distinct images in the *fovea centralis* being small enough for us to substitute here the ratio of the linear magnitudes of image and object in place of the magnification-ratios (which actually represent only the limiting values). Let α, β denote the linear sizes of object and image for the naked eye; then if the reduced *principal point angle* (ω_h) and the reduced *focal point angle* (ω_f) are defined to be the angles equal to αA and αL, respectively, the following expressions will be obtained:

$$\frac{\beta}{\omega_h} = b, \qquad \frac{\beta}{\omega_f} = \frac{1}{D};$$

which are better adapted for the case when the object is very far away, because then the magnification-ratio tends to become infinitely small. (In air the reduced angles are the same as the angles themselves, but they are employed here in order that the formulae may be applicable at once to vision under water.) These formulae state that *the ratio of the size of the retinal image to the focal point angle depends simply on the refracting power of the optical system, whereas its ratio to the principal point angle involves only the reduced length of the axis of the eye.* In employing them it should be noted that, by the definition of the angles, they are negative in the case of a real object, and the resulting negative sign of β simply means that the image on the retina is inverted. The visual angle formerly used which has its vertex at the anterior nodal point of the eye is not so well adapted for precise purposes; and indeed the use of the nodal points in physiological optics is rather unfortunate anyhow, because there is no real advantage in them, and, on the other hand, they are conducive to much that is imaginary.

When the eye is used in conjunction with an artificial optical system, the size of the image on the retina can be ascertained by the effect which this system produces on the magnitudes of the principal point angle and the focal point angle; that is, by investigating *the magnifying power of an optical system used in conjunction with the eye.* But here the question as to the absolute magnifying power of the instrument must be kept distinctly separate from that of the individual magnifying power. The capability of the instrument itself is measured by the absolute magnifying power, since this magnitude has nothing whatever to do with the eye; whereas the individual magnifying power differs from the absolute magnifying power simply because it does take into account the idiosyncrasies of the eye in question. If α_0, β_0 denote the sizes of object and image with respect to the optical instrument, then

$$\omega_h = \beta_0 A, \qquad \beta_0 = \frac{\alpha_0 A_0}{B_0}, \qquad B_0 = A_0 + D_0 = \frac{A}{1 + \delta A};$$

and hence by eliminating A_0, B_0, we find:

$$-\frac{\omega_h}{\alpha_0} = D_0 - A\,(1 - \delta D_0).$$

An optical instrument used in conjunction with the eye performs best when the accommodation is entirely relaxed. Only beginners, and especially young people, accommodate unnecessarily in using an optical instrument. Moreover, since the emmetropic eye is to be regarded as normal, we must put $A = 0$ in the above formula in order to obtain the *absolute magnifying power of an instrument.* Accordingly,

the absolute magnifying power is measured by the refracting power, and hence is expressed best by this value in dioptries. However, most people who work with optical instruments do not know what refracting power means, and so in order to get an abstract number without any dimensions, it is customary to multiply this value by the conventional "distance of distinct vision," namely 0.25 m. Thus, the refracting power in dioptries is four times as great as the conventional value of the magnifying power. The idea of the distance of distinct vision goes back to the time when it did not occur to people that the eye could be focused for infinity; and for this very reason it is an extremely unfortunate expression, because even nowadays an optician may be led to suppose that when an instrument is in action the image must be formed at this conventional distance from the eye. It is singular that ABBE,[1] who perceived the necessity of the notion of the absolute magnifying power of a lens system,—and called it "die vergrössernde Kraft"— was obliged to think of it in terms of a definite distance of the instrument from the eye, because at that time the distance of distinct vision was regarded as being something real. As a matter of fact, it is nothing more than simply a conventional distance of projection. When, employing this conventional mode of speech, we say that the magnification number of an instrument is n, all that we mean is that its refracting power is $4n$ dioptries; and hence an emmetropic eye, using this instrument without exerting any accommodation, obtains a retinal image of the object that is n times as large as it would be if the object was viewed by the eye alone at a distance of 25 cm from its anterior principal point; the reduced length of the eye not having been altered by the accommodation that is required in this latter instance.

In the case of an afocal instrument like the telescope, the absolute magnifying power cannot be obtained by the above formula. But it can be found immediately from the fact that in such types of instruments the values of the magnification-ratio are the same everywhere; and, consequently, an afocal instrument is completely determined as soon as we know the positions of a pair of conjugate points and the corresponding magnification-ratio. The reciprocal of this ratio is the reduced angular magnification-ratio; and if the latter is denoted by k, then

$$k = \frac{\omega_h}{\mathfrak{a}_0 A_0},$$

where the reduced convergence A_0 is measured from the point on the axis of the instrument that is conjugate to the anterior principal point

[1] ERNST ABBE, *Gesammelte Abhandlungen*. I. Jena 1904. S. 445.

of the eye. For an emmetropic eye the object is at infinity when the instrument is afocal. In this case the distance apart of the two points just mentioned is infinitesimal as compared with the distance of the object, and hence the number k gives the absolute magnifying power of the focal system.

The *individual magnifying power* in the various cases is to be found from the general formula above. With an ordinary magnifying glass, for example, both δ and D_0 are positive; and hence $1-\delta D_0$ is positive also, provided the posterior focal point of the glass is beyond the anterior principal point of the eye. Under such circumstances, the individual magnifying power is less than the absolute magnifying power for an hypermetropic eye, and greater for a myopic eye; and assuming that the reduced length of the eye is constant, the individual magnifying power increases with accommodation. Exactly the reverse is the case when the anterior focal point of the magnifying glass is nearer the glass than the first principal point of the eye; and if these two points coincide, the individual magnifying power and the absolute magnifying power are equal, no matter what the state of refraction of the eye is. If we put

$$\Delta = \delta - \frac{1}{D_0},$$

where Δ denotes, therefore, the reduced distance of the anterior principal point of the eye from the posterior focal point of the optical instrument, the following equation is obtained:

$$-\frac{\omega_k}{a_0} = D_0(1+\Delta A),$$

which is the same in form as that given by ABBE.[1] It shows that what was stated above in regard to a magnifying glass is true likewise for any optical system—for example, in case of a compound microscope for which D_0 is negative, and where δ may be negative for an instrument of low power; differences of magnifying power as above given being numerical differences.

Hence, the individual magnifying power as given by this formula is expressed in dioptries, since, after all, the scientific measure of magnifying power must be a measure of refracting power. In order to obtain an abstract number, this value must be multiplied either by the conventional distance of projection or by the distance which would be best for viewing the object in question by the naked eye, in each case the distance being given in metres. The first of these numerical values would not contribute anything useful, because nobody who is not

[1] Loc. cit.

already familiar with refracting power will be interested in individual magnifying power; but the latter number does give an expression for what may be termed the *individual effective power* of the magnifying instrument. Inasmuch as the displacement of the posterior principal point of the eye during accommodation is too slight to be taken in account, the individual magnifying power of an optical instrument as found by using the principal point angle is sufficient for obtaining the individual effective power; because the latter measure involves a comparison between two different states of accommodation, and the size of the retinal image is always proportional to the principal point angle, no matter what the accommodation may be.

Hence, *the essential facts concerning the magnifying power of an optical instrument used in conjunction with the eye* may be summed up as follows:

The expression

$$sD_0(1+\Delta A),$$

gives generally the individual magnifying power of the instrument in dioptries, provided we put s equal to the number 1; and for $A=0$, it gives also the absolute magnifying power. If, on the other hand, we put s equal to the value in metres of the distance of distinct vision in case of the given individual (counting this distance as positive), the above expression gives the number representing the individual effective power; and for $s=0.25$ m, which is the conventional distance of projection, and for $A=0$, it gives the number that expresses the conventional magnifying power of the instrument. Again, if we put s equal to the distance of the first principal point of the eye from the object, the above expression is identical with that of Eschricht and Panum[1], which is erroneously referred to the nodal point angle. A positive value for the expression means magnification without apparent inversion. Moreover, it should be noted that the method of derivation throughout implicitly assumes that the images are very small, and hence the formula applies only to the size of the image that is seen distinctly when the eye is stationary. There is really no difference between this formula and that of Abbe who used the tangent of the angle instead of the angle itself. However, nothing is gained by doing this, because, so far as we are concerned with the actual imagery, we have no right to substitute the magnitudes of object and image in place of the magnification-ratio unless the field is so circumscribed that the difference between angle and tangent is negligible. The contrary assumption is due to a fiction of the imagery's being collinear according to Abbe's notion.

[1] P. L. Panum, Die scheinbare Grösse der gesehenen Objekte. *Arch. f. Ophth.* V, 1. 1859. S. 1.

The peculiar contrivance of the organ of vision, with a very narrow field of distinct vision, which, however, may be shifted from place to place, is responsible for the circumstance that in inspecting an extended object the real factor after all is not the size of the image on the retina but the amount of the requisite angular movement of the eye. And hence when we come to consider *the magnification of an extended object*, we must employ the angle at the centre of rotation of the eye in place of the principal point angle, that is, the angle subtended at the centre of rotation by the image as seen in the instrument. And here we must keep in mind the fact that the values as given by the imagery-laws of the first order are merely approximate.

Again, when we wish to *compare the sizes of the retinal images in different eyes*, evidently, the principal point angles, cannot be employed, because the reduced length of the axis of the eye is not the same for different degrees of refraction. However, we know from ophthalmometric investigations that the differences that occur in the optical system of the eye are not connected with the state of refraction; and hence the focal point angle does afford a measure of the size of the retinal image that—at least in case of so-called axial ametropia—is independent of the refraction, and consequently is best suited for the purpose here. The refracting power of the optical system of the eye is equal to the ratio between the focal point angle and the size of the retinal image; but it is not practicable to compare the sizes of the retinal images in two different eyes unless the refracting power has the same value for both eyes. For most eyes this seems to be approximately the case, and such cases of ametropia as show no sign of any difference of this kind are included under what is known as *axial ametropia*, as distinguished from *curvature ametropia*, which is the name given to those forms of ametropia in which the length of the axis of the eye is normal. However, both in axially-ametropic eyes and in emmetropic eyes different values of the refracting power do occur, as we know from v. REUSS's[1] investigations. Hence the size of the retinal image as measured by the focal point angle represents merely its probable value, and the best we can obtain at present.

Finally, *in comparing the visual acuity before and after removing the crystalline lens of the eye*, the angle to be used (as BJERKE[2] found) is *the angle whose vertex is situated at the apparent place of the optical centre of the lens*. (With the degree of accuracy that is attainable in making this comparison it is quite permissible to use here the simplified schematic eye.) If this angle is denoted by ω', and if the magnification-

[1] Loc cit.
[2] Loc. cit.

ratio at this optical centre is denoted by κ, then

$$\omega' = \beta_0 A' = \kappa\beta B',$$

where A', B' are the reduced convergences. Now since $\kappa B'$ has the same value before and after the extraction of the lens, this must be true also with respect to the ratio $\frac{\omega'}{\beta}$.

The derivation of the formulae is precisely the same for the other angles as for the principal point angle; and hence the three following expressions,

$$-\frac{\omega_n}{\alpha_0} = -\frac{A_n A_0}{B_0} = D_0 - A_n(1 - \delta_n D_0) = D_0(1 + \Delta_n A_n)$$

are quite general, where ω_n denotes the angle which the image as seen in the optical instrument subtends at the point designated here by N; A_n denotes the refraction of the eye as measured at this point; and δ_n, Δ_n denote the distances of N from the second principal point and from the second focal point of the optical instrument, respectively.

Thus we are in the position to investigate now the variations of the visual acuity as dependent both on the ametropia and on its correction. The visual acuity of the eye is partly a measure of the sensitivity and efficiency of the retina itself; and consequently must be measured by some method which enables us to compare the sizes of the retinal image in different eyes. The principle of it consists, therefore, in ascertaining the smallest actual value of the focal point angle when accommodation is entirely relaxed (far point focusing). But the visual acuity is partly also a measure of the capacity of the individual eye, entirely apart from its state of accommodation; and from this aspect it must be measured by the smallest actual value of the principal point angle. The former measure is called the *absolute visual acuity*; and the latter *the natural visual acuity*, as directly ascertained, therefore, by finding the least value of the principal point angle for the unaided eye. In both of these measurements the retinal image is the smallest image possible, and for one and the same eye this smallest image is the same in both cases. Now the visual acuity is inversely proportional to the "angle of distinctness" or resolving power of the eye; and since for a given value of β we have always

$$\frac{\omega_h}{\omega_f} = \frac{B}{D},$$

we derive the following formula for the absolute visual acuity (S) in terms of the natural visual acuity (S_n):

$$S = S_n\left(1 + \frac{A}{D}\right).$$

If L denotes the correction-value measured at the anterior focal point of the eye, then since

$$\frac{1}{A}=\frac{1}{L}-\frac{1}{D},$$

the formula above may be expressed also in terms of L instead of A, as follows:

$$S_n=S\left(1-\frac{L}{D}\right).$$

If f denotes the numerical value of the anterior focal length of the eye in centimetres, the preceding formulae may be stated in words, thus: *The absolute visual acuity is found by adding to the natural visual acuity $f\%$ for every dioptry of ametropia; and the natural visual acuity is found by subtracting from the absolute visual acuity $f\%$ for every dioptry of the correction-value at the anterior focal point of the eye.* However, in using these rules it must be remembered that for a myopic eye both the ametropia and the correction-value are negative, and hence algebraic addition is equivalent to arithmetical subtraction, and *vice versa.* The formulae show how axial ametropia affects the natural visual acuity In a case of pure curvature ametropia, in which the reduced length of the axis of the eye is normal, the natural visual acuity is independent of the ametropia. On the other hand, for the same sensitivity of the retina, the absolute visual acuity, as in the case of curvature-anomalies of an emmetropic eye, is inversely proportional to the refracting power of the optical system of the eye.

If in measuring the visual acuity of the eye by the aid of an auxiliary optical instrument the vertex of the "angle of distinctness" used in the determination is taken at the anterior principal point of the interposed optical system, and if the apparent value of the visual acuity as thus found is called *the relative visual acuity* and denoted by S_r; the ratio of S_r to S is equal to $\frac{\omega_f}{\alpha_0 A_0}$, and the ratio of S_r to S_n is equal to $\frac{\omega_h}{\alpha_0 A_0}$; and hence these ratios may be obtained immediately from the above formulae. From the equations

$$S_r=S\cdot\frac{L}{B_0}=S_n\cdot\frac{A}{B_0}$$

we derive:

$$S_r=S\,(1+\delta_f L)=S_n\,(1+\delta A),$$

where δ_f denotes the distance of the anterior focal point of the eye from the posterior principal point of the auxiliary optical system. In

the ordinary determination with a spectacle glass this interval is so slight that it is negligible for the degree of precision with which the visual acuity can be measured; and the same thing applies also to the difference between the value of L and the correction value as found in this case. For $\delta_{\prime}=0$, the relative visual acuity of the unaccommodated eye is equal to the absolute visual acuity; and hence, with sufficient accuracy for practical purposes, we may say that *the absolute visual acuity is obtained immediately by making the measurement with a spectacle glass that abolishes accommodation;* and also that the ratio between the natural and the absolute visual acuity is the same as that between the correction value and the ametropia. According to the above, the distance at which the vision is tested is a matter of no consequence whatever; but the shorter this distance, the more powerful must be the instrument employed, and the more perfect must be the fulfilment of the condition that the distance from the anterior principal point of the instrument shall be the basis of the measurement of the visual acuity, and that its posterior principal point shall coincide with the anterior focal point of the eye.

Moreover, the formulae show that *the relative visual acuity of the unaccommodated eye is to the natural visual acuity generally in the same ratio as the ametropia is to the correction value measured at the posterior principal point of the auxiliary system;* the latter value being the same as the refracting power of the auxiliary system only when the object is very far away. But for a great distance of the object,

$$(1+\delta A)\,(1-\delta D_0)=1,$$

and hence

$$S_n=S_r\,(1-\delta D_0).$$

Consequently, if the interval δ is given in centimetres, we can say that *the natural visual acuity generally may be found from the relative visual acuity as determined with a distant object by subtracting from it $\delta\%$ for every dioptry of the power of the glass employed,* no matter whether the eye accommodates or not.

Instead of using the angle ω_h which is concerned in the measurement of the natural visual acuity, suppose we employ the angle $\omega_{\prime}$ with its vertex at the apparent place of the optical centre of the crystalline lens. Then, applying the formula above, we get the ratio of the relative *visual acuity after extraction of the transparent crystalline lens to that found before its extraction*, namely:

$$\frac{1-\delta_{\prime}D_{\prime}}{1-\delta_{\prime}D_{\prime\prime}},$$

where $\delta_{\prime}$ denotes the distance of the apparent place of the optical centre from the posterior principal point of the auxiliary system, and $D_{\prime},D_{\prime\prime}$

denote the refracting powers of the spectacle lenses used before the operation and afterwards, respectively. The same result is obtained, rather more circuitously, by comparing the absolute visual acuity of the eye without the crystalline lens to that of the eye with it, this ratio being equal to $D:D_c$.

When the eye is combined with an optical instrument in such manner that the posterior principal point of the instrument and the anterior focal point of the eye are united, it is possible to *calculate the change in the length of the eyeball corresponding to a given value of the axial ametropia.* According to the formulae on page 361, the refracting power of the compound system in this case is equal to that of the eye alone, and the anterior focal point of the system as a whole coincides with the anterior principal point of the auxiliary system. Since the focal lengths are not altered by inserting the latter system in this way, the reduced displacement of the posterior focal point is

$$H_t' = -\frac{D_0}{D^2};$$

and if the correction value is D_0, the posterior focal point lies on the retina of the eye. According as the basis of the calculation is the exact or simplified schematic eye, it follows that the length of the axially ametropic eye differs from that of the emmetropic eye by 0.389 or 0.374 mm, respectively, for every dioptry of the correction value. The same result is obtained by using the general focal point equation.

When the eye is combined with an instrument whose posterior focal point coincides with the anterior principal point of the eye or with its anterior focal point, the natural visual acuity in the former case, or the absolute visual acuity of the unaccommodated eye in the latter case, is obtained by measuring the angle $a_0 D_0$; as follows immediately from the formula

$$\omega_n = -a_0 D_0 (1 + \Delta_n A_n),$$

since in both cases $\Delta = 0$. Since, as was shown above, the refraction in the first instance, and the correction value in the second, is proportional to the distance of the anterior focal point of the auxiliary system from the object, these combinations have frequently been used in the construction of optometers.

In measuring the refraction of the eye, besides an auxiliary optical instrument, an indicator is needed or some means of telling whether the image in the eye is sharp or not. The best means will always be a measurement of the visual acuity that is under the control of the investigator and independent of the patient's opinion. The division of the pupil into separate parts would afford another indicator for the existence of blur circles; these parts being employed, simultaneously

or in succession, for optical projection, unless the accuracy of the result is impaired by the aberration-behaviour of the normal eye. Here may be mentioned the indicators based on the experiments of SCHEINER and MILE; together also with the method recently proposed by HOLTH[1] under the name of Kinescopia. If the connection between the size of the pupil and the ocular aberration were such that the laws of the first order were applicable, the theory of this latter process would be contained in the value of the linear projection coefficient (page 290), since the question involves the optical projection on the retina of a hole held in front of the eye. In this case the magnitude denoted by κ in the formula

$$C = \kappa\,(1 - \delta'\mathfrak{B})$$

is the magnification-ratio at the point conjugate to the hole with respect to the optical system of the eye; whereas δ' denotes the reduced distance of the retina from this point, and $\mathfrak{B}$ the reduced convergence of the bundle of object-rays measured at the same place. A positive value of C here indicates an optical projection in the direct sense (homonymous), that is, a crossed monocular diplopia in SCHEINER's experiment and an apparent movement opposite ("against") the real movement in MILE's experiment. Putting

$$\Delta' = \frac{1}{\mathfrak{B}} - \delta',$$

where Δ' denotes the reduced distance of the image-point that is conjugate to the object-point as measured from the retina, we may write the preceding formula as follows:

$$C = \kappa\mathfrak{B}\Delta' = \frac{\mathfrak{A}\Delta'}{K},$$

where K denotes the magnification-ratio at this image point, and $\mathfrak{A}$ is the convergence of the bundle of object-rays as measured at the place where the hole is. Hence, it follows that C does not change sign when the sign of κ changes in consequence of a shifting of the hole along the axis of the eye past its anterior focal point.

The same formulae give also the *size of the blur circle* in so far as this can be obtained by the laws of the first order; but the results are to be considered as being merely approximations to the real facts in the case when the pupil is small and the focusing error quite large. Under these circumstances $\mathfrak{A}$ and $\mathfrak{B}$ denote the reduced convergences of the bundle of object-rays as measured at the *entrance-pupil* before

[1] S. HOLTH, Nouveau procédé pour déterminer la réfraction oculaire. *Ann. d'Oculistique.* CXXXI. 1904. p. 1.

refraction and at the *exit-pupil* after refraction, respectively. The centre of the entrance-pupil is the apparent position of the pupil of the eye and the centre of the exit-pupil is the point conjugate to the centre of the entrance-pupil with respect to the optical system of the eye; and κ denotes the magnification-ratio for this pair of conjugate points. If the (static or dynamic) refraction of the eye as measured at the entrance-pupil is denoted by $\mathfrak{R}$, then

$$\frac{\kappa^2}{\delta'}=\kappa D+\mathfrak{R}, \qquad \kappa^2\mathfrak{B}=\kappa D+\mathfrak{A}, \qquad \frac{\kappa K}{\delta'}=\mathfrak{R}, \qquad KD=L;$$

consequently,

$$\kappa^2\left(\frac{1}{\delta'}-\mathfrak{B}\right)=\mathfrak{R}-\mathfrak{A}, \qquad \frac{\delta'\mathfrak{R}}{\kappa}=\frac{L}{D};$$

and hence

$$C=\frac{\delta'(\mathfrak{R}-\mathfrak{A})}{\kappa}=\frac{L}{D}\left(1-\frac{A}{\mathfrak{R}}\right);$$

where C denotes the ratio of the diameter of the blur circle to the diameter of the entrance-pupil, and δ' denotes the reduced distance of the retina from the exit-pupil. Neglecting in this formula the distance between the entrance-pupil and the anterior principal point of the eye and also the difference between the size of the pupil and that of the entrance-pupil, we may put $\kappa=1$ and consider δ' as representing the reduced length of the axis of the eye. The approximate formula derived in this way was obtained by SALZMANN[1] for the general case, and by NAGEL[2] for the special case when $\mathfrak{A}=0$. GLEICHEN,[3] using the magnification-ratio for the pupil, has shown that the latter formula holds exactly.

The laws of imagery and optical projection of the first order are not sufficient for investigating *blurred vision*. For with a pupil of medium size the aberration of the eye is such that focusing errors amounting to several dioptries are not enough to prevent the existence of curves formed by the intersection of the caustic surface with the retina, involving a single or multiple imagery of lines; whereas, on the other hand, even with the sharpest focusing, blur circles of considerable magnitude occur. With a smaller pupil phenomena of diffraction have play. Due to the first circumstance, the exact investigation of the *depth of focus* of the eye, CZERMAK's so-called accommodation-line,

[1] M. SALZMANN, Das Sehen in Zerstreuungskreisen. *Arch. f. Ophth.* XXXIX. 2. 1893. S. 83.

[2] A. NAGEL, Die Anomalien der Refraktion und Akkommodation des Auges. *Handb. d. ges. Augenheilk. von* GRAEFE und SAEMISCH. Bd. VI. Kap. X. 1880. S. 457.

[3] A. GLEICHEN, *Einführung in die medizinische Optik.* Leipzig 1904. S. 117. And: Über die Zerstreuungsfiguren im menschlichen Auge. *Arch. f. Optik.* I. 1908. S. 211.

proves to be a very complicated mathematical problem. These aberrations explain also the influence of practice in trying to interpret the image-patterns made by other sections of the caustic surface, as SALZMAN[1] has emphasized. This practice also is particularly important for explaining the peculiar ability of reading in the case of high uncorrected hypermetropia; although the ratio between the size of the blur circle and the size of the indistinct image has something to do with it also.

In general, *for the optical projection on the retina of objects that are not seen distinctly*, the linear projection coefficient C_0 may be found from the fact that the reduced angular magnification-ratio at the pupil is equal to the reciprocal of the linear magnification-ratio:

$$C_0 = \frac{\delta' \mathfrak{A}}{\kappa}.$$

When the object is real and has to be adjusted near the eye to be seen distinctly, both $\mathfrak{A}$ and C_0 are negative, whereas C and $\mathfrak{R} - \mathfrak{A}$ are positive. Hence, for this case:

$$Q = -\frac{C}{C_0} \qquad O = -\mathfrak{A},$$

where Q and O are the numerical values in question. The general formula

$$\frac{C}{C_0} = \frac{\mathfrak{R} - \mathfrak{A}}{\mathfrak{A}}$$

may then be put in the form:

$$Q = 1 + \frac{\mathfrak{R}}{O}.$$

Let p denote the diameter of the entrance-pupil and o the linear size of the object; then $\frac{pQ}{o}$ is the ratio of the diameter of the blur circle to the linear size of the image projected on the retina. It follows at once from the formula that Q diminishes as O increases, provided $\mathfrak{R}$ is positive; whereas when this latter magnitude is negative, the case is just opposite. This agrees with the fact that when an object approaches an hypermetropic eye, the size of the blurred image on the retina increases faster than the diameter of the blur circle; while with a myopic eye it is the other way. This explains how an hypermetrope, of say, 8 dptr can read fine print 5 cm away from his unaided eye. However, the fact that hypermetropia is really more unfavourable than other states

[1] M. SALZMANN, Das Sehen in Zerstreuungkreisen. *Arch. f. Ophth.* XL, 5. 1894. S. 102.

of refraction, can readily be understood by observing that for a positive value of the refraction, even when the object is nearest, Q is greater than unity; whereas for emmetropia, Q always has the value unity; and this value is not reached in myopia as long as the object is too near the eye to be seen distinctly. This particular effect of the state of refraction is illustrated best by taking an object whose length is equal to the diameter of the entrance-pupil. Then Q denotes the ratio between the diameter of the blur circle and the size of the image, and the sign of $(Q-1)$ tells whether the blur circles corresponding to the two ends of the object overlap or not. Evidently, they do in hypermetropia; but in emmetropia these two blur circles touch each other; and in myopia there is a space between them. It is apparent, therefore, that emmetropic refraction, and still more myopic refraction, is more favourable for the kind of reading here spoken of than hypermetropic refraction; and that the hypermetrope gets the best results by disregarding his error of refraction and accommodating as much as possible. This is exactly what he does, in the effort to reduce the size of the blur circle, not so much by changing the optical focus as by the accompanying contraction of the pupil. Emmetropes and myopes might therefore be able to read in this way even better than hypermetropes, provided they were able to contract the pupil to the same extent as uncorrected hypermetropes, and had, like the latter, practised the art from childhood.

However, for the size of pupil here under consideration the quality of the image produced by optical projection may not be gauged by the size of the blur circle, because *diffraction* at the edge of the pupil influences the distribution of light inside this circle.[1] It is easy to be convinced of this by using a stenopaic opening. By making himself ametropic with a high power glass in front of his eye and looking through a hole held opposite the centre of the pupil between the glass and the eye, a person may see, even with a 2 mm hole, but better still with a 1 mm hole, a concentration of light on the border of the blur circle corresponding to a small luminous point-source. That this is a diffraction-effect, and not due, say, to aberration of the glass, is shown partly by the fact that the phenomenon persists with larger opening, and partly also because the effect of the aberration would be just the opposite. Being a matter of diffraction, the inference cannot be drawn that the shadow cone corresponding to a black point on a white surface is analogous to the cone of light for the luminous point. The exact investigation of the effect of diffraction in a case of this sort would be a very difficult mathematical exercise. However, it is clear that it

[1] Accordingly, there would be no practical result in discussing the case when $C+C_0=K$, however interesting it might be from the geometrical point of view.

does distribute some light in the region of the shadow cone, and hence also does have some influence on the size of the shadow circle and the shadow distribution.

Since in the optical projection in the eye the apparent size of the object is increased by the blur circles, obviously, the magnification in the correction of high hypermetropia by convex glasses is less noticeable than the opposite effect in the correction of high myopia by concave glasses; and in the latter case the correction is not so advantageous, especially if on account of poor visual acuity small retinal images cannot be appreciated anyhow.

Chromatic resolution of the blur circles affords a method of investigating them that is comparatively unaffected by the aberration and diffraction. In the formula on page 371 Δ' has different values for different colours; and therefore when the focusing is sharp for a colour of short wave-length there must be a blur circle for a colour of long wave-length, and *vice versa*; and these blur circles must be seen more and more distinctly in proportion as the light consists of a mixture of two colours of long and short wave-lengths. A cobalt glass of sufficiently saturated colour illuminated by an ordinary candle flame affords practically a very useful light-mixture for this purpose. A very good indicator of the imagery, as described by HELMHOLTZ on page 176, is afforded also by observing the coloured edge of a hole placed in front of the source of light.

DONDERS' *method of finding the refraction* is still for the time being superior to all others. It consists in connecting the eye with various glasses, and using the visual acuity test with object far away as indicator of the focusing. The process therefore gives both the correction-value and the absolute visual acuity, the strongest positive or weakest negative lens being found for which the visual acuity of the eye is greatest. The advantages of this method consist in the objective control made possible by the visual acuity test and in the relaxation of the accommodation, which according to experience is easier for most people with spectacles and a distant object than when they have to look into an instrument. It would take us too far to enter here into the details of this method or of the other methods of optometry. It may be merely remarked in this connection, that just as a great distance of the object is favourable to relaxation of accommodation, similarly a short distance stimulates this power. Thus while DONDERS' method is best for finding the far point, *the determination of the near point* necessary for obtaining the amplitude of accommodation is best made by direct measurement, after having shifted it by inserting a suitable lens before the eye to a distance that is convenient for measuring.

Numerous investigations have established the fact that *the physiological refraction of the eye* at birth is hypermetropic. The contrary data of JAEGER,[1] as HESS[2] and ELSCHNIG[3] have shown, are due to the fact that in his measurements the pupil was not dilated artificially, and that the refraction of babes is essentially diminished by accommodation, temporarily anyhow. Congenital hypermetropia, amounting on the average to 2 dioptries according to STRAUB's[4] findings, decreases even in childhood, so that from school-age upwards the normal state of refraction is emmetropia or very feeble hypermetropia, returning again after fifty years of age to something like that of infancy and approximately reaching this state at a very advanced age. STRAUB is of the opinion that the greatest part of the hypermetropia continues throughout the entire life, being masked by a tonic of the ciliary muscle; but the material in support of this view is perhaps hardly sufficient to establish its correctness.

The change of refraction that occurs in earliest childhood is the result of the change of the length of the eyeball and of the optical system in growth. As to the optical system, the radius of curvature of the cornea is indeed rather smaller in the infant than in the adult—the values are around 7.0 mm. But the chief difference is in the lens, whose form is nearly spherical, and which therefore must have a very high total index of refraction. The attainment of approximately emmetropic refraction by the vast majority of eyes would evidently imply a regulating mechanism operating in growth. Thus we know that in hypermetropia efforts of accommodation are necessarily of much longer duration than in other states of refraction, and that in continuous accommodation a force is at work which with the growth of the eye tends to lower the refraction. By the tension of the choroid produced by the contraction of the ciliary muscle, a statical moment is always brought into play which might have some effect on the growth of the axis of the eyeball, and whereby the change of form of the lens in accommodation may undoubtedly have some influence during growth on the arrangement of the iso-indicial surfaces, and accordingly also on the total index. It is quite possible that other regulating forces, as yet unknown, may be present also.

The recurrence of hypermetropia in extreme old age is explained partly by the change of form due to the increased resistance of the

[1] ED. V. JAEGER, *Über die Einstellungen des dioptrischen Apparates im menschlichen Auge.* Wien 1861.

[2] Loc. cit. S. 284.

[3] Bemerkungen über die Refraktion der Neugeborenen. *Zeitschr. f. Augenheilk.* XI. 1904. S. 10.

[4] M. STRAUB, Die normale Refraktion des menschlichen Auges. *Zeitschr. f. Physiol. d. Sinnesorg.* XXV. 1901. S. 78. — Über die Ätiologie der Brechungsanomalien des Auges und den Ursprung der Emmetropie. *Arch. f. Ophth.* LXX. 1. 1909. S. 130.

coating of the eyeball and the decreased pressure of the surrounding tissue, as established by ophthalmometric investigation of the cornea; but partly also by senile changes in the crystalline lens. Whether the first of these causes produces a measurable increase in the radius of the cornea, perhaps, as remarked above, cannot yet be positively asserted; but there is hardly any doubt that it might be responsible for shortening the length of the ocular axis a few tenths of a millimetre. The other influence may cause a reduction of the total index, not only by augmenting the cortical index but also by changing the form of the iso-indicial surfaces of the crystalline lens. Last of all, the effect of the senile contraction of the pupil has to be considered also, which with normal aberration must produce at least some slight increase of the refraction; because an eye that was emmetropic with a pupil of medium size would have to show hypermetropia amounting to 1 dptr with an infinitely small pupil.

The change of the lens, that is senile in its last stage, begins in earliest childhood. This is connected with the fact that the lens is a closed epithelium structure with no outlet for discharge; in which indeed, throughout the whole life a constant influx occurs, as is shown by the investigations of PRIESTLEY SMITH.[1] This change is manifested in the living eye by a steady increase of fluorescence[2] that is perceptible without difficulty by suitable arrangements. Another sign of it is the appearance of reflex images on the periphery of the lens core known as HESS's "Kernbildschen"; indicating a surface of discontinuity in the variation of the index, which might constitute, so to speak, the boundary between a core and a cortical layer. And, finally, this process in the lens is shown by an increasingly diffused reflection of the light and by a colour of the light reflected from the core, which with advancing years gets more and more yellow; frequently too by a doubling of the PURKINJE image reflected in the posterior surface of the lens, when this image is being observed in a direction that borders on the equator of the lens core. Correspondingly in the dead lens, along with the steady increase of dimensions, there is found also a progressive sclerosis of the central portions and differentiating of the core. Functionally, this change in the lens is manifested by a progressive *decrease of the amplitude of accommodation* which begins as soon as this magnitude can be measured. The following are the numerical values as given by DONDERS[3] in his fundamental researches:

[1] The growth of the crystalline lens. *Brit. Med. Journ.* I. 1883. S. 112.

[2] A. GULLSTRAND, Die Farbe der Macula centralis retinae. *Arch. f. Ophth.* LXII, 1. S. 43.

[3] Loc. cit.

Age in years	Amplitude of accommodation in dioptries
10	14
15	12
20	10
25	8.5
30	7
35	5.5
40	4.5
45	3.5
50	2.5
55	1.75
60	1
65	0.5
70	0.25

However, in considering these numbers it should be borne in mind that the method employed gives the limit of perceptibility of the smallest blur circle, and hence the measurement as made depends on the focus depth and is affected by the size of the pupil. Perhaps, therefore, the senile decrease of amplitude of accommodation may proceed rather more suddenly than the table indicates, and for every 5 years after the age of forty it may be estimated at 1 dptr with sufficient accuracy.

Presbyopia is the recession of the near point beyond the conventional distance of distinct vision, which means here a distance of 22 cm according to the earlier value that was assigned to this arbitrary measure. (In calculating magnifying power the value of 25 cm was taken by the followers of ABBE as the measure of the distance of distinct vision.) In the case of an emmetropic eye presbyopia begins after forty years of age. But from a practical standpoint the customary size of the pupil has much to do with this matter. This magnitude is very often affected by an illness that has lowered the general vitality or by a neurasthenic condition; and thus owing to an increase in the size of the pupil there is a development, familiar to every ophthalmologist, of sudden presbyopia in an emmetrope fifty years old. Corresponding to the idea of presbyopia, the corrected ametrope usually becomes presbyopic at the same age as the emmetrope, but the uncorrected hypermetrope become presbyopic sooner, and the uncorrected myope later or perhaps never.

It would lead us too far at the present stage of this science to take up more in detail *the anomalies of refraction*. Its enormous development since the appearance of the first edition of this treatise is apparent when it is recalled that there for the first time in a supplement the difference between hypermetropia and presbyopia was noticed. DONDERS'[1] great

[1] Loc. cit. Translated in German by O. BECKER: *Die Anomalien der Refraktion und Akkommodation des Auges.* Wien 1866.

work was followed by books by MAUTHNER,[1] NAGEL[2] and LANDOLT;[3] and the entire subject was essentially revised by HESS[4] by employing the exact dioptry-idea, and its treatment elevated to the requirements of the time.

Hypermetropia is congenital in the typical cases and comes under the head of axial ametropia. In its higher degrees it is to be regarded as a structural hindrance, frequently accompanied by other structural faults such as astigmatism, asymmetry, abnormal form of the papilla, etc.; whereas in its milder form it should perhaps be explained as some anomaly of growth. Of all the unusual (or atypical) forms of hypermetropia the most frequent by far is that which occurs after extraction of the crystalline lens, which is also the most remarkable example of a curvature ametropia.

On the other hand, typical *myopia* is an acquired anomaly, belonging, however, like typical hypermetropia under the head of axial ametropia. The mildest degrees of myopia, as of hypermetropia, are probably due to some simple growth anomaly, whereas its highest manifestations are undoubtedly symptoms of disease. Just as to a certain extent the line between growth anomaly and diseased condition is arbitrary, so also the opinions of ophthalmologists are not at all unanimous as to the degree of myopia that is most common. The original cause of it is known to be a disposition, either constitutional or acquired by weakness of some kind, and the effect of over-exertion with close work. The disposition is generally attributed to too much yielding of the coating of the eyeball, but possibly too the static relations in growth may have something to do with it, similar to the predisposition of *dolichocephali* or long-headed folks. As to how and why over-exertion with close work is injurious, there has come to be a wide diversity of opinion. The view that accommodation has a tendency to increase the intraocular pressure to which ophthalmologists nowadays attribute the prejudice against complete correction of myopia on the part of the laity and many factory workers, has been upset by careful investigations. Moreover, the idea that the convergence necessary for close work should bear the blame, can hardly be any longer maintained in those cases where convergence is normal and effected without any abnormal strain of the external muscles of the eye. On the other hand, it is evident that a steady fixation involving exertion

[1] Loc. cit. *Vorlesungen über die optischen Fehler des Auges.* Wien 1876.

[2] Loc. cit. *Die Anomalien der Refraktion und Akkommodation des Auges.* In *Handb. d. ges. Augenheilk. von* GRAEFE und SAEMISCH. Bd. VI. X. Kap. 1880.

[3] E. LANDOLT, *La réfraction et l'accommodation de l'oeil.* In *Traité compl. d'ophth.* par WECKER et LANDOLT. T. III. Paris 1887.

[4] Loc. cit. *Die Refraktion und Akkommodation des menschlichen Auges und ihre Anomalien.* Leipzig 1902, and in *Handb. von* GRAEFE und SAEMISCH. 2. Aufl. II. T. XII. Kap.

on the part of all the external muscles, together with the customary knitting of the eyebrows as in sickness, may have a tendency to elongate the axis of the eye. And supposing the steady accommodation that goes on in the hypermetropic eye of a child is a process of regulation inducing emmetropic refraction, may it not likewise be probable that an excessive accommodation strain tends to promote myopia? Certainly, clinical experience testifies to a comparatively frequent conjunction between myopia and those conditions dating back to childhood, which, due to poor visual acuity, require the work to be brought excessively close to the eye; such as lamellar cataract, corneal specks, astigmatism, abnormal decentration, cases where the accommodation by means of the accompanying contraction of the pupil improves vision disproportionately, and is for that very reason apt to be used excessively.

However, while opinions may differ as to the manner of the effect of "near work," the necessity of trying to prevent unnecessarily strenuous "near work" is recognized to be one way of combatting the spread of myopia, as is attempted, for instance, in the modern system of school-hygiene. The glorious victory which Sweden has won by this method is a tribute to WIDMARK's[1] classification. By abandoning Gothic type and the so-called German handwriting, Germany might probably no longer enjoy the unenviable distinction of being known as "Myopia-land."

From what has been said it is evident that the supposition, especially prevalent among uncorrected myopes, that myopia is a sort of adaptation to the needs of civilization, is completely mistaken, since, rather than identifying civilization with close work, myopes are to be considered as people who have to pay for civilization by being disabled.

Cases of unusual or *atypical myopia*, coming under the category of curvature ametropia, occur with abnormal form of the cornea and of the lens surfaces. In the same class, though strictly speaking representing "indicial myopia," belongs the myopia that is found in connection with senile changes of the lens in advanced age, and likewise the temporary myopia discovered first by HIRSCHBERG[2] in case of diabetes, by MOAURO[3] in case of jaundice, and by SCHIECK[4] in case of sleeping sickness; whereas the temporary myopia occurring in case of iritis,

[1] J. WIDMARK, Über die Abnahme der Kurzsichtigkeit in den höheren Knabenschulen Schwedens. *Mitt. a. d. Augenklinik d. Carol. Med.-Chir. Inst. zu Stockholm.* X Heft. 1909. S. 41.

[2] J. HIRSCHBERG, Diabetische Kurzsichtigkeit. *Centralblatt f. prakt. Augenh.* XIV. 1890. S. 7.

[3] G. MOAURO, Di alcune alterazioni oculari in malattie epatiche. *Lavori della clin. oculist. di Napoli.* III. 1893. S. 100.

[4] F. SCHIECK, Über temporäre Myopie. *Klin. Monatsbl. f. Augenh.* XLV. 1907. S. 40.

regarded by SCHAPRINGER[1] as an indicial myopia, may be explained by tension of the zonula fibres in case of swelling of the ciliary processes. Senile myopia is always accompanied by a characteristic variation of the aberration and is due to an increase of the total index of the lens. An increase of the core index may tend in this direction, but myopia is possible even without such a change, because with progressive sclerosis the form of the iso-indicial surfaces may be changed. The final stage, known as "false lenticonus," consists in the loosening of the transparent core from the cortical substance or in a cataract formation usually proceeding with pronounced core-sclerosis. The tendency to cataract in diabetes renders it likely that cases of permanent myopia in connection with diabetes among old people are similar. MOAURO and SCHAPRINGER connected temporary myopia with increase of index of the aqueous humor, but as HESS[2] has shown, this is impossible. It might be more reasonable to attribute it to a variation of the index of the substance of the lens caused by a change of composition of the surrounding fluid, because this is the explanation also of temporary hypermetropia as observed in case of diabetes. Thus if the new surfaces corresponding to the change of index do not coincide with the iso-indicial surfaces, the arrangement of the latter will be changed, and the total index for the individual difference in the procedure of the iso-indicial surfaces may be raised thereby as well as lowered.

Astigmatism cannot be included under ametropia proper, because it does not imply a peculiar focusing fault but rather an image fault that defies exact focusing. Whereas in the case of ametropia in the strict sense it is possible to speak of a single image in the *fovea centralis*, this is no longer permissible with astigmatism, but the two imageries have to be kept separate. In the former case there is a certain point-to-point correspondence between object and image, but not so in the latter case. Only those lines are reproduced that are parallel to the principal sections when the latter are planes of symmetry, which however may not be at right angles to each other; but the image-lines on the retina are perpendicular. With astigmatism of sufficient amount its existence can therefore be established even in the most regular cases, and the orientation of the principal sections can be found, by making the patient name the two meridian lines of a circle, as marked on a chart, that are seen distinctly with different focusing, no matter whether the change of focus is produced by accommodation or by lenses from the trial case. However, in the ordinary degrees of this trouble

[1] A. SCHAPRINGER, The proximate cause of the transient form of myopia associated with iritis. *Amer. Journ. of ophth.* X. 1893. p. 399.

[2] Loc. cit. S. 341.

the shortness of the focal length as compared with the amount of the aberration is such that for normal size of pupil the relations are much more complicated, because different sections of the caustic surface cause the imageable lines to lie in different directions for different focusings or produce a double image. The only way, therefore, to find the directions of the object lines corresponding to the principal sections is to contrive in such fashion that the entire focal interval (as measured by the astigmatic difference) shall lie in front of the retina and at the same time coincide with the edge of the caustic surface; so that the rear edge, being its only line of section, is all that can affect the retina, as follows from Fig. 121. This form of the caustic surface shows the erroneousness of the prevalent opinion that the amount of astigmatism can be found by using the fan-shaped chart with the meridian lines. This can only be done by using the visual acuity as indicator, since it is the maximum visual acuity—and not just merely the "normal" acuity—that corresponds to the proper correction of astigmatism. Astigmatism is so common and so often complicated with abnormal decentration of the eye that there is perhaps no other scientific investigation that has a greater claim on scientists than a conscientious measurement of refraction and visual acuity.

The limit of the normal astigmatism of the eye probably should be given as 0.5 dptr, the direct form being more common in youth and the inverse form ("against the rule") in more advanced life. Consequently, an inverse astigmatism of 0.5 dptr in youth is to be regarded as a pathological symptom. Clinical experience shows that such a condition may be harder for the patient than a direct astigmatism amounting to 1 dptr or more.

In the typical cases of abnormal astigmatism the trouble is congenital and evidently dependent on static conditions in the development and growth of the eye. Acquired astigmatism (except that beginning in old age and the inverse astigmatism due to increase of pressure in glaucoma) occurs generally after illnesses and operations through the cornea.

IV. The Mechanism of Accommodation

For a dioptric study of the accommodating lens it is necessary to know the nature of the changes in the form of its surfaces and of the increase of its thickness. The other changes which have been shown to be associated with accommodation are of less importance for the

dioptrics of the accommodating lens than for the mechanical processes of accommodation.

In order to begin the discussion of the problem of the mechanism of accommodation from the optical standpoint, supposing the change of focus of the eye due to the variations above mentioned is known, we have now merely to ascertain the data of the accommodating schematic eye that corresponds to the schematic eye as previously found. However, owing to the difficulties involved in the decentration of the refracting surfaces, and to lack of knowledge as to the form of the peripheral parts of the surfaces of the lens, the requisite data are not sufficiently well known at present. There have been plenty of researches on these subjects, but, so far as the forward displacement of the anterior pole of the lens and the change of curvature of the anterior surface are concerned, the results can be considered as only partially reliable. On the basis of the earlier work,[1] HELMHOLTZ selected the schematic values 0.4 and 6 mm for the change in position of the anterior pole of the lens and the radius of the anterior surface of the lens, respectively, in the accommodating eye; the amount of the optical change of focus being found by calculation. However, the change of focus as found by observation should be greater than appears from this calculation, as up to this time no investigator has found that the radius was reduced to one-half the size, and HELMHOLTZ's accommodating schematic eye shows a change of refracting power amounting to only 6.5 dioptries. The recent investigations of TSCHERNING and BESIO[2] agree very well with these values, provided the relative change of curvature is considered, and allowance is made for the sources of error incident to the methods of investigation. In the cases which TSCHERNING studied, instead of a forward displacement of the anterior surface of the lens, he found a backward movement of the posterior surface, resulting in an increase of 0.3 mm in the thickness of the lens. He attributed this to individual differences, at the same time pointing out that the determination of the position of the surfaces of the lens is not very accurate. In this connection it may be remarked that the results of such measurements are decidedly more certain when HELMHOLTZ's precautions are followed; which consist in repeating the experiment after exchanging the positions of the telescope and the source of light. Furthermore, the cornea must not be regarded as a spherical surface.

The writer also has obtained similar results. He had the opportunity of examining repeatedly, over a long period of time, an intelligent

[1] See the works already mentioned of KNAPP, ADAMÜK and WOINOW, MANDELSTAM and SCHÖLER, REICH, together with compilation in 2nd edition of *Handb. d. phys. Opt.*, S. 147.

[2] Loc. cit. Compiled by TSCHERNING, loc. cit. *Encycl. fr. d'ophth.* T. III. p. 266.

young man, nineteen years old, who was an unusually good marksman and who could fixate well. The determination of the depth of the anterior chamber was made by HELMHOLTZ's method. The radius of the anterior surface of the lens was measured directly with his ophthalmometer, using an ophthalmometric NERNST lamp. The calculation was made by the exact formulae applicable to imagery obtained with an oblique illumination. Values between 0.3 and 0.4 mm were found for the displacement of the pole of the lens when the eye was accommodated for a needle placed 10 cm from the cornea. Results between 10.34 and 10.42 mm were found for the radius of the anterior surface of the lens with relaxed accommodation; and between 5.5 and 5.9 mm. in case of accommodation for a point 10 cm away. The closer agreement of the figures for the anterior surface of the passive lens as compared with the results for the accommodating lens might be taken to show how accurate the measurements were; or it might indicate also the difficulties of exact accommodation in the region of the near point. For this reason the smaller value probably is the more correct.

Hitherto, nothing has been definitely known concerning a change of position of the posterior surface of the lens during accommodation. To be sure, experiments have often indicated a slight movement one way or the other, but the methods used have not been exact enough to exclude the possibility of this result's being due to inherent sources of error, even if the lens itself were actually homogeneous. As a matter of fact, the properties of the heterogeneous medium constitute additional sources of error, inasmuch as the configuration of the isoindicial surfaces is altered by the change of form of the lens, as shown, for example, by a variation of the total index. Now this would not be of so much importance provided the investigation of the posterior surface of the lens could be made by studying the procedure of rays along the axis, because then it would be easier to apply a correction. But inasmuch as obliquely falling light has to be used, the necessary means for the correction are lacking as long as the exact form of the anterior surface of the lens in both the passive and the accommodating eye is unknown. Besides, owing to the concomitant contraction of the pupil in accommodation, it is often impossible to examine reflex images in the posterior surface of the lens with the axis of the eye maintained in the same direction as it was adjusted for the images in the anterior surface (unless the result is vitiated by an artificial dilatation of the pupil entirely beyond the limits of strictly physiological conditions). And, moreover, the centering of the eye is influenced by accommodation. In view of these sources of error, it is not surprising that quite different results have been obtained for the variation of curvature of the posterior surface of the lens in accommodation. KNAPP, who was

one of the earlier investigators, found a difference of from 0.5 to 1.5 mm between the radius of the passive lens and that of the accommodating lens. In the case measured by TSCHERNING, the radius of the posterior surface of the lens was diminished from 5.7 to only 5.3 mm by accommodation, whereas the corresponding values of the radius of the anterior surface were 9.7 and 5.4 mm, respectively. BESIO's results, on the other hand, differed by 1.0 mm.

Taking into account the above sources of error, the only definite conclusion that can be drawn from these investigations is, that *as yet there is no proof of a change of position of the posterior surface of the lens in accommodation*, and that *the curvature of the posterior surface of the lens increases in accommodation, though to a very slight extent.*

By an original method of his own, GERTZ[1] has recently come to the same conclusion. He investigated the conditions under which the spot of light mentioned on page 223 appeared as a sharp image, that is, the secondary catadioptric image in the eye. To be sure, owing to the unavoidable oblique incidence of the light in this experiment, there are probable sources of error due to astigmatism that render the method useless for this purpose; but otherwise it is adapted for checking schematic eyes. GERTZ also drew no conclusions in regard to this matter, but he found that in this particular case the posterior pole of the lens exhibited no noticeable axial movement during accomodation, and that the curvature of the posterior surface of the lens increased by the generally accepted amount during accommodation. Inasmuch as the question here is simply one of the comparison of the results of different investigations on one and the same eye, under the influence of similar sources of error, perhaps these errors should in the main counteract each other, and the result be correct.

In trying to construct a schematic accommodating lens, it appears to be best to represent the relations as they exist when the power of accommodation is greatest, that is, in youth before the appearance of surfaces of discontinuity or cleavage in the lens. Accordingly, the writer has chosen the focusing for a point approximately 10 cm distant from the cornea. With reference to his numerical results above mentioned, he has likewise assumed that the radius of curvature of the anterior surface of the lens becomes reduced in accommodation from 10 to 5.33 mm. But the value 0.4 mm, taken by HELMHOLTZ for the amount of displacement of the anterior pole of the lens in accommodation, has been retained. Owing to the very small value that must be ascribed to the change of form of the posterior surface of the lens in accommodation, the curvature of this surface in the accommodating

[1] H. GERTZ, Über das sekundäre katadioptrische Bild des Auges. *Skand. Arch. f. Physiol.* XXII. 1909. S. 299.

eye is largely a matter of choice. The selection of a symmetrical form seems to be justified, because the schematic eye is intended to have maximum accommodation, and, as far as can be determined from the measurements, the lens approaches this form as closely as possible during accommodation. The accommodative change of curvature of the posterior surface of the lens as here assumed is between the values found by HELMHOLTZ and his pupils, on the one hand; but it is also between the values found by TSCHERNING and his pupils, on the other hand; and in each instance is nearer the value found by the teacher. But as to the reliability of this datum, all that can be said is that with our present knowledge it is not possible to be more accurate. The choice of the definite value 5.33 for the radius of curvature was made, because this number gives the value of the total index that corresponds to MATTHIESSEN's law, and so makes it possible to give a mathematical proof of the law. At the same time, the discrepancy between it and the value obtained by the writer's measurement is no more than might be due to the sources of error of the methods. Since the anterior surface of the lens is subjected to a greater change in accommodation than the posterior surface, it may be supposed that in the axial enlargement the portion of the lens lying in front of the point of maximum index undergoes more variation than the portion beyond it. Consequently, a symmetrical structure of the accommodating lens is probable also for this reason. The refracting power of the lens may be calculated in the usual way from the optical focusing, the data of the corneal system, and the length of the passive schematic eye. By using an optical centre for the lens that is to be calculated, an approximate value of about 33 dioptries is obtained. The refracting power of each surface of the lens is 9.375 dptr. Now since the refracting power of the core lens which comes into the calculation must be somewhat greater than its exact value, and since the sum of the refracting powers of the individual systems again exceeds the total refracting power, the approximate value of 15 dptr is found for the refracting power of the core lens to be used in the calculation. In addition to these data, the other two equations needed for finding the indicial equation of the accommodating lens are dependent on the two conditions, that during the change of form, (1) there is no compression at the centre of the lens, and (2) that the volume included within the largest closed indicial surface remains unaltered. These conditions are due to the fact that the forces operative in accommodation are too feeble to produce by compression any appreciable alteration of the volume or the indices of refraction of the various parts.

Accordingly, the data to be used in the calculation of the accommodating schematic lens are:

$$-x_1 = x_2 = 2, \qquad \rho_1 = -\rho_2 = 5.333\ldots, \qquad D = 0.015,$$

which give the following constants for the indicial equation:

$$m = 0.0025031, \qquad n = 0.0023443,$$

$$p_m = 0.0224907, \qquad p_o = 0.0021085, \qquad p_n = 0.0008399.$$

Owing to the symmetrical structure, the coefficients M and N are both equal to zero. Exactly as in the case of the unaccommodated eye, these results have been used by the writer to calculate a number of coördinates of the points of intersection of a meridian plane with the iso-indicial surfaces corresponding to the values 1.386 and 1.404 for the indices of refraction. The results are exhibited by the curves in Fig. 134; and for the sake of comparison, the corresponding curves for the unaccommodated lens are reproduced in Fig. 133. The outlines of the surfaces of the lens as shown in these diagrams have been constructed by making them parabolic as far as possible towards the equator; but the connecting branches have been put in arbitrarily and estimated so as to fulfill the condition that the volume contained between the surfaces of the lens and the greatest closed indicial surface should remain unaltered during the change of shape The writer particularly wishes to emphasize the fact that these figures are entirely accurate only with respect to the iso-indicial surfaces and only on the assumption of the symmetrical structure of the core lens for maximum accommodation. Their purpose is to represent the optical mechanism of accommodation objectively so fas as this question is concerned with the imagery laws of the first order along the axis, and to illustrate diagramatically the relationship of this optical mechanism to the dynamics of the accommodative changes of the lens. With respect to the first of these objects, it should be noted that a slight difference of curvature, such as might well be present in many an eye if sufficient weight were attached to the ophthalmometric measurements, would make N have only a very small value. The effect of this on the form of the iso-indicial surfaces would be hardly noticeable; as can be readily seen, even without calculation, by considering that the comparatively

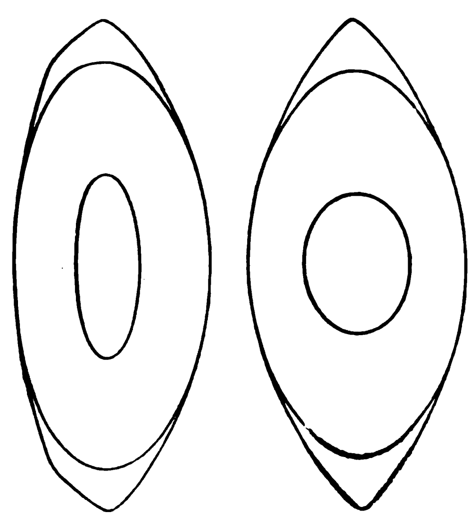

Fig. 133.

high bending of the unaccommodated lens does not produce any more asymmetry in the arrangement of the iso-indicial surfaces than is shown in Fig. 133. Again, as to the connection between the optical mechanism and the dynamics of accommodation, this will be discussed more at length presently; but here attention is directed simply to the fact that, if, in accordance with BESIO's researches, the peripheral parts of the anterior surface of the lens are flatter than those of the posterior surface, the result might be an asymmetry of the external form of the lens without any asymmetry of the core lens.

A comparison of the distribution of the iso-indicial surfaces in the passive and in the accommodating lens shows at once that the shifting of the individual parts in the direction of the axis during increase of thickness of the lens is greatest in the equatorial plane, and that here the parts lying nearer the axis of the lens are shifted more than those nearer the equator. The mathematical investigation shows that this latter condition is an expression of the alteration of the total index with the change of shape of the lens. As this might have been postulated *a priori* from the anatomical structure of the lens, the conclusion is that the change of the total index accompanying accommodation is necessarily connected with the anatomical structure.

In order to understand this relationship, it is simply necessary to remember that the lens fibres are attached both anteriorly and posteriorly, describing in their course arcs which are convex toward the equator. When the points of attachment of the fibres are separated from one another by the increase of thickness of the lens, the arches must be spread, involving the greatest amount of dislocation of particles in the parts of the fibres farthest from the points of attachment. If the lens were always symmetrical, a centripetal shifting would have to occur at the equator. If the point of maximum centripetal shifting on each lens fibre were determined, and a surface passed through all these points, this surface of maximum accommodative shifting would coincide with the equatorial plane. But since the passive lens is asymmetrical, and the change of shape is particularly marked on the anterior surface, the surface of maximum accommodative shifting must be concave towards the front. This conclusion, drawn entirely from the anatomical structure of the lens with respect to its asymmetrical accommodative change of shape, may also be deduced directly from the figures given above, as a result of mathematical analysis. The slight change of form of the posterior surface of the lens demonstrates that the points of attachment of the lens fibres adjacent to this surface must, on the average, be less separated from one another during accommodation than those of the fibres lying on the anterior surface. Since, on the whole, the fibres of

the posterior surface have their points of attachment situated more towards the periphery on the anterior surface, and towards the centre on the posterior surface, and as these conditions are reversed in the case of the fibres of the anterior surface, the distance of the posterior pole of the lens from the anterior point of attachment of the *Zonula Zinnii* must be relatively less changed during accommodation than the distance of the anterior pole from the posterior point of attachment. As a result, the shifting at the anterior point of attachment must occur in a direction approximately corresponding to a tangent to the surface. Consequently, it follows from the anatomical structure of the lens, that *the increase of curvature of the anterior surface of the lens during the accommodative change of form is accompanied by an axipetal shifting of the anterior point of attachment of the zonule.* As may be deduced from the figures given above, mathematical investigation demonstrates the presence of a corresponding shifting in those parts of the largest closed iso-indicial surface that are nearest to the point of attachment.

As the surface of maximum accommodative shifting contains cross or slighty oblique sections of the lens fibres, the rapidity of the centripetal movement of these sections during accommodation must be greater at a point nearer the axis than in the vicinity of the equator. If, for example, a centripetal movement of 0.1 mm occurs 4 mm from the axis, the superficial area of the fibrous cross sections contained within the circle of radius 4 mm is equal to 0.8π square millimetres; and exactly the same area of fibrous cross sections would be contained in the circle of radius 2 mm for a centripetal movement of 0.2 mm. It is true, this mechanism might be impeded by the fact that the fibres lying nearer the axis would be cut obliquely by the surface of maximum shifting in the passive state and perpendicularly during accommodation, provided the centripetal movement could occur to a sufficient extent. But in order to compensate the suggested difference of centripetal shifting, the oblique section must make an angle of 60° with the perpendicular section; and this is manifestly impossible. Another consequence, therefore, of the anatomical structure of the lens is that the equatorial diameters of the smaller iso-indicial surfaces must be proportionately more shortened in accommodation than those of the larger. But according to the mathematical investigation, this is an expression of an increase of the total index; and hence *the increase in total index during accommodation*, as proved by physiological-optical investigations, *may be deduced directly from the anatomical structure of the lens.* The so-called S-shaped curvature of the lens fibres is inferred from the fact that the projection of such a fibre on the equatorial plane is not a straight line; and the reason why in this discussion of the anatomical structure of the lens the possibility of a change in this curvature has

not been mentioned, is because the only thing that could modify it would be radially directed elevations and depressions. This is a necessity due to the mode of attachment of the lens fibres in rows, so that any mutual shifting of the individual fibres at these points is impossible. On the other hand, it follows again from the anatomical arrangement of the lens fibres, that during the accommodative change of form, such elevations and depressions must either originate in the iso-indicial surfaces or must be reversed there, if they are already present. Else, they would undergo a reduction of superficial area during accommodation. This might perhaps be possible if the lens were composed of freely movable particles; but is actually impossible because the capability of movement is restricted by the arrangement of the fibres. However, a necessary mathematical consequence of this accommodative change of the iso-indicial surfaces is the variation of the star-shaped appearance of a luminous point.

A slight increase of the index at any given point may result from the interpenetration of individual fibres between others, even though the physical indices of the individual fibres are not altered. This explains why the smaller iso-indicial surface shown in the diagrams is apparently a little nearer the anterior pole of the lens during accommodation, because the superficial extent of that portion of it which is nearest the axis is augmented by the forward displacement, and this must involve an interpenetration of fibres from the central region.

Thus, the dioptric investigation of the lens in accommodation has resulted in finding out the accommodative variations that occur in the substance of the lens. At the same time, it appears that these changes, which for convenience may be grouped together under the name of the *intracapsular mechanism of accommodation*, are not only in complete agreement with the anatomical structure of the lens, but also establish and explain the casual connection between this structure and the variation of the total index of the lens as proved by the change of refraction that occurs when the lens is removed or during the process of accommodation.

The necessary data for the calculation of the schematic accommodating lens, that is, the refracting power of the core lens and the positions of the principal points, in terms of the millimetre as unit of length, are:

$$D_L = 0.01496, \qquad -H = H' = 0.012566,$$

the distances being measured from the centre of the lens, as in the case of the lens of the unaccommodated eye. In finding the *equivalent core lens*, it develops that a mathematically exact equivalent is not possible, because the principal points of the real core lens are too far apart.

The values

$$r_1 = -r_2 = d = 2.6551,$$

corresponding to the maximum possible interval between the principal points of the equivalent core lens, are, however, exact enough for the schematic lens, since the difference between these intervals in the real core lens and the equivalent core lens does not amount to as much as 0.007 mm. The combination of the three component systems of the lens gives the following data for the entire system, in terms of the metre as unit of length:

$$D_l = 33.055, \qquad 1000 n_2 H_l = -1000 n_2 H_l' = 1.9419.$$

These are the values that are obtained with the real core lens; whereas for the equivalent core lens they become 33.056 and 1.9449, respectively.

The total index is 1.426 in the exact schematic eye; and is 1.424 in the simplified schematic eye when the distance between the principal points is neglected.

The table on the following page exhibits the data, side by side, of the exact schematic eye and the simplified schematic eye, as calculated by the writer on the basis of the equivalent core lens, both for relaxed accommodation and for maximum accommodation. The refracting powers are expressed in dioptries and the linear dimensions in millimetres.

Although the exact schematic eye in the passive state has an hypermetropia of one dioptry along the axis, so as to represent the actual normal emmetropic eye, this effect of aberration cannot be taken in account in the case of the accommodating eye, because the amount of aberration under these circumstances has never yet been ascertained. However, owing to the pupillary contraction that is concomitant with accommodation, the influence of any residual aberration that might be present would certainly be considerably reduced. On the other hand, however, a necessary result of the same pupillary contraction is to draw the practical near point of the schematic eye rather nearer the eye than the exact near point, because the depth of focus has to be added in.

It should be noted with regard to the simplified schematic eye, that the difference between the total index of its lens and that of the lens of the exact schematic eye as given above is due to the fact that, instead of the principal points of the lens, an optical centre is assumed, and because in the passive state the effect of aberration is left out of account.

SCHEMATIC EYE

	Exact		Simplified	
	Accommodation relaxed	Maximum accommodation	Accommodation relaxed	Maximum accommodation
Refractive Index				
Cornea	1.376	1.376		
Aqueous humor and vitreous body	1.336	1.336	1.336	1.336
Lens	1.386	1.386	1.413	1.424
Equivalent core lens	1.406	1.406		
Position				
Anterior surface of cornea	0	0	0	0
Posterior surface of cornea	0.5	0.5		
Anterior surface of lens	3.6	3.2		
Anterior surface of equiv. core lens	4.146	3.8725		
Posterior surface of equiv. core lens	6.565	6.5275		
Posterior surface of lens	7.2	7.2		
Optical center of lens			5.85	5.2
Radius of curvature				
Anterior surface of cornea	7.7	7.7		
Posterior surface of cornea	6.8	6.8		
Equivalent surface of cornea			7.8	7.8
Anterior surface of lens	10.0	5.33..	10.0	5.33..
Anterior surface of equiv. core lens	7.911	2.655		
Posterior surface of equiv. core lens	− 5.76	− 2.655		
Posterior surface of lens	− 6.0	− 5.33..	− 6.0	− 5.33..
Refracting Power				
Anterior surface of cornea	48.83	48.83		
Posterior surface of cornea	− 5.88	− 5.88		
Equivalent surface of cornea			43.08	43.08
Anterior surface of lens	5.0	9.375	7.7	16.5
Core lens	5.985	14.96		
Posterior surface of lens	8.33..	9.375	12.833.	16.5
Corneal System				
Refracting power	43.05	43.05	43.08	43.08
Position of first principal point	− 0.0496	− 0.0496	0	0
Position of second principal point	− 0.0506	− 0.0506	0	0
First focal length	−23.227	−23.227	−23.214	−23.214
Second focal length	31.031	31.031	31.014	31.014
Lens System				
Refracting power	19.11	33.06	20.53	33.0
Position of first principal point	5.678	5.145	5.85	5.2
Position of second principal point	5.808	5.255	5.85	5.2
Focal length	69.908	40.416	65.065	40.485
Complete optical system of eye				
Refracting power	58.64	70.57	59.74	70.54
Position of first principal point	1.348	1.772	1.505	1.821
Position of second principal point	1.602	2.086	1.631	2.025
Position of first focal point	−15.707	−12.397	−15.235	−12.355
Position of second focal point	24.387	21.016	23.996	20.963
First focal length	−17.055	−14.169	−16.740	−14.176
Second focal length	22.785	18.930	22.365	18.938
Position of Fovea centralis	24.0	24.0	24.0	24.0
Axial refraction	− 1.0	− 9.6	0	− 9.7
Position of near point		−102.3		−100.8

It would be idle to make a comparison here with previous schematic eyes that have been proposed; because although these schemes are frequently based on the refracting power of the lens as found by the change of refraction after extraction of the lens, the connection between the change of shape of the lens and the amount of accommodation is usually left entirely out of consideration. The reason of this was evidently because it was not possible to construct a schematic eye corresponding to the facts without knowing the dioptrics of heterogeneous media. Merely with respect to the values of the refracting power, it should be noted that they cannot be compared directly with the data of TSCHERNING and his pupils, because the latter did not always employ the scientific idea of refracting power, and magnitudes as expressed in terms of the dioptry are not commensurable.

A comparison of the lens system of the exact schematic eye in the passive state and for maximum accommodation, as given in Fig. 135, indicates, schematically of course, the intracapsular mechanism of accommodation that was proved above.

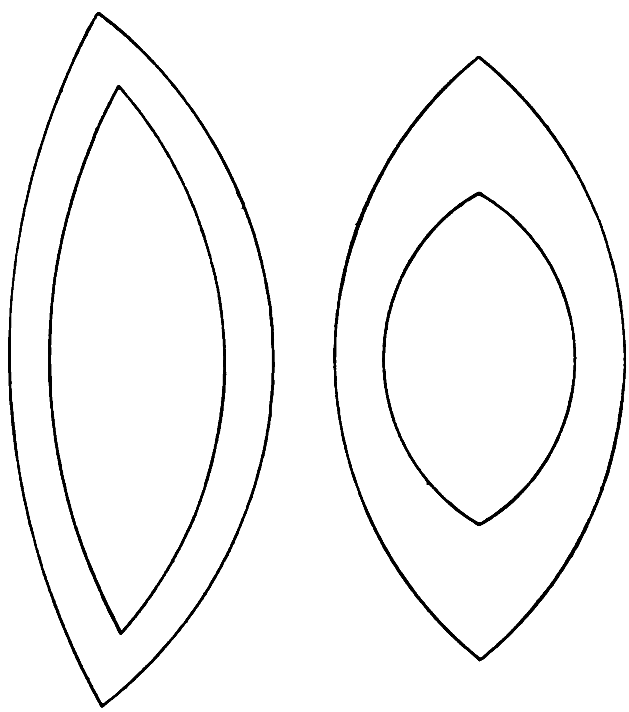

Fig. 135.

While this mechanism is well known for the eyes of very young people, the best we can do in the case of the lens of people of middle age or older is to get an approximate idea of it, because the form of

the surfaces of discontinuity in the lens is unknown. As long as the lens shows a small axial cleft, as is the case with the eyes of children (see the schematic drawing of BABUCHIN, Fig. 136), the intracapsular mechanism will probably remain unaltered, since this cleft must be shortened by the tension of the zonule, and its meridional section must assume the form of either a small cross or of a radial cleft. If the central portions become more homogeneous with advancing age, the mutability of form in the centre of the lens begins to decrease, so that the maximum change of curvature of the surfaces of the lens cannot produce the same increase of the total index. Hence it follows that the amplitude of accommodation begins to decrease before the change of curvature of the surfaces of the lens influences the changes in the core. As soon as the central portion becomes less mobile, strains must arise during the changes of shape, and these lead to the formation of surfaces of discontinuity. This is proved by the doubling of the

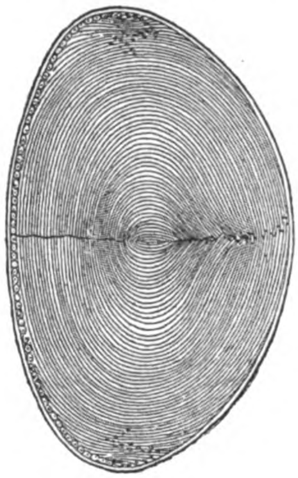

Fig. 136.

reflex image that occurs in the posterior surface of the lens, as recently described by ZEEMAN.[1] This phenomenon, as was stated above, may be observed in many senile lenses by looking in a direction nearly along the equator of the core, but it was erroneously attributed by ZEEMAN himself to a supposed posterior lenticonus. The investigations of ADAMÜK and WOINOW[2] have shown that the change of curvature of the surfaces of the lens, at least of the anterior surface, is finally checked by increase of sclerosis of the core.

[1] W. P. C. ZEEMAN, Über die Form der hinteren Linsenfläche. *Klin. Monatsbl. f. Augenheilk.* XLVI. 1908. S. 83.

[2] Loc. cit.

In the investigation of the *extracapsular mechanism of accommodation* it is well first to review the known facts that have been ascertained for the human eye.

Next to the *change of curvature of the surfaces of the lens* and the *increase of thickness of the lens*, HELMHOLTZ attached most importance to the *contraction of the pupil* accompanying accommodation. It has been a question of debate whether this phenomenon is associated with accommodation or with convergence of the lines of vision. Concerning this, clinical observations on eyes in which accommodation was completely paralysed as a result of diphtheria have shown, that contraction of the pupil occurs in convergence, and that accommodative contraction of the pupil may be unhampered when convergence is absent. It seems most probable that the three innervations are associated with one another and are released by the impulse for clear vision of near objects. (Simply because the pupil becomes contracted by mechanical means when the anterior chamber is drained is no reason for inferring, as TSCHERNING does, that the contraction of the pupil in accommodation is produced by mechanical means.) The function of the contraction of the pupil during accommodation is to be seen in the increase of the depth of focus, since the changes of accommodation requisite for seeing near objects are essentially diminished by the increased depth. This has nothing to do with aberration, as will be shown further on. What advantages this contraction of the pupil offers, might have been better understood before the invention of spectacles, because in those days it afforded the only recourse against normal presbyopia.

The best way at present to observe the *changes in the form of the anterior chamber* during accommodation is with the binocular magnifying glass invented by CZAPSKI. The experiments described by HELMHOLTZ do not always give positive results, for reasons presently to be discussed. In the stereoscopic image obtained with the instrument above mentioned, it is easy to establish definitely not only the protrusion of the central parts, but also (at least in the case of young people with sufficient accommodation) the recession of the periphery. The latter effect is a necessary consequence of the former, since the volume of fluid contained within the anterior chamber cannot be reduced by flowing over into the posterior chamber on account of the physiological adherence of the iris to the anterior surface of the lens capsule. But the periphery of the iris does not move as much as the centre, because the fluid that escapes from the central region takes up in the periphery a space of larger diameter and consequently of less thickness. This does not indicate that the mechanism of the peri-

pheral increase of depth of the anterior chamber assumed by HELMHOLTZ may not function, but simply that it might be superfluous for this particular purpose. That his mechanism is certainly not sufficient, is demonstrated by one of ULBRICH's cases,[1] in which there was a small hole in the iris covered by a thin membrane which was invaginated during accommodation. As the decrease in capacity of the anterior chamber produced by the protrusion of the central part of its posterior wall is compensated by the recession of the peripheral portions of the iris, a dilatation of the pupil must result if the superficial area of the iris remains the same; or, to express it differently, *the mechanical effect on the iris of the change of form of the lens during accommodation is to widen the pupil.* In confirmation of this, sudden traumatic decrease of space in the anterior chamber results either in the tearing loose of the iris from its attachment or in its prolapse, if it has had time to relax.

The contraction of the entrance-pupil in accommodation does not take place concentrically. This was demonstrated in the early days of ophthalmometry, both by KNAPP[2] and by ADAMÜK and WOINOW,[3] whose work was done under the direction of HELMHOLTZ. They found that there was invariably a shifting of the centre of the pupil to the nasal side during accommodation. As the ophthalmometer was set up on the axis of the osculating ellipse in these observations, so that the centre of the entrance-pupil was on the nasal side of this axis, this eccentrical contraction of the entrance-pupil could not be attributed to the asymmetry of the cornea; but there might be a very great probability that the anatomical pupil behaved in the same way. Even if this were not the case, it is easy to understand how the entrance-pupil must show a decentration with respect to the line of sight. For, of two rays parallel to the line of sight directed towards the nasal and temporal ends of the horizontal diameter of the entrance-pupil, the former meets the cornea at a greater angle of incidence than the latter. The apparent shifting due to refraction increases with the size of the angle of incidence and with the distance of the point of incidence from the plane of the pupil. Owing to the oblique incidence of the line of sight and on account of the asymmetry of the cornea, both the angle of incidence and the distance of the point of incidence from the pupillary plane will vary asymmetrically with changes in the size of the pupil; which is the same as saying that the entrance-pupil cannot expand and

[1] H. ULBRICH, Zur Lehre von der intraokulären Flüssigkeitsströmung. *Ber. über die 34. Vers. d. Ophth. Ges.* Heidelberg 1907. S. 105.

[2] Loc. cit.

[3] Loc. cit.

contract concentrically, if this is what happens in case of the anatomical pupil. In confirmation of this, the angle of incidence of the line of sight varies with the size of the pupil, as was noted above. The same condition occurs in contraction of the pupil during accommodation, as the writer has verified by special investigations.

For a definite line of sight, accommodation is often accompanied by a *change of the direction of the optical axis of the eye.* If the experiment, described by HELMHOLTZ in connection with Fig. 60, is performed with an ophthalmometric NERNST lamp with vertical slit, the lamp, however, being placed far enough back for the light to fall on the inner surface of the sclera in front of the point of origin of the iris, the small spot of light will be observed to be rather farther back in accommodation than it is with accommodation relaxed; provided the eye is made to look past a needle-point fixed 10 cm away at a more distant point, and provided also the subject understands that, whether he is fixating on the distant point or on the needle, he must get a sharp image. It is impossible to follow the movement, because the eye executes lateral movements during the change of focus, but the change in position of the bright spot may be gauged by its altered distance from the corneal limbus. There is a slight movement of the eye towards the temporal side during accommodation. This motion, along with the shifting of the pupil in the nasal direction, may clear up the confusion that in many instances renders it difficult to harmonize the results of HELMHOLTZ's two experiments concerning the change of shape of the anterior chamber during accommodation.

A partial explanation of these experiments may be found by the laws of imagery of the first order. If one tried to explain the phenomenon on the basis of the erroneous notion of the meaning of the nodal points, exactly the opposite conditions would have to be present. For the nodal points move forward during accommodation, and then the line of sight makes a smaller angle with the optical axis of the eye. Consequently, the eye would have to move towards the nasal side if the line of sight were unaltered. But not only observation itself, but exact mathematical investigation of the procedure of the line of sight, shows that what happens is the reverse of this. Suppose the abscissae of the principal points of the optical system of the cornea are denoted by h_c, h_c' and those of the optical system of the whole eye by h, h'; and the corresponding refracting powers by D_c and D, respectively. Moreover, let d, p, p' denote the abscissae of the centres of the pupil of the eye, the entrance-pupil, and the exit-pupil, respectively; and let κ denote the magnification-ratio with respect to the entrance-pupil and exit-pupil. Finally, let n denote the index of refraction of

the acqueous and vitreous humors. Then the positions of the pupils and the magnification-ratio can be found by the following formulae:

$$\frac{n}{d-h_c'}=\frac{1}{p-h_c}+D_c, \quad \frac{n}{p'-h'}=\frac{1}{p-h}+D, \qquad \kappa=\frac{p'-h'}{n(p-h)}.$$

The distance of the fovea (denoted by l) being projected from the axis, the angle between the line of sight and the optical axis is found by using the reduced coefficient of angular projection. As this is equal to the reciprocal of the magnification-ratio, that angle is directly proportional to $\frac{\kappa}{l-p'}$, the value of which in the exact schematic eye is 44.67 with relaxed accommodation, and 45.25 for maximum accommodation; and in the simplified schematic eye 44.7 with relaxed accommodation, and 45.1 for maximum accommodation. Thus, in accommodation there is an increase of the angle between the line of sight and the optical axis, and, consequently, for a constant line of sight there must be a movement of the eye towards the outside in accommodation. However, the amount of this movement might not be sufficient for it to be perceived in the manner mentioned. On the other hand, in the accommodative changes of the asymmetry-values along the line of sight, to which we shall refer again in the proper chapter, lies the cause of a movement of the eye which may be added to that just discussed.

In case of a strong innervation of accommodation the tension of the zonule is diminished, and a decentration of the lens occurs in the direction of gravity. Coccius[1] had described the oscillations of the image in the posterior surface of the lens; and Tscherning[2] had noted a downward displacement of this image. Neither of them, however, understood the real cause of the phenomenon. The oscillations were supposed to be due to action of the "Musculus tensor choroideae," and the connection between the displacement and gravity was not discovered. A strictly scientific explanation and definite solution of this problem was given first by Hess.[3] In the case of a small granular spot in his lens, he verified an entoptical parallactic movement of it towards the pupil, by exertion of maximum accommodation. A very small hole, set up 12 mm in front of the eye, served as the source of illumination. With vigourous effort of accommodation, towards the end of the

[1] A. Coccius, Über die vollständige Wirkung des Tensor chorioideae. *Ber. d. VII. intern. Ophth.-Kongr.* Heidelberg 1888. S. 197.

[2] Théorie des changements optiques de l'œil pendant l'accommodation. *Arch. de physiol.* VII, 1. 1895. p. 181.

[3] C. Hess, Über einige bisher nicht gekannte Ortsveränderungen der menschlichen Linse während der Akkommodation. *Ber. über die XXV. Vers. d. Ophth. Ges. Heidelberg* 1896. S. 41. Also: Arbeiten aus dem Gebiete der Akkommodationslehre. *Arch. f. Ophth.* XLII, 1. S. 288 and 2. S. 80. XLIII, 3. S. 477.

time when the pupil was contracting, the spot on the lens as seen entoptically had an upward displacement in the blur circle. Now this displacement invariably occurred in a direction opposite to that of gravity, no matter how the head was held, provided the pupillary plane was vertical, whereas, when the pupillary plane was horizontal, without changing the position of the far point, there was an increase of the amplitude of accommodation when the head was bent forwards, and a decrease when it was bent backwards. The first experiments proved that with strong accommodation, the spot on the lens was displaced with respect to the pupil in the direction of gravity; and the last experiment showed that this is the case with the entire lens. That the entire lens sunk in the first experiment, was shown objectively by the opacities present in the lens. HESS went on to show that, when a powerful effort of accommodation is being made, the lens shakes with movements of the eye (indeed, in many people the iris can be seen to share this trembling); and that both the sinking of the lens sagittally in the frontal plane and its trembling are increased by the use of eserin. If the eserin drops are instilled after dilatation of the pupil by homatropin, the process of accommodation can be observed through the large pupil in the early stages of the action of the eserin. In the reflex images in the lens the appearance is quite different; only the one in the posterior surface appears to tremble and it either sinks by itself or more than the reflex image in the anterior surface. HESS pointed out that the lens may sink down without any dislocation of the reflex image in the anterior surface, because the apparent position of the centre of curvature may be relatively unaffected by the movement. That the sinking of the lens must actually occur in just this manner, is a consequence of the anatomical structure of the zonule, as will be explained below. HESS[1] has also shown that sinking of the anterior surface of the lens may be observed by means of the marking of the epithelium, which is visible with the CZAPSKI magnifying glasses, inside the reflex image in this surface, and also that the anterior surface shares in the trembling in cases where there are little isolated points on it that can be watched.

For strong, arbitrary accommodation, the results of the entoptical measurement of the sinking of the lens were from 0.3 to 0.35 mm. If the head were turned from the right to the left shoulder, the accommodating lens was shifted by twice this amount, and under strong influence of esserin by almost 1 mm. When the head is moved downwards from above, the lens was displaced forwards 0.15 mm. The

[1] Beobachtungen über den Akkommodationsvorgang. *Klin. Monatsbl. f. Augenheilk.* XLII. 1904. S. 1.

earlier measurements have been confirmed by HEINE[1] by measuring the apparent shifting of visible objects that occurs with decentration and the parallax of the reflex images in the posterior surface of the lens and in the cornea.

The mobility of the lens under great strain of accommodation proves unequivocally that there is *no difference of pressure* on its two surfaces in the accommodating eye; whereas in the passive state a slight difference of pressure is possible corresponding to the tension of the zonule. Any increase of pressure in the vitreous body and in the anterior chamber due to accommodation is excluded *a priori* by reason of the viscosity of the fluids. For in order for the tension of the choroid to produce an increase of pressure during the act of accommodation, the suprachoroidal space would have to be replenished without hindrance. Also, according to the investigations of HESS and HEINE[2] referred to above, no *increase of pressure* can be observed. In the case of a freshly enucleated child's eye, HEINE[3] succeeded in verifying an earlier observation of BEER which showed that the mechanism of accommodation goes right on unhindered by a fenestrated sclera and without the slightest movement of the drops of vitreous humor that had welled up through the openings of the sclera.

Direct observations as to *the movement of the ciliary processes* in accommodation cannot be made, of course, on the normal uninjured eye, though they may be made in many instances following an iridectomy or in traumatic or congenital irideremia. With the use of eserin, HESS[4] verified a forward movement of the ciliary processes in iridectomised eyes in which they moved out in front of the equatorial plane of the lens. And in a case of congenital irideremia, GROSSMANN[5] succeeded by means of eserin in seeing ciliary processes that were before invisible, but he conceived the shifting as taking place in a direction towards the axis of the eye, and not towards the cornea. The difference is not of fundamental importance for the mechanism of accommodation and might perhaps be due to an anomalous topography of the ciliary muscle in GROSSMANN's case, or to the fact that an iridectomy, which extends far enough peripherally to make the ciliary processes visible, must involve the origin of the iris in the ciliary body, so as to affect

[1] L. HEINE, Akkommodative Ortsveränderungen der Linse. *Ber. über die XXVI. Vers. d. Ophth. Ges. Heidelberg* 1897. S. 20.

[2] C. HESS und L. HEINE, Arbeiten aus dem Gebiete der Akkommodationslehre. *Arch. f. Ophth.* XLVI, 2. S. 243.

[3] Ein Versuch über Akkommodation und intraokularen Druck am überlebenden Kinderauge. *Arch. f. Ophth.* LX, 3. 1905. S. 448.

[4] Loc. cit. Die Refraktion und Akkommodation usw. S. 222.

[5] KARL GROSSMANN, The mechanism of accommodation in man. *Ophth. Review.* XXIII. 1904. p. 1.

the mechanical movement of the ciliary processes by the operation. (In GROSSMANN's case the trembling began after the administration of the eserin drops, and the changes in form of the lens during accommodation could be confirmed. An excessive increase of thickness and an unusual shortening of the equatorial diameter of the lens, together with a displacement of it upwards and inwards, however, cause one to wonder whether the lens and zonule were normal. The lens showed granular opacities at both poles.)

The *edge of the lens* has been observed to become broader in the iridectomised eye during accommodation and after the administration of drops of eserin, but no definite conclusions can be drawn from this fact. Up to the present time, nothing positive is known as to a decrease in the equatorial diameter of the normal lens during accommodation. According to HESS, the margin of the lens in an eye treated with atropin usually appears as a delicate wavy irregular line; whereas, after the introduction of eserin drops, the line is regular and more like a circle. Low swellings and tent-like protuberances, which may be seen in the eye, treated with atropin in the region of the point where the zonule fibres are attached to the capsule, may disappear or become less prominent after the infusion of eserin.

Except for CZERMAK's *phosphene of accommodation*, all the changes during accommodation that have been observed in the living eye have now been duly considered. The phosphene may be due to a mechanical stimulation of the retina on account of the sudden decrease in the tension of the choroid in rare and especially sensitive eyes.

The *dynamics of the contraction of the ciliary muscle* was explained to a very great extent by the investigations of HENSEN and VÖLCKERS.[1] Inasmuch as most of the muscle fibres (the meridional bundle) run almost parallel to the sclera, the most important question is, whether the contraction wave passes from the posterior end forwards or from the anterior end backwards. It has been demonstrated that, if a needle is inserted at the equator of the eyeball of a dog, electrical stimulation of the ciliary muscle causes the outer end of the needle to move backwards, and that there is no place where the needle can be inserted that makes it move forwards. The inference is that the contraction of the ciliary muscle causes the anterior portion of the choroid to move forwards, and that no point of the choroid moves backwards during contraction. The forward movement of the inner coats of the eye may also be observed through a window in the sclera. On the other hand, if all the cornea is removed except a peripheral border 2 mm wide, electrical stimulation of the ciliary muscle is accompanied by a back-

[1] V. HENSEN und C. VÖLCKERS, *Experimentaluntersuchung über den Mechanismus der Akkommodation.* Kiel 1868.

ward movement of this border, thus indicating that what is known in anatomy as the anterior point of origin of the muscle behaves physiologically as such. Moreover, the relaxation of the zonule on contraction of the ciliary muscle is definitely established. The mechanism of accommodation of a dog is different in some respects from that in man. Thus, Hess and Heine[1] showed that the contraction of the ciliary muscle produces only a slight change of the optical focusing of the eye. Consequently, Heine's[2] microscopic examination of specimens fixed in the act of accommodation, first for doves and then for monkeys,[3] constitutes a valuable addition to our sources of information. The latter investigation illustrates in the most clear-cut manner the function of the muscular fibres which are attached to the strictly meridional bundles and which in the anterior inner portion have a more radial course. Were it not for them, the increase of thickness of the muscle during contraction would result in a component acting towards the centre of the eyeball; whereas the attachment of this bundle to the inner surface of the meridional bundles of fibres causes the resultant of the shortening and thickening to act on the inner surface of the ciliary body in the direction of the tangent to it. In the first case, a cross section of the muscle would have a radial direction, but, in the actual anatomical arrangement of the muscle bundle, the line that cuts orthogonally the fibres which are in a meridional section, is represented by a circle as a first approximation, with its centre in the vicinity of Schlemm's canal. The component corresponding to the increase of thickness coincides at every point with the tangent to this circle, which in turn, except for the most posterior and thinnest parts of the muscle, forms everywhere an acute angle with the inner surface of the ciliary body. Another thing to be noted here is that the full increase of thickness is towards the inner side, because the sclera lies in contact with the outer surface, and that its effect is increased by its concave shape. If a ring-shaped portion of the ciliary body can be imagined, the diameter of this ring must be diminished by a movement forwards and inwards, and the ring must become correspondingly thicker, even if none of the fibres in it are thickened. Hence, the total action of the meridional and radial fibres, which indeed form one common system, is to produce a relatively *uniform displacement of the inner surface of the ciliary body in the direction of its tangent*; whereas, if there were nothing but meridional fibres, the posterior portion would

[1] Loc. cit. *Arch. f. Ophth.* 1898.

[2] Mikroskopische Fixierung des Akkommodationsaktes. *Ber. über die XXVI. Vers. d. Ophth. Ges. Heidelberg* 1897. — Physiologisch-anatomische Untersuchungen über die Akkommodation des Vogelauges. *Arch. f. Ophth.* XLV. 3. 1898. S. 469.

[3] Die Anatomie des akkommodierten Auges. Ibid. XLIX, 1. 1899. S. 1.

be shifted more than the anterior, and the effect would necessarily be offset by the increase in thickness. These relations, which are a mathematical necessity of the anatomical structure of the ciliary muscle, are illustrated in Figs. 137 and 138, reproduced from HEINE. That the circular fibres, if present, will act in harmony with the others, is what might be expected. Inasmuch as they lie on the inner anterior angle, their contraction can only produce a component at this place directed towards the axis; which will cause a rotation of the resultant in the same sense as the tangents to the meridional fibres will have to be rotated in order to become parallel to the tangent to the inner surface of the ciliary body. Whether this muscle exists or not, is therefore of secondary importance. The bundles

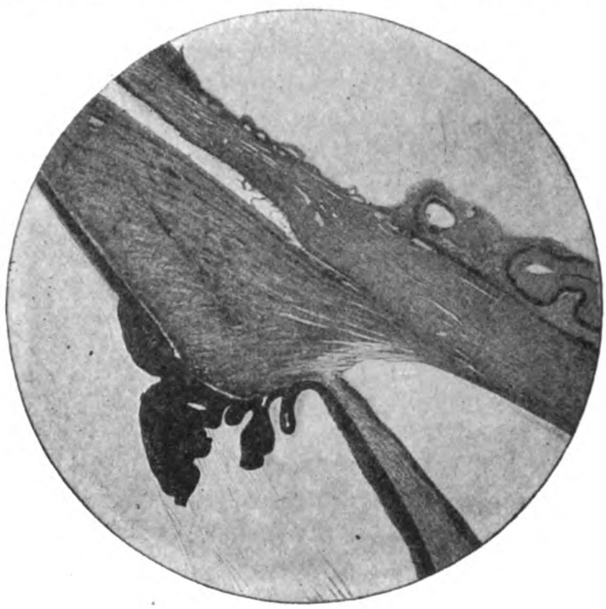

FIG. 137.
Ciliary muscle not contracted.

of the main mass of the muscle do not lie exactly in the meridional plane, but are interlaced and bend back both at their posterior end in the choroid and at the anterior inner angle. Thus, the impression of a circular muscle may be gained much more readily when the fibres are not stretched. In accordance with this, many more oblique sections are to be seen in the ciliary body during accommodation than when it has been treated with atropin. The illustrations also show the opening of SCHLEMM's canal and the widening of the angle of the anterior chamber during accommodation. Since the most anterior radial fibres insert on the inner side of the canal, this action on the canal is as easy to understand as the combined action of the circular

and radial fibres. To just how great an extent the enlargement of the anterior chamber is produced by the contraction of the ciliary muscle, cannot be readily explained at the present time, inasmuch as the tension of the iris during the act of accommodation, as proved by ULBRICH's case, must act in the same direction.

Very recently HESS[1] published new results of comprehensive studies of the mechanism of accommodation. These have shown, in the first place, that accommodation in reptiles and birds takes place in an essentially different way; for the peripheral portions of the anterior surface of the lens are flattened by the pressure of the intrinsic musculature on the part of that surface in front of the equator, whereas the

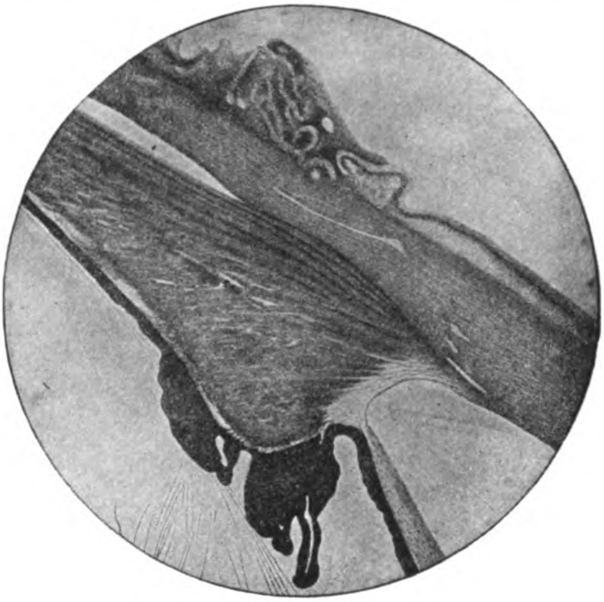

FIG. 138.
Ciliary muscle contracted

portions around the anterior pole become more highly curved, so that in the enucleated eye an accommodative increase of pressure takes place. The shifting forward of the anterior portion of the choroid can be easily seen for some time after enucleation by cutting the eyeball in half along the equatorial plane and watching the movement from behind; it occurs in a manner similar to that found in mammals.

[1] C. HESS, Untersuchungen zur vergleichenden Physiologie und Morphologie des Akkommodationsvorganges. *Arch. f. Augenheilk.* LXII. 1909. S. 345. — Vergleichende Untersuchungen über den Einfluss der Akkommodation auf den Augendruck in der Wirbeltierreihe. Ibid. LXIII. 1909. S. 88. — Also, from personal communications to the writer presumably to be published before the appearance of the first volume of the third edition of HELMHOLTZ's *Phys. Opt.*

At last Hess succeeded also in fixating the act of accommodation in the human eye, with two patients just before death, by using eserin in one eye and atropin in the other. In one case, the eyes were enucleated ninety minutes after death and hardened for eighteen hours in formalin. In the other case, they were enucleated twelve hours after death and studied fresh. After sectioning in the equatorial plane, it was shown, by observation and measurements from the side of the vitreous humor, that both the diameter of the lens and the diameter of the ring formed by the summits of the ciliary processes were smaller, and the notching of the edge of the lens less noticeable, in the eye treated with eserin than in that treated with atropin.

The form of the lens is influenced by the ciliary muscle through the medium of the *zonule*. The fibres of the latter arise over the entire inner surface of the ciliary body as far as the *ora serrata*, so that they appear to cross through it, because the bundles passing to the anterior and posterior surfaces of the lens alternate. As a rule those going to the anterior surface, originate more to the back and in the depressions between the cilliary processes, while those going to the posterior surface of the lens and to the equator arise more to the front and on the summits of the processes. As a result of this arrangement, the bundles of the zonule passing to the anterior surface of the lens, during the contraction of the ciliary muscle, make greater excursions on the average in their long direction than the others do, because, whereas the former lie in the direction of the movement, the latter make an angle with it. This condition is particularly clearly illustrated in Fig. 139 reproduced from Retzius. Indeed, it is possible to decide by this figure that the bundles arising farthest forward and passing to the posterior insertion point on the lens are not noticeably relaxed by the contraction of the ciliary muscle, but merely act in a new direction; whereas when the bundles attached to the anterior surface of the lens are relaxed, the points of insertion on the capsule of the lens can move in the direction of their tangents to an extent which must be approximately as much as the movement of the posterior portion of the inner surface of the ciliary body.

What change of form the anterior surface of the lens will undergo on relaxation of the anterior fibres of the zonule, no one can say *a priori*, except that it is certain that the curvature of the central portions must increase. But whether the entire form tends then to become more spherical or hyperbolical, defies conjecture at the present time. It is a fact that a closed elastic membrane containing such a large volume of an incompressible, freely mobile substance that it does not get limp even while it is still spherical, tends to return to the form of a sphere after each deformation. But these conditions do not occur at

all in the case of the lens, as is shown by the form of a young person's lens when it is freed from its attachments. This being the state of affairs, when the anterior fibres of the zonule are relaxed, the form of the anterior surface of the lens depends on the size of the surface and the elasticity of the capsule of the lens, and on the distribution of tension in it, which can neither be calculated nor estimated. Hence, it is indeed quite possible that the peripheral portions of the anterior surface do become flatter in accommodation, as BESIO[1] thought he had proved;

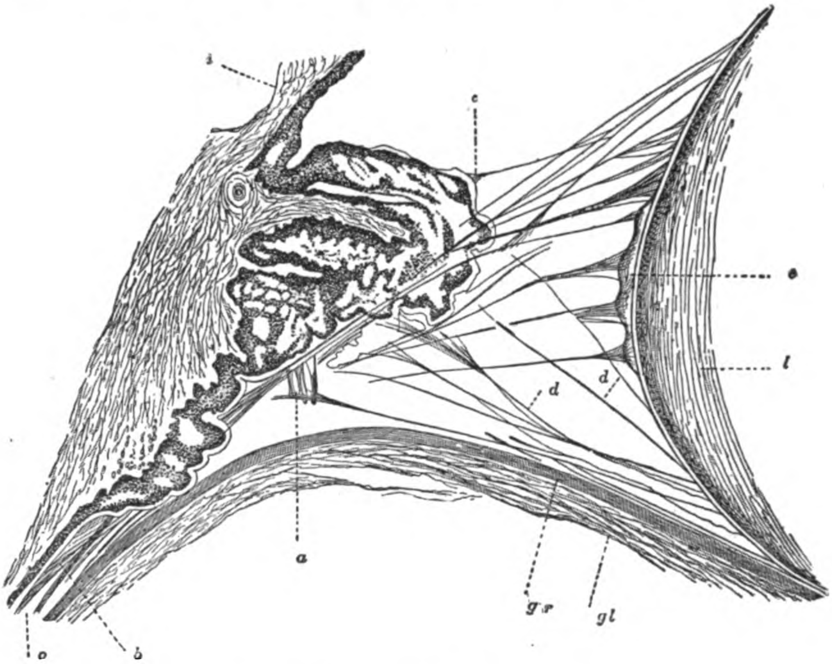

FIG. 139 (reproduced from RETZIUS).

l, lens—*gl*, vitreous humor—*gr*, anterior boundary layer—*o*, orbicular space—*i*, iris root—*a*, short, strong fibrous attachments of the posterior ligament of the zonule—*b*, fibres of the zonule arising behind from the hyaloid membrane—*c*, fibres of the zonule arising anteriorly from the ciliary processes—*d*, fibres of the zonule arising from the ciliary processes which cross the ligaments of the zonule and partly attach to them—*e*, space between the capsule of the lens and the pericapsular membrane.

although in the writer's opinion it cannot be considered as proved, because the approximate methods of calculation may involve considerable sources of error, and the methods of measurement were not very accurate anyhow. Moreover, his measurements appear to have been made on eyes dilated with cocaine, which might influence the mechanics of contraction of the ciliary muscle even if the amplitude of accommodation were not reduced.

[1] Loc. cit.

Since during contraction of the ciliary muscle the fibres of the zonule passing to the anterior surface of the capsule are relatively more relaxed than the others, the useful surface of attachment on both the lens and the ciliary body continually shrinks, because it contracts to the equatorial and posterior points of attachment on the lens and to the anterior parts of the ciliary body. It is just this mechanism which the writer has tried to represent schematically in an intentionally exaggerated way in the contours of the lens surfaces in Figs. 133 and 134. The indicated change of accommodation would correspond to a complete relaxation of the fibres of the zonule proceeding to the anterior surface of the lens and to a forward shifting of the ciliary points of attachment of the other fibres. The writer does not suppose that the lens has just this form in maximum accommodation. For if the relaxation of those fibres is not complete, the result of the mechanism may very well be a somewhat bent form; and if the fibres passing to the equator of the lens are already appreciably relaxed, a decrease in the equatorial diameter may occur. Perhaps, too, the equatorial portions of the lens should be more rounded in both the relaxed and contracted condition of the zonule than is shown in the figures, because the fibres of the zonule are inserted superficially on the lens. But that the bundles proceeding from the anterior portion of the ciliary body to the equator and to the posterior surface of the capsule, are the last to be relaxed, may be inferred from the mode of the sinking and trembling of the lens under maximum effort of accommodation. For this is the only reason why during these motions the lens turns around an axis that passes approximately through the center of curvature of the anterior surface, so that the reflex image in this surface is practically motionless. This means, for example, that part of the margin of the lens which approaches the optical axis during the displacement of the lens must be inclined forwards by an accurately functioning mechanism; and there is no other mechanism except the tension of the fibres above mentioned that could function in this way. This implies that these fibres are the least relaxed.

Thus, anatomical and physiological investigations prove beyond doubt, that, *with contraction of the ciliary muscle, the ciliary point of origin of the fibres of the zonule, particularly of those going to the anterior surface of the lens, is displaced towards the lens in the direction of the bundle; until, when the contraction is greatest, there is a relaxation of the zonule; and that this contraction is accompanied by an increase of the thickness of the lens and of the curvatures of its surfaces, especially of the anterior surface.* Since the relaxation of the zonule does not begin

until the contraction is greatest, it must be kept under tension during normal contraction by an axipetal movement of the points of insertion on the lens, especially of those on the anterior surface. Since there is only one force present which can maintain this tension, namely, the elasticity of the lens capsule, *the extracapsular portion of the mechanism of accommodation consists essentially in an axipetal movement of the points of attachment of the zonule on the lens, especially on the anterior surface, this movement being due to the elasticity of the lens capsule.*

The dioptric investigation of the process of accommodation has shown that an *intracapsular mechanism of accommodation* is a mathematically necessary consequence of the change of form and of the increase of refracting power. Also, in accordance with the histological structure of the lens, it demands *an axipetal movement of the portions of the substance of the lens that are nearest to the points of attachment of the zonule, particularly on the anterior surface, and that are contained within the greatest closed iso-indicial surface.*

Moreover, taking into consideration that there are no known facts which could in any way refute this mechanism (see the discussion of TSCHERNING's theory presently), it is doubtful whether there is a more complete chain of proof in all the medical sciences. Modern investigations have established that the theory of the mechanism of accommodation remains unchanged, in all essential features, just as HELMHOLTZ, by a real inspiration of genius, considering the state of knowledge of that time, conceived it.

In the light of the modern theory of antagonistic action, what is known as *double antagonism* is of particular physiological interest. The form of the lens is determined by two antagonistic elastic forces; and at the same time the muscular force and the stronger of the two elastic forces act antagonistically. Now it is readily seen that this arrangement is beautifully adapted to protect the lens from the action of too strong external forces and from sudden variations of these forces. The force that produces the change of form of the lens in accommodation is the weakest of the three that are present in the system, and, like all elastic forces, constantly diminishes during the development of its effect, so that the movement terminates without any jerk, and the potential energy accumulated can never exceed a certain maximum amount dependent on the elasticity of the zonule. This effect is promoted still more by the fact that, with increasing contraction of the ciliary muscle, the elastic resistance of the choroid is reduced by its stretching. In the relaxation of accommodation, the greatest force in producing the change of form depends then on the elasticity of the choroid, and this force diminishes steadily during the movement, and at the same time the resistance of the lens capsule is

continually increased by dilatation. The advantages of this arrangement may be easily demonstrated by mechanical models. For example, one of them is shown in Fig. 140. The two springs are united by a cord which represents the zonule. The upper one represents the lens, its shortening corresponding to the change of form during accommodation, and its force indicating the elasticity of the lens capsule. The lower spring illustrates the elasticity of the choroid, and the tension of the chord passing over the pulley, produced by putting weights in the pan, corresponds to the force of the contracting ciliary muscle. The upper spring must be the weaker of the two, and must not be stretched enough for the connecting cord ever to get slack when the pan reaches the level of the table. Obviously, the upper spring cannot be injured by suddenly throwing the heaviest weights into the pan, nor is it in the least danger from a sudden removal of the weights, provided the height of the fall is not too great and the lower spring not too strong. On the other hand, supposing we wished to illustrate a fictitious mechanism of accommodation where the accommodative change of form would be produced by muscular action and the recovery by the elasticity of the lens capsule, the pan should be hung directly from the lower end of the upper spring; and then the elongation of the spring would correspond to the change of form of the lens in accommodation. In this arrangement the spring might be injured by a sudden addition of heavy weights. Of course, this could be avoided by suitable adjustment of the height of fall, and in this way it might be possible for the lens of a young person to be protected by this fictitious mechanism of accommodation. However, this means of protection would fail with decrease of ability of the lens to change its form, which could be represented by exchanging the spring for a more brittle one. Every strong tendency to accommodation would result in a jerk on the structure of the lens, any sudden heavy loading of the pan would produce a breaking of the spring. When it is realized that opacities can originate in a previously entirely transparent lens of an older person as a result of an insignificant mechanical force, such as slight draining of the chamber in an iridectomy, one is disposed not to underestimate the *protective arrangement* provided by the *double antagonism of the forces acting in accommodation.*

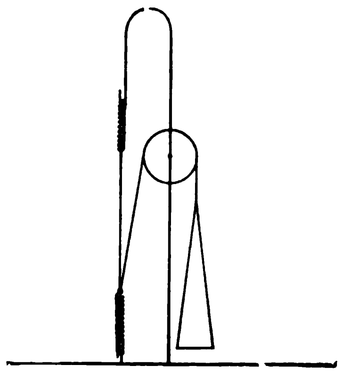

Fig. 140.

Since the contraction of the ciliary muscle is accompanied by a change of form of the lens merely until, in consequence of the shifting of the parts of the lens, this change reaches its maximum extent, whereas the muscle itself, as we know from the pupillary contraction and the trembling that occurs in the end, may be contracted beyond this point; therefore only a part of the contraction of the ciliary muscle is *manifest*, the rest is *latent*. According to HESS, the transition begins when the actual near point is reached, whereas, during the latent contraction of the ciliary muscle, there is merely a small apparent displacement of the near point further towards the eye, which depends on the concomitant contraction of the pupil.[1] Consequently, an accommodation of two dioptries, at an age when this is all the power of accommodation available, does not make any higher demands on the ciliary muscle than the same amount of accommodation at a younger age. This is a very important fact for the symptomatology of presbyopia; but in connection with it the contraction of the pupil due to latent contraction of the ciliary muscle, especially if the habitual size of the pupil is not very small, is another factor to be taken into account, because this by itself often decoys an uncorrected presbyope into trying to accommodate beyond his strength. HESS called attention to another important consequence of the mechanism of accommodation, namely, that we have the right to conclude from a normal amplitude of accommodation that the action of the ciliary muscle is normal also, because a paresis of the ciliary muscle does not become apparent until the restriction of movement extends to the region of manifest contraction. So far as physiological optics is concerned, this fact is significant as showing, as was remarked before, that TSCHERNING's opinion, as to the performance of the ciliary muscle not being impaired by cocaine, cannot be proved by its effect on the amplitude of accommodation, and therefore is without any basis at present.

There has been quite a lot of discussion in ophthalmological literature of an astigmatic accommodation. Without going into this subject in detail, it may simply be stated that no known facts indicate the possibility of a voluntary change of astigmatism by accommodation or by the practice of astigmatic accommodation. However, it may be possible that the normal inverse lenticular astigmatism or the lenticular astigmatism that is present in the higher degrees of ocular

[1] ¶The *physical* or *manifest near point* (what is usually meant by the "near point of the eye") is the point that is conjugate to the point where the optical axis meets the retina when the surfaces of the crystalline lens are most curved, the zonule being relaxed. The *physiological* or *latent near point* (whose position cannot be ascertained by any method at present available) is the point on the optical axis for which the eye would be focused when the ciliary muscle is contracted to its utmost power, supposing there were no limit to the effort of the crystalline lens to become spherical. (J. P. C. S.)

astigmatism may be altered to some slight extent by the accommodative change of form of the lens. But there is no sufficient evidence to show that this actually happens. Owing to the frequent occurrence of vertical asymmetry of the eye, whereby the apparent amount of astigmatism may change with the size of the pupil, it would probably be very difficult to obtain the data necessary to prove that anything of this sort does take place.

After what has been stated in the preceding argument, an exhaustive criticism of the various hypotheses that have been proposed since HELMHOLTZ's discovery of the mechanism of accommodation might seem to be superfluous, because they all presuppose a tension of the zonule during accommodation, and therefore have been actually refuted by HESS's investigations. Hypotheses have been framed by MANNHARDT,[1] SCHÖN[2] and TSCHERNING; and TSCHERNING's theory has stirred up so much controversy and been responsible for so many contributions to the literature of the psychology of science (in the broadest sense of the word), that a brief discussion of it seems to be advisable. As it has already passed through two essentially different phases and appears now to be about to enter a third, the best plan is to take up the first and better known of these phases first; and for that purpose let us consider TSCHERNING's own presentation of his theory as given in his text-book on Physiological Optics, which has been referred to above. The hypothesis consists essentially of three propositions, namely, (1) the assumption that accommodation consists in a temporary formation of a "lenticonus anterior," (2) the assumption that the tension of the zonule produces a structure of this kind, and (3) the assumption that the tension of the anterior fibres of the zonule may be produced by the contraction of the ciliary muscle.

The basis of the first assumption is the argument, that the aberration of the eye during accommodation varies in sign, that the refracting power increases more at the centre of the pupil than it does at the periphery, and that the distortion of the reflex images in the anterior surface of the lens is correspondingly altered by accommodation. But the aberration was investigated by unreliable methods. The writer has shown that the aberroscope does not give the aberration but a distortion value. The distribution of the light in the blur circle as described in TSCHERNING's experiment with the luminous point is such that by the nature of the caustic surface it could not possibly be due to aberration, although it might be an interference phenomenon of

[1] J. MANNHARDT, Bemerkungen über den Akkommodationsmuskel und die Akkommodation. *Arch. f. Ophth.* IV, 1. 1858. S. 269.

[2] W. SCHÖN, Zur Ätiologie des Glaukoms. *Arch. f. Ophth.* XXXI, 4. 1885. S. 1; and in other publications.

the kind described above. Finally, all that the experiments with YOUNG's optometer give is the peripheral total aberration; which is true also with respect to the skiascopic phenomena[1] which were subsequently included. The latter experiments do not give constant results, because in many cases the change of the skiascopic aberration phenomena cannot be observed at all in accommodation; and HESS, for instance, got a negative result in his investigation with YOUNG's optometer. Consequently, all that can be concluded is that in many cases the peripheral total aberration of the eye decreases during accommodation, and that this phenomenon is unessential so far as the mechanism of accommodation is concerned. (For further information on this subject, see the proper chapter below.) The writer's own experience all goes to show that the phenomenon of the distortion of the reflex images in the anterior surface of the lens cannot be seen with sufficient clearness unless the pupil has been treated with cocaine, and, in the first place, this gives no information concerning normal accommodation. In the second place, this distortion is changed without alteration of the flattening of the reflecting surface by a change of its curvature and of its distance from the cornea; and hence even when the angles of incidence were very small, a calculation would be necessary to determine whether the change of the distortion indicated a change of the "Abflachungswert" of the surface. In the third place, such large angles of incidence are necessary to verify this phenomenon, that the asymmetrical flattening of the cornea cannot be left out of consideration, and this makes the calculation certainly quite complicated.

TSCHERNING's first assumption, therefore, was entirely without foundation, although let it be stated once more that there are no proofs of its impossibility. It was used for a false conclusion. Under the title of "The Author's Theory of Accommodation," TSCHERNING says that the "hypothesis" of HELMHOLTZ appears to be no longer tenable; at least he himself cannot see how such a mechanism can produce a flattening of certain parts of the lens and at the same time an increase of curvature of other parts. The only conclusion that can really be drawn from this statement is, that the cause of this lack of comprehension must be sought either in the "hypothesis" of HELMHOLTZ or in TSCHERNING himself. HELMHOLTZ's own words are:[2] "Stretched elastic membranes which contain an invariable volume of an incompressible fluid and which are attached by a circular margin, as the zonule is to the choroid, tend to approach the form of a segment

[1] Le Mécanisme de l'accommodation. *IX. Congr. internat. d'Utrecht. Compte rendu.* Amsterdam 1900. p. 244.

[2] *Handbuch der Phys. Opt.* 2. Aufl. S. 138.

of a sphere in proportion as its tension increases. In the relaxed state, as is the case in near vision, the anterior surface of the lens is curved forwards in front of the flat curvature of the anterior ridges of the zonule. In the stretched state, as in far vision, this occurs to a much less extent. However, the radius of curvature of the anterior surface of the lens which is about 10 mm is always a little less than that of the zonular arching, which may be estimated at about 14 mm." Thus, he states that the entire curvature made up of the zonule and the anterior capsule of the lens must approach the spherical form when the zonule is contzacted. In the supposititious final state when the spherical form was attained, the anterior surface of the lens would be, therefore, a segment of a sphere with a radius of about 14 mm. To infer from this that the anterior surface of the lens, as it bulges forwards with a decrease of tension in accommodation, is obliged to approach the form of a sphere, or that an increase of curvature can be produced by the increase of tension, involves conceptions that are incompatible with the mathematical knowledge of a HELMHOLTZ. Nowhere at all in his writings can the author find any intimation as to the probable form of the surfaces of the lens in the state of accommodation. This is not very surprising because, as already stated, there is no way of either calculating this form or of estimating it. All that can be said about it is HELMHOLTZ's own statement concerning the increase of curvature of the surfaces and of the thickness of the lens. As to the distribution of the increase of curvature over the various parts of the surface, and as to the possibility of a peripheral flattening during this process, nothing is stated; nor can anything be deduced from the relaxation of the zonule.

TSCHERNING's second assumption (that the result of the tension of the zonule is the formation of a "lenticonus anterior") is again simply a false conclusion which he made from experiments that prove that in the extirpated lenses of animals a traction exerted on the zonule may have this effect. Crucial experiments were carried out by EINTHOVEN,[1] HESS[2] and DALÉN.[3] EINTHOVEN exposed the lens and zonule of a calf's eye from above, and found that by pulling with two forceps at opposite ends of a diameter, he could make the curvature of the anterior surface of the lens increase or decrease, according as he pulled more backwards or forwards, respectively. HESS's experiments were made on the freshly enucleated eye of an ape, from which a portion of the sclera was removed without injuring the choroid, and the cornea

[1] W. EINTHOVEN, Die Akkommodation des menschlichen Auges. *Ergebnisse der Physiologie.* I, 2. 1902. S. 680.

[2] Loc cit. *Klin. Monatsbl. f. Augenheilk.* 1904.

[3] Loc. cit.

and iris taken away; and he showed that a pull on the zonule produced a decrease of the curvature of the anterior surface of the lens. DALÉN experimented with the dead eye of a human being in which the lens had been laid bare by removing the cornea and iris with special precautions; and he verified ophthalmometrically that there was an increase of the anterior surface of the lens after severing the zonule.

The third assumption (that a tension of the anterior fibres of the zonule may be produced by the contraction of the ciliary muscle) is based on ideas of the anatomy of the ciliary body that are not clearly enough expressed, but that seem to point to the conclusion that there is an innermost muscle layer, which by its contraction would pull the anterior inner end of the ciliary body backwards. These are notions that have no objectively demonstrable connection with the known anatomy of the ciliary body. The subjective connection is concerned with the fibres inserted on the inner side of SCHLEMM's canal.

In Fig. 141, reproduced from TSCHERNING, he has shown the first phase of his idea of the mechanism of accommodation. It may be of some interest to know that the writer has shown that such a mechanism of accommodation is mathematically impossible unless the total index decreases in accommodation. TSCHERNING's assumption that for an accommodation of 7.5 dioptries the radius of the anterior surface of the lens might be reduced to 4.8 mm (which in itself is in startling contrast to the results obtained in all previous ophthalmometric measurements) would not be enough, because the radius would have to be even smaller.

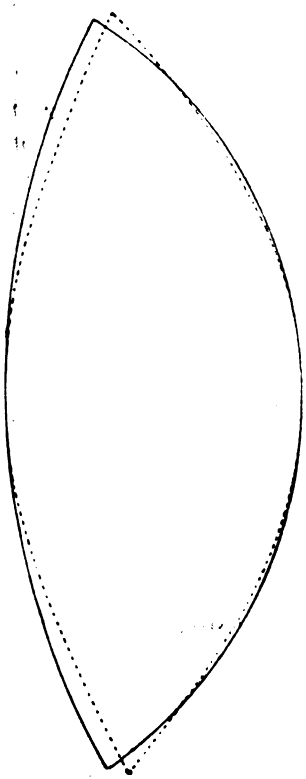

Fig. 141.

The second phase of TSCHERNING's ideas on this subject appears in the work above cited published in the *Encyclopédie française d'ophthalmologie*. The real difference here is that the decrease of depth of the anterior chamber in accommodation, previously proved beyond a doubt by the more accurate methods of HELMHOLTZ, MANDELSTAM and SCHÖLER, and BLIX, is now admitted to be a fact, in consequence of its having been observed in BESIO's researches even with the less accurate methods of the Sorbonne Laboratory. Now, therefore, he represents the process of accom-

modation as shown in Fig. 142 (reproduced from this work) in which the continuous line indicates the contour of the lens at rest and the dotted line that of the lens with an accommodation of 7 dptr. It would be idle to discuss the new anatomical conceptions, which enable him to derive this accommodation form of the lens from the passive shape by a tension of the anterior fibres of the zonule. Here we shall merely emphasize the fact that, leaving out of account the sources of error in the method, BESIO's researches support TSCHERNIGN's first assumption (as we designated it above) for the case of an eye that is *under the influence of cocaine.*

The third phase of TSCHERNING's ideas is indicated in the *Thomas Young Oration* before the *Optical Society* in London in 1907.[1] The part with which we are concerned reads: "v. PFLUGK has recently succeeded in fixing the dead lens in its accommodative shape; he has found that the posterior surface frequently becomes a little concave in its peripheral parts. This concavity increases during accommodation. One of my pupils, Dr. ZEEMAN, has since observed this concavity in the living eye."

VON PFLUGK's experiments[2] consisted in freezing the lens with liquid carbonic acid. However, the value of such experiments for purposes of demonstration should not be very highly estimated, since the action of forces developed by the freezing cannot be overlooked. FISCHER[3] too has shown that the freezing method may give rise to accidental changes of form of the lens; and, besides, the mechanism of accommodation of the eye of a bird is essentially different from that of the human eye, as shown by HESS's investigations mentioned above. ZEEMAN's report, alluded to above, merely proves the presence of the surface of discontinuity. The existence of a posteriorly directed concavity on the posterior surface of the lens can be proved only when two images are seen moving in opposite directions, as has been observed by the writer in the case of true lenticonus posterior.

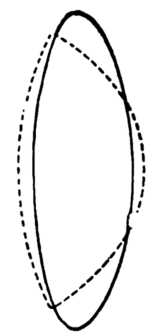

Fig. 142.

[1] *The development of the science of physiological optics in the nineteenth century.* Reprint from *The optician and photographic trade journal.* Nov. 1, 8 and 15. 1907.

[2] *Über die Akkommodation des Auges der Taube.* Wiesbaden 1906.

[3] F. FISCHER, Über Fixierung der Linsenform mittels der Gefriermethode. *Arch. f. Augenheilk.* LVI. 1907. S. 342.

V. The Monochromatic Aberrations of the Eye[1]

It is not practicable to obtain an actually homocentric bundle of rays by refraction in an optical system except in certain quite singular cases, in which invariably the rays must emanate originally from a point on the axis of a system of revolution. Thus, so far as real optical imagery is concerned, this requirement has no significance whatever. However, this fact was not fully realized for a long time; and until the constitution of the general bundle of optical rays was known more in detail, the only description that could be given of it was in terms of the deviations or so-called aberrations of a ray from the ideal procedure of rays that were homocentric. According as magnitudes of different orders were included in the computation, various values were obtained for the aberrations of a ray, each representing, as we know now, certain geometrical relations that are characteristic of the bundle of rays. In this way we may speak of monochromatic aberrations of a certain order. For example, *astigmatism* is an aberration of the first order, and the *asymmetry-values* are aberrations of the second order. But at present what is meant by *monochromatic aberration* in the restricted sense usually includes simply aberrations of order higher than the first—and so does not include astigmatism. The term *aberration* by itself is applied specially to the aberrations of the third order defined by the aberration-values mentioned in a previous section. In physiological optics this is the best term to employ for the deviations or aberrations of order higher than the third. In technical optics they are also called zonal errors with respect to the axis of a system of revolution.

The fact that the line of sight is not normal to the cornea involves a slight degree of inverse astigmatism in the eye, which, little as it is, tends to compensate the normal direct corneal astigmatism. At the same time, there are finite asymmetry-values along the line of sight, and hence after refraction in the eye the bundle of rays is singly asymmetrical with respect to the ray corresponding to the line of sight. However, along another ray, as shown by investigations with a luminous point, the bundle of refracted rays is anastigmatic and without asymmetry. This is the important ray for the imagery, because the convergence along it is of higher order; and consequently, *as a matter of fact, the asymmetry-values of the bundle of rays refracted in the eye are equal to zero.* But between this state characteristic of the best formed eyes and those degrees of *pathological asymmetry* which may cause a considerable lowering of visual acuity or a bad sort of asthenopia,

¶See experimental investigation by A. Ames, Jr., and C. A. Proctor, Dioptrics of the eye, *Jour. Opt. Soc. Amer.*, 5, 1921, 22-84. (J. P. C. S.)

there are all possible intermediate stages. The methods of studying the asymmetry are both subjective and objective; and as the former are the same as those used for studying the aberration, they will be considered first in the following discussion. The objective methods are ophthalmoscopic; and therefore treatment of this subject will be confined here to the investigation of the asymmetry of the cornea and pathological decentration.

Obviously, sufficient data would be obtained by making a complete ophthalmometric investigation of the anterior surface of the cornea, together with a determination of the position of the optical axis of the eye (in both cases by methods fully described above); but on account of the labour involved, this is practically out of the question. Very good data can be obtained by finding the position of the optical axis and making the ophthalmometric measurements in four directions symmetrically taken in the two principal sections at angles of 10° to the line of sight. However, the majority of cases of pathological or of unusually high physiological asymmetry are found incidentally by the ophthalmologist during the determination of the refraction of the eye; and, therefore, a simpler method of getting the required data is needed. This is supplied by using keratoscopy instead of keratometry.

In keratoscopy the form of the cornea is estimated by the distortion of the reflex image. Now experience proves that this judgment is most reliable when, in the absence of any deformation of the cornea, the reflex image is a square. Accordingly, the target whose reflex image is to be observed in the cornea is made by the writer in the form represented in Fig. 143. When it is held at the right distance or attached at the end of a properly focused telescope, its image in a spherical mirror consists of four squares, each separated from the next by a distance equal to the length of the side of the central square. If the form of the reflecting surface differs from that of a sphere, the image will be correspondingly deformed, in such fashion that the intervals between

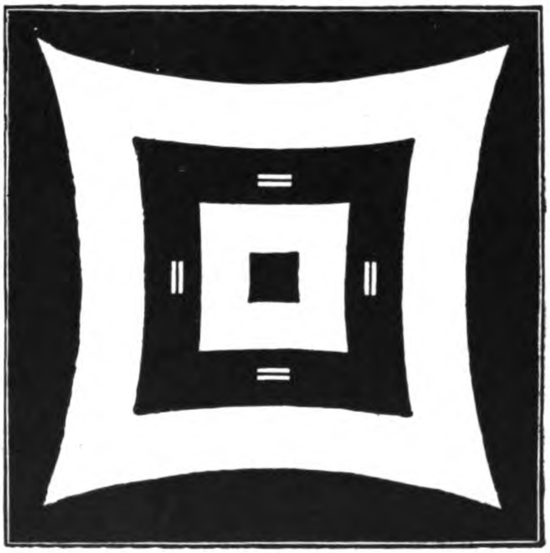

Fig. 143.

the sides are proportional to the radii of curvature of the corresponding elements of the surface. The patient looks, first, straight into the objective of the telescope; and then in four other directions in succession, above and below, and to the right and the left, as determined in each instance by a fixation-mark. For each of the four peripheral adjustments the two points of the cornea that correspond to the reflexes of the middle points of the sides of the two interior figures must be exactly the same as corresponded to the middle points of the sides of the two outer figures when the eye was looking straight in the objective of the telescope. Thus, for the two planes in which the eye looks, the keratoscope gives exactly the same data as the photographic method of keratometry described above.

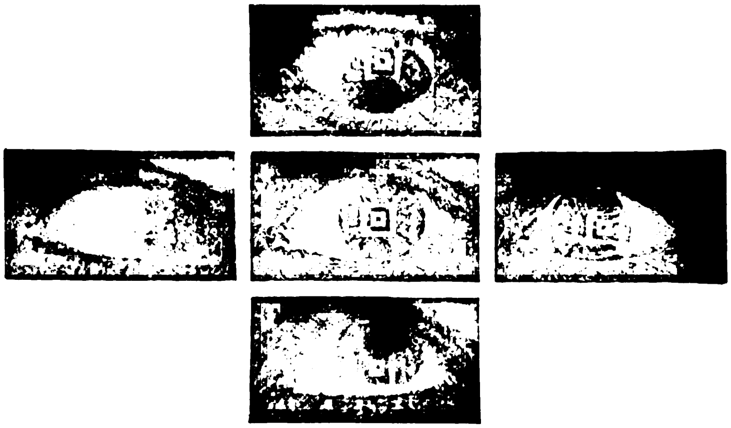

Fig. 144.

Fig. 144 shows the appearance of the reflex images in a typically normal cornea, the same cornea, in fact, as the one for which the ophthalmometric measurements were given in Appendix II. The central image approximately covers the optical zone of the cornea; and the squares there are perfectly regular, except that the top line, being shaded by the lids, is therefore not visible. The image is midway between the upper and lower margins of the cornea, but appreciably nearer the inner margin than the outer. With a magnifying glass one can see plainly, on the plate anyhow, that the pupil is displaced laterally with respect to the centre of the image. The two images in the upper and lower parts of the cornea are symmetrical with respect to each other, and indicate a considerable flattening of the cornea, becoming more marked towards the periphery. In both of them, but particularly in the lower one, the oblique angles show that the cornea in these places is not symmetrical with respect to the vertical median line of the reflex image, but that the vertex of the cornea, that is, the

optical zone, is situated on the outside of the ophthalmometric axial point. In accordance with this, the two side images show the normal horizontal asymmetry with decidedly more flattening. These images are fairly symmetrical with respect to the horizontal median line, and yet they seem to indicate in some measure, especially the outer one, the slight vertical physiological asymmetry that was found by the ophthalmometric measurements. The pupil is situated symmetrically with respect to the horizontal median line in each of the side images.

Although, therefore, in this instance a distinct vertical asymmetry is not established, yet there are other eyes which seem to be perfectly normal in their functioning, but in which the keratoscopic images are considerably different from this type. If the notion of asymmetry and decentration is referred to the ophthalmometric axial point, *eyes that are perfectly normal from a clinical standpoint may be classified in three groups*, as follows:

1. In the most regular cases, normal horizontal asymmetry only.

2. In the less regular cases, a combination of vertical and horizontal asymmetry, so as to give the appearance of a normal asymmetry in an oblique direction.

3. In more irregular cases, normal asymmetry in the horizontal meridian of the cornea, combined with pronounced abnormal asymmetry in the vertical meridian, but with vertical decentration of the pupil in the direction of least flattening.

Just as the transition from the second to the third group is a gradual one, so also there is no sharp line of demarcation between the latter and the pathological region. There are cases, for example, which with vertical asymmetry and compensating decentration of pupil show symptoms of asthenopia which can be made to disappear by corresponding correction of the inverse total astigmatism that is ordinarily present. Vertical asymmetry of the cornea with opposite decentration of the pupil may more certainly be regarded as pathological. According to the writer's experience this condition is invariably accompanied by asthenopia or some other symptoms of illness or by myopia.

The best way is to consider the pathological cases as a special kind of anomaly of refraction called *asymmetry* or *decentration*. In the first place, then, this will include pathological *vertical asymmetry* as manifested either by opposite decentration of the pupil or by an inverse total astigmatism and sometimes by an unusually big difference between the corneal astigmatism and the total astigmatism. Frequently, without thorough investigation, these cases are mistaken for a mild myopia that does not disappear until the inverse astigmatism, which is sometimes hard to discover, has been corrected; and that is the explanation of the name "latent" inverse astigmatism. TSCHERNING'S

eye (which will be referred to again) shows an unusually high vertical asymmetry which certainly is on the border of the pathological region, if not beyond it, and must be considered as abnormal.

In the second place, there is also an abnormal *horizontal asymmetry*. In rare instances it occurs as an increase of the normal asymmetry, but this is not very troublesome and does not have to be corrected except with large pupil. In other cases, particularly with myopia, there may be more flattening of the cornea on the outside than on the inside; or the investigation of the peripheral refraction by the ophthalmoscope and skiascope indicates a difference of several dioptries, when the eye looks first in a nasal direction and then equally far in a temporal direction.

In the cases mentioned the bundle of rays refracted in the eye may be singly asymmetrical, which is generally the way it is; so that usually with proper correction good visual acuity can be obtained. But this is not so often the case with what is known as *oblique asymmetry*, because the bundle of refracted rays then is apt to be doubly asymmetrical. Accordingly, the caustic surface is not so favourable in form, and the visual acuity is lowered. The existence of an astigmatism whose principal sections make angles of from 35° to 55° with the horizontal plane usually indicates some oblique asymmetry, just as is indicated in ophthalmometric measurements of the cornea when the two meridians for which the "Denivellation" vanishes are not at right angles to each other; or as is shown by a marked difference between the principal sections of the corneal astigmatism and of the total astigmatism.

Although the keratoscope enables us to find merely the corneal asymmetry and the decentration of the pupil, it is a good way of finding an asymmetry of the bundle of rays refracted in the eye. When the asymmetry is very great, the entire eye is affected by the deformation, as shown by ophthalmoscopic investigation of the papilla of the optic nerve. In the typically normal cases the latter is symmetrical around the horizontal line; whereas in abnormal vertical asymmetry and in oblique asymmetry very often there is a corresponding deformation of the optic nerve (ultimately with conical structure downwards or obliquely). The perverse structure of the papilla usually means an abnormal horizontal asymmetry that can be found by the ophthalmoscopic and skiascopic investigation above mentioned.

Thus, while the asymmetry of the bundle of rays refracted in the eye is to be considered practically as a pathological effect, the *aberration* of the bundle is a physiological condition. Indeed, this is something that we might have known in the beginning, because the absence of aberration is an unusual state of affairs, which would be of no use for

the eye. Owing to the size of the pupil, the faults of higher order are bound to be of such importance that the aberration along the axis will certainly be a minor matter. Thus, the investigation of the aberration involves a thorough knowledge of the constitution of a wide-angle bundle of rays. The best way is to make a direct investigation of the section of the bundle made by a plane screen, because there the lines of intersection of the caustic surface come out clearly, and enable us to ascertain its form. Fortunately, the retina itself is an excellent screen for this purpose. The only drawback is that the method is subjective and dependent on the patient's powers of observation. Consequently, it is not adapted for use with a large number of subjects. The bundle or rays is created by looking at a small, bright, luminous point. The different cross sections of it are brought on the retina by changing the optical focusing of the eye with the aid of spectacle glasses. The complete exploration of the caustic surface in this way is what the writer calls the method of *subjective stigmatoscopy*, as distinguished from the aimless investigations with a luminous point that are described in the literature.

The necessary mathematical relations having been ascertained, this was the only method that enabled the writer to explore the constitution of the bundle of rays refracted in the eye. And, in fact, now that this constitution is known, there is no other method of showing completely what it is. The reason for this is the extraordinarily complicated form of the caustic surface. Not only are there three cusps in its meridian section, as shown in Fig. 120, but the form of the curve steadily varies as the meridian section is revolved around the axis, in such fashion that the distance of the two symmetrical cusps from the axis becomes alternately a maximum and a minimum. Corresponding to these greatest and least distances of the cusps, the cuspidal edges of the secondary caustic surface are so contrived that, with respect to this alternation, there is a certain analogy with what is known as diagonal astigmatism of the aberration; but with this difference, namely, that whereas there are only two maxima and minima in this effect, the number is greater in the eye. It can be seen from Fig. 145 that the form of the section of the caustic surface is quite complicated even in the case of diagonal astigmatism of the aberration; as this illustration represents the cross section of such a bundle of rays as obtained by a certain form of bicylindrical lens.[1] The meridian sections for which the aberration has its maximum value correspond to the four points of the star-shaped figure; while the bright diagonal lines indicate the sections of the secondary caustic surface where it is bent

[1] A. GULLSTRAND, Demonstration eines Instrumentes zur Erzeugung von Strahlengebilden um leuchtende Punkte. *Ber. ü. d. XXX. Vers. d. Ophth. Gesellsch. Heidelberg* 1902.

over at the edges. By screening off a part of the lens, it may be demonstrated that the former are due to rays of light that have already crossed the axis, whereas the latter are made by rays that cross the axis beyond the place where they are intercepted by the screen.

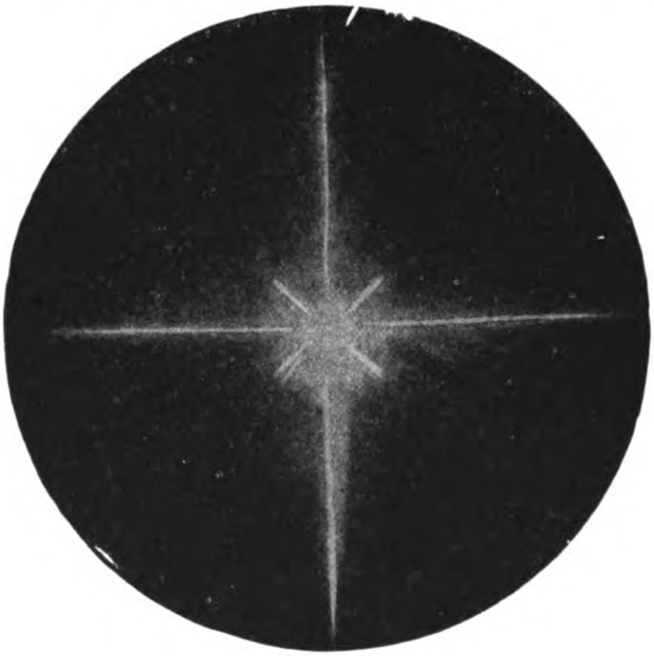

Fig. 145.

Now by subjective stigmatoscopy we find that the bundle of rays refracted in the eye shows precisely the same characteristics as those just described, except that the starry points and the intermediate bright flecks are more numerous and not always arranged perfectly regularly. Just as in HELMHOLTZ's description, when a portion of the pupil of the eye is screened off by the partial interposition of an opaque card, the starry rays that are seen around a bright point vanish first on the same side as that on which the pupil is covered. Since the image of the star pattern appears exactly opposite to the actual inverted arrangement of the blurred image as projected on the retina, the rays of the star are due to rays of light which have crossed each other before getting to the retina. Chromatic effects, such as HELMHOLTZ used, enable us to show this; cobalt glass being particularly suited for the purpose, because it makes the central bright point look purple, while the rays of the star are blue. These star-points, as HELMHOLTZ states, are visible with a sufficiently bright luminous point even when the eye is focused most sharply, provided the pupil is not excessively contracted by the illumination in the room. In a complete investiga-

tion by the method of stigmatoscopy, the luminous point is 4 metres away, and its diameter is 2 mm. Accommodation being entirely relaxed, amyopia of 4 dptr is produced at first by the aid of a suitable spectacle glass. The refraction of the compound system is steadily increased, a half dioptry at a time, by changing the glass. Initially, a bright blur circle will be seen, which, perhaps, may appear to be punctuated by brighter points, but which does not admit of any distinct separation into brighter and darker portions. As the far point begins to recede, the first thing that is noticed in the way of change of appearance is the development of a more or less uniformly bright point in the centre, around which the rays of the star begin to be visible; corresponding to the section of the bundle of rays that is shown by HELMHOLTZ in Fig. 72, *b*. The illustration in Fig. 72, *a*, which represents the image seen by HELMHOLTZ in his right eye, is not so well suited to show the simplest case, because it indicates a vertical asymmetry in that eye (possibly due to some small opacity in the lens), which makes it a little hard to explain. Indeed, the drawings Fig. 72, *b*, *d*, that relate to the left eye, do not indicate an eye that is one of the most regular in its structure, but they are sufficiently typical to be used for demonstration. In the best formed eyes the star figure has eight points, and its fundamental form is that of a vertical cross with diagonal rays, one of which however, may be split in two. Evidently, HELMHOLTZ's figure may be explained as a variant of this form in case of an oblique cross. If by increasing the refraction the far point of the reinforced eye is made to recede farther and farther, until at last it becomes virtual, a darker centre surrounded by a brighter serrated line will be seen. With bigger pupil these serrations appear to lengthen out into rays of the star. When the pupil is partly covered, the serrations vanish on the opposite side. Viewed through cobalt glass they are red, implying therefore that they are due to rays of light which have reached the retina before intersecting the central ray. These star-patterns are more numerous than in the ordinary star-figure. The orientation is certainly different from that of the ordinary star-figure, because there are no rays in these patterns in the directions where they are clearly perceived in the ordinary star-figure. When the pupil is artificially dilated, there is a certain focusing for which both kinds of star-patterns may be seen together, and the alternation verified. The serrations lengthened into rays are to be seen in HELMHOLTZ's diagram Fig. 72 *d*, where the dark centre is plainly visible, and where, too, the writer's statement about orientation can be verified. The fact that the number of serrations in HELMHOLTZ's case is less in this section of the bundle of rays than in that shown in Fig. 72 *b*, is due to the fact that he got the different sections of the bundle, not by using different spectacle lenses, but by varying the

distance of the luminous point. For instance, when this point is brought closer to the eye the visual angle which it subtends is so large that the star-points next one another merge together.

The cross section of a bundle of rays with diagonally astigmatic aberration, as represented in Fig. 145, exhibits not only the rays corresponding to the ordinary ocular star-figure but also the corners between the rays corresponding to the serrated line that contain the brighter, radially directed spots of light. A cross section of this sort is obtained with a wave surface which has four planes of symmetry, and which is represented by a certain equation of the fourth degree. By varying the bicylindrical combination used in producing this figure, the symmetry may be destroyed and very complicated effects obtained. However, if the equation of the wave surface is one of the eighth degree of corresponding form, and if there are eight planes of symmetry, there will be eight rays in the star-figure and eight indentations with brighter, radial medians, instead of the four corners shown in the figure. And if the symmetry about the eight planes is not perfect, there will be apparent irregularities in the arrangement of the rays and indentations due to the complicated form of the caustic surface. In the organ of vision the contrasts are heightened, because the minimum that is perceptible is lowered in the vicinity of a brightly illuminated point of the retina, which owing to the brighter central parts, must have the effect of making the indentations appear as serrations. When this is taken into consideration, it is not difficult to understand that the appearance of the star-patterns in the eye does not at all imply real irregularities, such as might be produced by edges or cusps in the refracting surfaces themselves or by discontinuities in the variation of the index of the crystalline lens; but that the phenomenon is just as regular as any effect that is dependent on an equation of the eighth degree or higher degree.

A characteristic feature of the wave surface of a bundle of rays of this sort, as found by mathematical investigation, is that out towards the periphery the flattening is different in different meridian sections; the result being that there are just as many minima corresponding to the rays of the star as there are maxima corresponding to the indentations. A cylindrical surface whose axis coincides with the axial ray will cut the wave surface in a line which, when the cylinder is unrolled on a plane, has a sinuous form; the sinuosity being more pronounced in proportion as the section is farther from the centre. This means that the wave surface contains ridges and valleys running radially along it and getting shallower in towards the centre. We might speak of them as "creases" ("Faltenbildungen"), provided the term is used by way of analogy without being taken too literally. Now there

is nothing that can be responsible for this idiosyncrasy of the wave surface except a similar peculiarity of the refracting surfaces of the crystalline lens or of the iso-indicial surfaces, because the star-figure around a luminous point disappears when the lens is removed from the eye. This form ought to be manifested in the surfaces of the lens by jerky movements of the reflex images during movements of the eye. As a matter of fact such twitchings are often perceived in the reflexes in the anterior surface of the lens. But so far as the writer has been able to discover, this phenomenon occurs only at the periphery of the lens, and this would not be sufficient for the explanation of the "creases." Besides, the reflex image in question is produced not simply by the anterior surface of the lens, but also in the most anterior portions of the lens substance. This is why it is so vague. The only other hypothesis left, is that the forms of the iso-indicial surfaces of the lens correspond to that of the wave surface. Taking into account the anatomical structure of the lens, we reach the same conclusion from the study of the dioptrics of the lens. During the accommodative change of form of the lens, the iso-indicial surfaces must contain constant volumes; and hence for the different focusings of the eye their superficial areas must be different, supposing they are surfaces of revolution. But this would not be possible unless either the particles of the lens were perfectly mobile, or the substance of the lens had considerable elasticity. Now since neither of these is the fact, the different forms assumed by the iso-indicial surfaces cannot be surfaces of revolution; which means that the change of form must be accompanied by the production of "creases" or by the variation of such "creases." Owing to the concomitant contraction of the pupil in accommodation, a variation such as that just mentioned is difficult to investigate in a perfectly satisfactory manner. However, in case of accommodation that occurs in the first stage of the action of eserin on a pupil that is dilated by homatropin, it is easy to verify the fact that there is some change of this nature.

The fact that the star-figure is due to the iso-indicial surfaces of the lens and their variations during accommodation, plainly shows that this particular peculiarity of these surfaces is influenced by the tension of the zonule. That the effect cannot depend on the anatomical structure of the lens, which in its embryonic stage has the shape of a three-point star, is shown both by the number and by the disposition of these points; whereas in the most regular cases the star-image itself consists of eight rays, its fundamental form being that of a cross with diagonal rays. On the other hand, in the mechanism by which the lens is suspended there is an anatomical contrivance that must produce alternate maximum and minimum tensions of the zonule in the different meridian

planes, not simply because the mechanical relations for the ciliary processes are different from those for their interstices, but also because these relations must be modified by the way the fibres of the zonule cross each other in proceeding to the anterior and posterior sides of the capsule of the lens. It is true, the number of these maxima and minima is much greater than the number of rays in the star-pattern. But since the tension for the various maxima cannot be mathematically and precisely the same, the lines of force will have to merge together towards the centre. The star-figure illustrates the same thing. With a larger pupil, the rays of the star are often seen to separate from each other at a certain distance from the bright nucleus. As the lens is composed of fibres, the peculiar form of the iso-indicial surfaces ought to be manifested by a corresponding characteristic arrangement of these elements. Now it is even probable that with steady growth of the lens the anatomical arrangement of the fibres is affected by the tensions that occur, and thus the stellar appearance seen on the anterior surface of the lens by oblique illumination may represent this structure as acquired under the influence of the tension of the zonule. The best way to observe this starry form is with dilated pupil, by the same sort of experiment as was used for inspecting the reflex image in the anterior surface of the lens, where the light was concentrated on this surface by a convex lens. This figure indicates real discontinuities in the variation of the index; but without a very complicated mathematical study, it can hardly be regarded as conclusive on this point, inasmuch as the "creases" in the iso-indicial surfaces certainly seem to be qualified to produce the reflex phenomenon in question.

Owing to the property of the wave surface of the bundle of rays refracted in the eye which has just been proved, it is a mathematical impossibility for any cross section to cut the caustic surface in a smooth curve in the form of a circle concentric with the pupil. On the contrary, this section must be serrated everywhere or must consist of separate isolated points. Accordingly, the serrated curve described above is a section of the caustic surface, and the concentration of light at the centre that is seen at the beginning of the stigmatoscopic investigation is the vertex or cusp of the same surface. Now this means that the aberration along the axis is positive, because the cusp points in the direction in which the light goes. Pursuing farther the stigmatoscopic investigation with dilated pupil, by steadily increasing the hypermetropia of the reinforced eye, we find that in the last section of the bundle of rays where the line of intersection of the caustic surface can still be seen, it does not coincide with the boundary line. The inference is that a meridian section of the caustic surface has three cusps, like the curve drawn in Fig. 120, where the serrations that are the last

to be seen in the stigmatoscopic investigation of the given meridian plane are represented by the two symmetrical cusps. By measuring the difference of refraction between this section of the bundle and that which contains the cusp lying on the axis, the distance between the two sections can be found. In the writer's own case this difference is 4 dptr, and apparently this value is never exceeded. A higher value is frequently obtained with persons who are less expert, probably due to their inability to relax the accommodation completely. The diameter of the line $R=0$ corresponding to the symmetrical cusps of Fig. 120 is measured by the diaphragm held in front of the eye. The serrated curve in the section of the caustic surface nearest the refracting system can be made to coincide in this way with the boundary line of the cross section of the bundle of rays. By this means the writer has obtained a diameter of 4 mm. Let d denote this diameter, D the difference of refraction to be used in the calculation, f the posterior focal length of the eye, and n the index of refraction of the vitreous humor; then the aberration-value is found by the formula:

$$A = \frac{8 f^4 D}{1000 n d^2},$$

the distances being expressed in millimetres and the value of D in dioptries. For $f=20$ mm and $n=4/3$ (as in Donders' reduced eye), the formula gives 240 mm for the aberration-value; whereas on the assumption that the refracting surface is spherical, the calculated aberration-value for this eye is 540 mm. The amount of the aberration-value found in the living eye proves at once that the bundle of refracted rays acquires a positive aberration in traversing the crystalline lens. This could be demonstrated by Helmholtz's schematic eye; because by giving the surfaces of the lens in this eye a form such that the lens itself does not contribute to the aberration one way or the other, with spherical cornea and emmetropic focusing the value found for the aberration is 162 mm. Hence the writer infers that the variable index of refraction of the lens has little effect on the refraction of paraxial rays in the eye, and that, consequently, the chief significance of this peculiarity of the lens is probably in connection with the change of form of the lens in accommodation and possibly also for peripheral vision. This reasoning is completely sustained by the dioptrics of the lens; which likewise indicates that the lamellar structure of the lens is merely in the interest of the change of form in accommodation, since, as was proved above, the effect of this structure is to augment the astigmatism of an oblique bundle of rays.

Again, if the posterior focal length of the exact schematic eye and the index of refraction of the vitreous humor are substituted in the

formula above, the aberration-value thus obtained is 403.5 mm. Now if the aberration-value of the exact schematic eye with the real core-lens is calculated by the writer's formulae, it is found to be

$$A = 691.17 + 75854\,\Phi_1 - 7511.5\,\Phi_2 + 6113.9\,\Phi_3 - 3264.4\,\Phi_4,$$

where the symbols Φ are used to denote the "Abflachungswerte" of the four refracting surfaces; each of them being found by the equation

$$\Phi = -\frac{3\epsilon^2}{\rho^3},$$

where ρ denotes the radius of curvature, and ϵ is the eccentricity of the surface of the second degree that has contact of the fourth order with the refracting surface in question. When this value calculated for the exact schematic eye is compared with the result 403.5 mm, as obtained in the investigation of the living eye by using the focal length of the schematic eye and the index of refraction of the vitreous humor, it should not be overlooked that the formula by which the latter value was computed is merely approximate, because the effect of the aberration-values of higher order cannot be taken into account. However, since there is in this case a curve $R=0$, these values must be negative. Hence, the aberration-value calculated for the living eye from the results of the investigation is too small; but how much too small, cannot be estimated at present by investigations. However, in the calculation of the schematic eye, HELMHOLTZ's ellipsoid was used for the corneal refraction, so that even in the schematic eye the calculated value is smaller and, consequently, the effect of negative aberration-values of higher order is taken into account. Suppose, therefore, that the error due to the above cause is compensated by the fact just stated; then if the surfaces of the lens are considered as parabolic; and if the value of the eccentricity of the cornea, as calculated by MATTHIESSEN (as above stated) from the writer's measurements of the cornea, namely, the value $\epsilon = 0.551$, is employed, the aberration-value turns out to be 476.16 mm. As a matter of fact, the surfaces of the lens are probably flatter towards the periphery than the paraboloid; and the peripheral increase of thickness of the cornea may indicate a positive value of Φ for the posterior surface of the cornea, which would lower the calculated aberration-value still more. If these latter considerations are taken into account, the agreement between the value calculated for the exact schematic eye and that found by the investigation of the living eye could not be better.

This result is all the more important because it proves the correctness of using MATTHIESSEN's law along the axis of the lens. The greatest part of the aberration, as calculation shows, originates in the

lens and depends on the value of p_n. Now this coefficient can be made smaller by assuming an hyperbolic indicial curve along the axis, but this is contrary to the result of all refractometer measurements. On the other hand, an increase of the value of p_n would result in still greater positive aberration in the schematic eye, which, according to the above, must certainly appear very unlikely.

The region of the pupil enclosed by the curve $R=0$ is what the writer has called the *optical zone* of the pupil. The name seems to be all the more appropriate because the region coincides approximately with that comprised by the optical zone of the cornea. Within this area the aberration of the normal eye is invariably positive. This can be shown by finding the refraction of the eye at the different parts of the pupil. Everywhere it is less than it is at the centre; and the difference increases out from the centre along any meridian section, more rapidly at first, and then rather more gradually, until it amounts to 4 dioptries at the boundary line. However, in this connection it must be kept in mind that, according to ordinary usage, myopic refraction is negative, and that what is meant by the refraction at a point of the pupil is the state of refraction that the eye would have if an infinitely small stop were situated at this place. This refraction is determined, therefore, by the distance of the point of intersection of the given ray with adjacent rays, not with the axis. Inside the optical zone, along every ray except the axis, there is astigmatism, because the meridional focal distance is shorter than the equatorial. If a meridian section is followed beyond the border of the optical zone, the astigmatism steadily diminishes along the rays meeting this curve and vanishes entirely for the ray that is tangent to the caustic surface at the place where it crosses the axis. Leaving out of account the so-called "creases" in the wave surface, the totality of all such rays for the dilated pupil forms a conical surface with its vertex on the axis. And since along each one of these rays there is a perfect ray-convergence of the first order, the convergence of the rays on the axis at the vertex of the cone is extraordinarily good, and consequently this point in a cross section of the bundle of rays stands out sharply from its surroundings. In vision with a wide pupil it is this point, therefore, that serves for the imagery, the eye being focused for the cross section here indicated. By subjective stigmatoscopy it is easy to prove that with dilated pupil the sharp focusing is for a cross section of the bundle nearer the refracting system than is the case for a pupil of moderate size. Thus in the former case when the eye is accommodated as sharply as possible for the luminous point, the only star-points that are visible around it are such as appear red through cobalt glass and vanish from the opposite side when the pupil is partly screened. But when the far point

is brought closer to the eye by means of a convex lens, the result is that with sharp accommodation and with a pupil of moderate size the ordinary star-figure is seen. When, with dilated pupil, the far point is brought so near the eye that the cusp of the caustic is visible, the section of the part of the caustic that is bent over will be seen at the same time, appearing along its boundary as a clear, serrated, bright ring.

In order to make this explanation of the convergence of the rays more intelligible, the writer has had to leave out of account the peculiarity of the bundle of rays that is due to the "creases of the wave surface." This feature involves a difference of the aberration-value in different meridian planes. However, in this method of investigation the measured aberration is its maximum value. The writer has not succeeded in finding any way of measuring the minimum value; and for the present it must suffice to know that in the corresponding meridian planes the aberration-value is smaller, and the point of best convengence of rays lies farther away from the refracting system. The distance of this point from the cusp of the caustic surface is correspondingly smaller, and a ray that goes through it will meet the pupil in a point closer to its centre, and therefore this point of the bundle of rays is used for sharp imagery when the size of the pupil does not surpass that of the optical zone. But there seems to be no reason why, with a pupil of the smallest size, the focusing may not be made for a cross section of the bundle that is still nearer the cusp of the caustic surface.

In the stigmatoscopic investigation, the difference of refraction between the point of most advantageous convergence of the rays and the cusp of the caustic surface may be measured. In this way the writer obtains a difference of 1.5 dioptries, which gives, therefore, the degree of hypermetropia along the axis of an emmetropic eye for pupil of moderate size. But the measurements do not give results that are very accurate; and so in the schematic eye the writer has used the value one dioptry, which is certainly not too large. The difference between the optical focusings of the eye for a pupil of moderate size and for one as large as possible can be satisfactorily obtained in the above manner. In many instances, but not always, the result of clinical investigation by Donders' method of refraction may reveal a slight myopia of the emmetropic eye when the pupil is dilated. The simple explanation is that the visual acuity is reduced by the large pupil.

Evidently, from what has been said above, all that can be implied when we speak of the aberration of the eye is the maximum value of this magnitude as it occurs in the particular meridian section under consideration at the time. By more exact methods of investigation,

we find that usually there is an astigmatism of this aberration as manifested by an oblique oval form of the section of the caustic surface made by the retina, which is seen also in HELMHOLTZ's Fig. 72 *d*. This means that even when the visual acuity is maximum there is not complete absence of astigmatism along the central ray. Practically, however, this is of entirely secondary importance. In the bundle of rays that falls on the retina of the writer's right eye no traces can be detected of the horizontal asymmetry due to oblique incidence of the line of sight; and hence the asymmetry-values vanish, not indeed along the ray passing through the centre of the anatomical pupil, but along the ray that goes through the centre of the exit-pupil. The ray that is tangent to the cusp of the caustic surface is the controlling ray so far as the imagery is concerned; and, according to the above system of nomenclature, may be called, therefore, the *central ray*. The point where it crosses the plane of the pupil may be named the *optical centre* of the pupil. In the writer's own case this optical centre is in the vertical diameter of the exit-pupil, and the vertical line that divides the pupil in half is a line of symmetry with sufficient accuracy for practical purposes. This is not the case with the horizontal line. For even with the ordinary star-figure seen around a luminous point a vertical asymmetry may be recognized, the ascending rays appearing shorter than the descending ones. With reference to a vertical asymmetry, suppose the upward direction is taken as positive. Accordingly, there is a direct negative asymmetry of the bundle of refracted rays along the ray that goes through the centre of the pupil, which means that *the optical centre of the pupil is above the anatomical centre.* In the absence of a vertical decentration of the pupil with respect to the ophthalmometric axial point, this statement might have been made *a priori*, provided the asymmetrical flattening of the vertical section of the cornea is not compensated by some mechanism operating in the opposite sense. In keratoscopy and ophthalmometry the ophthalmometric axial point is the point of reference. Simiarly, in stigmatoscopy the optical centre of the pupil is the point of reference for what is called the "decentration." Thus the vertical asymmetry that has just been found may be spoken of as a downward decentration of the pupil. This normal vertical decentration may occur to any extent, until the optical centre is at the border of a pupil of medium size or even beyond it, in which case the limit of the physiological region is exceeded. TSCHERNING's drawings of the appearances in his eye[1] indicate a vertical decentration of this nature, which undoubtedly is in the neighbourhood of this limit and is certainly abnormally great.

[1] Loc. cit. *Encyclopédie Française d'ophthalmologie.* T. III. p. 207.

Besides this decentration of the pupil, a decentration of the optical zone has also to be taken into account. The former is computed from the relative upward and downward extent of the star-points that are visible in the section of the bundle of rays that contains the cusp of the caustic surface; and the latter is calculated from the difference of refraction when the different parts of the edge of the caustic surface corresponding to the curve $R=0$ are focused. Thus, in the writer's eye there is a difference of one dioptry between the upper and lower parts of this edge; and in the right eye the calculated decentration of the optical zone amounts to one-eighth of a millimetre downwards and is about the same as the decentration of the pupil in the same direction, which for a diameter of 6 mm has the value of one-seventh of a millimetre.

The writer has not been able by stigmatoscopic investigations to trace the transition from this physiological decentration of the optical zone to the pathological decentration, because these experiments require considerable practice and involve much waste of time. Where there is vertical asymmetry of the cornea and contrary decentration of the pupil, it is extremely likely that the pupil would exhibit also the opposite decentration with respect to its optical centre, and that with a pupil of medium size the curve $R=0$ would not be a closed curve, that is, there would be no optical zone. In the case of TSCHERNING's eye, with a pupil of the size for which he has drawn the blurred patterns, a closed curve $R=0$ is already lacking; and there is no indication in these drawings that there usually is an optical zone in the pupil. If the vertical asymmetry of the cornea of an eye of this kind should be augmented, the effect of the increase of flattening of the cornea towards the periphery would be to make the direct asymmetry-value in the vertical direction increase more along a ray in the vertical plane of symmetry that passed through the lower part of it. The influence of the aberration under these circumstances as compared with that of the asymmetry would be relegated still more in the background, and the result would be a better convergence of the rays along a ray going through the lower part of the pupil. The upward decentration in an eye of this kind would therefore impair the convergence of the rays, since in TSCHERNING's eye, as the drawings show, the asymmetry-values are negative.

In the normal eye, inside the positive zone, *the aberration is positive even with powerful accommodation.* This is proved by the fact that when the luminous point is moved up in the vicinity of the near point of the eye and fixated, the first thing that is seen is the line of intersection of the focal surface, and it is not until still greater effort of accommodation is exerted that this first appearance is transformed into the

star-figure or into the sharp image obtained with undilated pupil. Mathematical investigation of the dioptrics of the core lens shows that there is an accommodative decrease of the value of p_n to such an extent that if this were the only factor that affected the aberration-value during accommodation, the latter would have to be reduced to about two-thirds of its original value. However, the writer has not succeeded in verifying this variation by subjective stigmatoscopy in a perfectly satisfactory way, because the concomitant contraction of the pupil hinders the investigation unless mydriatics are used, and the phenomena with pupil dilated either by cocaine or by homatropin after treatment with eserin do not seem to be unique as to their interpretation.

The physiological importance of the constitution of the bundle of rays refracted in the normal eye as ascertained in the above fashion can be properly estimated only in the light of the general laws of optical imagery. The magnitude of the blur circles as represented by the star-points that are visible around a luminous point would, for instance, completely prevent getting an imagery of the same quality as obtained by the visual acuity of the normal eye, if the blur circles were as important for the imagery as they are supposed to be. Instead of them, what we actually have are the sections of the caustic surface, and therefore any conflict between the degree of visual acuity and the structure of the bundle of rays is cleared up at once. Another matter which likewise receives a satisfactory explanation is the ability of considerably enhancing the visual acuity by practice for different optical focusing or for different degrees of congenital or artificial astigmatism; which is a well known fact of clinical experience. For just as long as sections of the caustic surface continue to fall on the retina, there is always a question of imagery of some kind, and the strain experienced by trying to read with faulty cylindrical correction is simply an expression of the greater difficulty of construing sections of the caustic surface that have an extraordinary form and are less suited for maximum visual acuity. Hence in near work the strain might be felt more because the particular form of the section of the caustic surface, along with the chromatic aberration, might be the controlling factor in the continual changes of accommodative focusing. The mere fact that the cross sections of the bundle of rays on the retina have different appearances, depending on the degree of accommodation, may constitute a factor of this kind.

The magnitude of the aberration is of fundamental importance also for comprehending the constitution of the bundle of rays refracted in the eye in the ordinary cases of astigmatism, because it requires an astigmatism of more than four dioptries in order that sections of the two caustic surfaces may not fall on the retina at the same time.

Thus in the most important practical cases of astigmatism the caustic surfaces are of the type shown in Fig. 121, where there are two rays going through the pupil of medium size along which there is no astigmatism. The sections of a bundle of this sort are shown in Fig. 146, which is a photograph[1] made with a telephoto-lens, the different cross sections being focused on the plate by varying the distances of the parts of the lens; and hence the dimensions of the sections relative to each other are not correct. The phenomena in the astigmatic eye are similar, as can be verified by using cylindrical glasses. The only difference is in a cleavage of the blurred figures depending on the "creases" of the wave surface. As to the arrow-head effects, seen in two opposite points in the fourth cross section in Fig. 146, where the two anastigmatic focal points are, the same phenomenon is easily verified in the stigmatoscopic investigation of the artificially astigmatic eye. With abnormal vertical asymmetry and not too high a degree of astigmatism, there is always a point of this kind on the caustic surface; and in TSCHERNING's drawings of the cross sections of the bundle of rays refracted in his eye this point is easily recognized. The artificial astigmatism affords a means of eliminating the influence of accommodation in the stigmatoscopic measurement of the aberration, but that is a matter that cannot be discussed here.

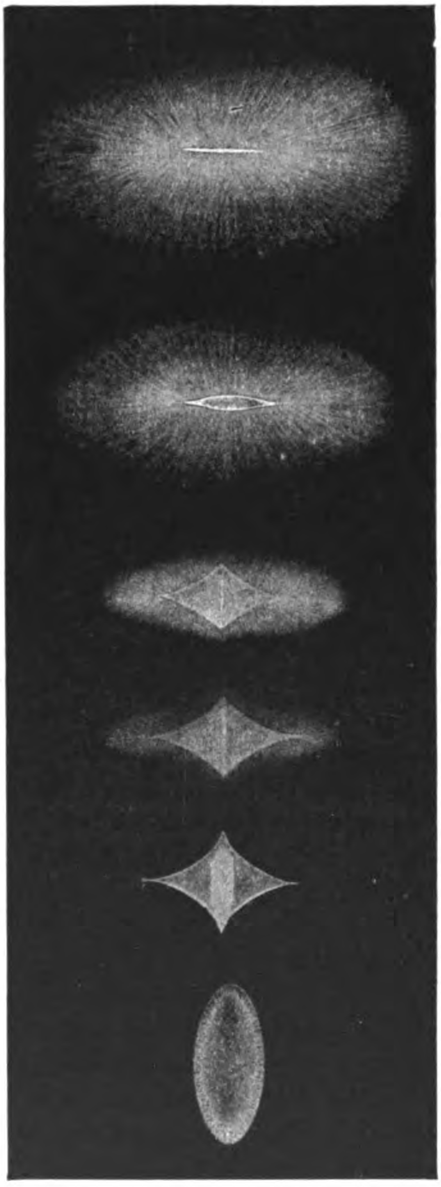

Fig. 146.

Objective stigmatoscopy (as the writer ventures to call it) is the most

[1] For which the writer is indebted to Mr. A. ODENCRANTS, candidate for the degree of doctor of philosophy.

important of all the other methods of investigating the aberration of the eye. The slit in the ophthalmometric NERNST lamp is replaced by a small hole; in front of which at an angle of 45° a cover-glass is set up vertically. The size of the latter is such that no light falls on its edges. The reflex image in the glass plate of the portion of the incandescent filament seen in the hole makes a very bright luminous point, which can be made to coincide with the pupil of the experimenter's eye. Then when the reflected light is projected into the pupil of the patient's eye at a distance of from 30 to 50 cm, the observer can investigate the convergence of the rays by moving his head in different directions. Suppose that the eye under examination is able to fixate sharply the reflex image of the luminous point with this arrangement of apparatus; then during fixation an image is formed on the retina which, with the diffuse reflection due to the distribution of light in it, may to a certain extent be regarded as a punctual image of the source of light of the bundle of rays to be investigated. For a distance as great as 50 cm the cross sections of the bundle of rays are of such magnitudes that the size of the pupil of the observer's eye does not enter into the problem at all. With his eye properly centered he looks into the patient's eye, whose pupil is supposed to be of the ordinary medium size in a dark room. At its centre he will see a bright luminous point, in many instances surrounded indeed by a perceptible star-figure. This point, which owing to the brightness appears more yellowish, is surrounded by a darker, that is, a more reddish looking, zone, which in turn is surrounded by a brighter, more yellowish ring. According to the size of the pupil, this bright annular zone extends to the margin of the pupil or is surrounded again by a darker, more reddish zone. Now suppose the observer moves his eye, say, in the horizontal direction; the central bright point in the pupil of the patient's eye will execute a movement in the same direction, but the vertical parts of the bright annular zone will be displaced in the opposite direction. This shows that along the central ray the rays converge behind the pupil of the observer's eye; but that the rays that meet the pupil of the patient's eye where the bright ring is seen are intersected by adjacent rays at places in front of the pupil of the observer's eye. In other words, the aberration is positive. Hence, when a luminous point is sharply fixated, a cross section of the bundle of rays that is in front of the cusp of the caustic surface is focused on the retina of the patient's eye. So far as physiological relations are concerned, the writer has found not a single exception to this behaviour. Apart from subjective stigmatoscopy, this is the only method of investigating the aberration within the optical zone. However, it cannot be used for finding the cuspidal edge of the caustic surface, because the image on the retina is not good enough

for this purpose. For the objective investigation of the refraction and astigmatism and of the pathological forms of asymmetry and aberration, it is superior to any other method. But it would take us too far to go here into the details of these investigations. It is what may be called a delicate, skiascopic method, but it is really much more than this, since it is a way of investigating the refraction at the fovea itself. Moreover, so far as the asymmetry and aberration are concerned, in the skiascopic method the size of the light-source, the distance between the observer's pupil and its reflex image, and the hole in the mirror are all just so many sources of error that make the phenomena uncertain and affect the accuracy of the conclusions derived from them. Consequently, in its results the method of objective stigmatoscopy is distinctly different from that of skiascopy, although the technique of the latter may be regarded as being a less precise modification of that of the former. But the method of objective stigmatoscopy absolutely requires a thin unsilvered mirror and a specific brightness such as cannot be obtained without the NERNST lamp or an arclight.

In cases of abnormal asymmetry or of negative peripheral total aberration (the latter of which not unfrequently leads to formation of cataract), the different states of refraction of the eye along different rays going through the pupil can be shown without difficulty by ophthalmoscopic measurement of refraction by the method of the erect image. However, in this determination the measurement must be made in a tiny portion at right angles to the meridian section in which the variation of refraction is being investigated, and the diameter of the peephole of the ophthalmoscope may not be more than from 1.5 to 2 mm. In finding the refraction through different parts of the pupil another matter that is important is not to let the peephole be dislocated with respect to the pupil of the observer's eye. The astigmatism that arises and the change of form of the papilla that is caused by it may have the same result. When the refraction varies, the radial principal section is always subject to a greater change of refracting power; and so the rule is, that, if for a displacement of the mirror with respect to the pupil of the patient's eye the papilla appears to be relatively more extended in the direction of the displacement, an increase of refracting power is indicated; and *vice versa*. It is true, this method of showing the physiological aberration does not always succeed. This is undoubtedly because the place of greatest refracting power is so little removed from the centre; and the transversal asymmetry, which, unlike the direct asymmetry, does not vanish here, is conducive to indistinctness. On the other hand, it can often be shown that this indistinctness increases at first with decentration of the hole in the mirror, and then decreases, as the constitution of the bundle of re-

fracted rays demands. With well dilated pupil the relative extension of the diameter of the papilla parallel to the direction of the displacement of the hole in the mirror, that is characteristic of positive aberration, can very often be observed.

Other methods are based on SCHEINER's experiment, which is the principle of YOUNG's optometer and TSCHERNING's aberroscope. The results obtained with this last instrument have much to do with TSCHERNING's theories; and therefore it might be worth while to give some space to its discussion. The instrument consists of two systems of perpendicular opaque lines ruled on the flat side of a planoconvex lens, which is held in front of the eye, from 10 to 20 cm away, and towards a luminous point. The shadows of the lines are seen in the blur circle produced by the artificial myopia. TSCHERNING argues that the aberration of the bundle of rays refracted in the eye is positive or negative according as the curvature of the lines is concave towards the centre or convex, respectively. However, evidently the phenomenon here is one of optical projection. The writer has shown that while it does depend in some measure on the aberration, it involves also another magnitude that cannot be calculated in the living eye. Moreover, the aberration on which the curvature of the lines in the aberroscope depends is not that which is characteristic of the bundle of rays used in vision, but is the aberration that has been imparted to the bundle of rays by the refraction in the glass lens, which is all the more significant in this case, because the aberration generally changes with the convergence of the bundle of incident rays.

That TSCHERNING's interpretation of the phenomena observed with the aberroscope is mathematically impossible, the writer has shown in the following way that anybody can understand. Consider the reduced eye, and suppose that the bundle of rays that issues from the lens of the aberroscope is free from aberration; then the shadows of the ruled lines as formed on the refracting surface will be simple curved lines, lying in planes that intersect each other in the image of the luminous point made by the planoconvex lens. But in order that these shadows should form straight lines on the retina, these planes would have to pass through the image of the luminous point in the eye, on the supposition that the bundle of rays was free from aberration after being refracted through the eye. Now this is impossible unless the image-point in the lens is at the centre of curvature of the refracting surface of the reduced eye.

What is actually found in the investigation with the aberroscope is the sign of the distortion-value for the optical projection, on the assumption that the eye is a system of revolution. Provided there is no refracting surface between the lines to be projected and the screen,

that is, between the ruled lines on the aberroscope and the retina, it is true that the sign of this magnitude does depend simply on the sign of the aberration; and, consequently, there would be no objection to the conclusions derived from investigations with the aberroscope, provided the instrument was set up in the vitreous humor! However, there is another factor besides an intervening refracting surface between the lattice of lines and the screen; and this factor cannot be calculated in the optical system of the eye, because the laws concerning it are not known for heterogeneous media. But it does depend on the distance of the lattice; and when this distance is considerable, it is appreciably affected by the aberration of the lens of the aberroscope. Thus, the value of the distortion whose sign is found by the aberroscope contains two terms, one of which depends on the aberration; and the sign of this term changes when the image of the luminous point falls beyond the retina, whereas the sign of the other term does not change. Consequently, the sign of the aberration of the bundle of rays refracted in the eye can be obtained by the investigation with the aberroscope, provided the curvature of the shadows changes sign according as the image of the luminous point falls on one side of the retina or on the other. Now, as a matter of fact, this does happen as a rule with the normal eye, and therefore the normal positive aberration may be established in this way, but not in the way TSCHERNING explains it. On the other hand, the aberroscope, like YOUNG's optometer in this respect, is by no means sensitive enough to bring out the real complicated form of the caustic surface, because, on the whole, no conclusion can be drawn from the curvature of the shadow-lines in the peripheral parts of the blur circle. Perhaps, it might be exceedingly well adapted for diagnosis of abnormal asymmetry, because it tells the sign of the transversal asymmetry, provided the sign of the curvature of the central shadow-line does not change when the image of the luminous point crosses from one side of the retina to the other.

The change of sign of the aberration during accommodation as reported by TSCHERNING has not thus far been proved by investigations with the aberroscope. Undoubtedly, as the writer himself has shown, the curvature of the shadow-lines can be seen to diminish during accommodation. But a change of this kind is bound to occur to some extent from the displacement of the image of the luminous point that is in front of the retina. It may be due partly also to the variation of the component of the distortion-value that is independent of the aberration-value. In order to show that the aberration varies during accommodation, it is absolutely necessary to prove first, by the method given above, that there is positive aberration in the case of emmetropic refraction (which means, of course, that the eye is properly

corrected); and then to cause accommodation by using more and more concave glasses, with corresponding variation of the original correction; the aberroscope being placed directly in front of the correction-glass. Now if during accommodation under these circumstances the following appearances should take place in the order named: shadow-lines convex towards the centre, then the point of light, and finally shadow-lines curved the other way, meanwhile nothing being altered with respect to the correction of the eye and the distance of the aberroscope;—then, and only then, could the investigation with this instrument be said to prove that the normal positive aberration becomes negative during accommodation. However, in spite of the publication of the necessary mode of arrangement of the experiment[1], no such proof has been adduced.

By an experiment with the luminous point also, TSCHERNING thinks he has shown that the aberration changes sign during accommodation. His method consists in comparing the appearance of the blur circle when the eye is made myopic by accommodation with its appearance when the eye is made myopic to the same extent by means of a convex lens. In the first case he finds a peripheral bright line parallel to the boundary line. However, owing to the peculiarity of the caustic surface due to the so-called "creases" in it, it is a mathematical impossibility for a negative aberration to produce a blur circle with the appearance noted by TSCHERNING. But in the same way a serrated section of the caustic surface should have been found beyond the axial focal point (such as occurs in the relaxed eye in front of this place), as proved, in one way, by the appearance of the bent-over part of the caustic surface that is seen when the pupil is dilated. On the other hand, as was mentioned above, the same appearance of the blur circle can be obtained as in accommodation by imitating, not simply the accommodative change of refraction, but also concomitant pupillary contraction, and combining the convex lens with a correspondingly small hole. Any one conversant with the phenomena of diffraction will recognize immediately the nature of the dark line that is seen between the bright zone and the boundary line. TSCHERNING's experiment might perhaps be used as a popular illustration of the phenomenon of diffraction. As was explained above, subjective stigmatoscopy, employed as a scientific method, demonstrates a positive aberration inside the optical zone even with the most powerful accommodation. And, therefore, the accommodative contraction of the pupil cannot be for the purpose of diminishing the effect of the aberration, since this effect is not troublesome even when accommodation is relaxed, although it decreases anyhow during accommodation.

[1] Loc. cit. *Arch. f. Ophth.* LIII, 2. 1901. S. 239.

The skiaskopic investigation shows that in many instances a change of sign occurs in the peripheral aberration during accommodation. Owing to the inherent sources of error in the method, this result cannot be said to be absolutely certain. But if it is established, it seems to be an expression of the change of aberration due to the accommodative change of form of the core-lens; because the latter must probably be accompanied by corresponding variations of the higher derivatives of the indicial equation, and these in turn affect the peripheral total aberration.

During accommodation the asymmetry-values along the line of sight may vary, involving also a variation of the angle between the central ray and the line of sight, although this variation cannot be calculated. However, the central ray alone is responsible for the direction of the optical axis, and, so far as the monochromatic aberrations are concerned, its role is the same as that of the line of sight when the aberrations are left out of account. Obviously, therefore, in consequence of what has been said, there may be an *accommodative change of the direction of the optical axis*, such as is actually observed.

HELMHOLTZ's famous dictum, that the monochromatic aberrations of the eye are such as would not be tolerated in any good optical instrument, is sometimes construed to mean that the eye is a very badly constructed optical affair—which HELMHOLTZ never said and certainly did not mean. But another question that this statement raises is whether these aberrations are not serviceable and what is their purpose. First of all, it should be noted, as HELMHOLTZ pointed out, that a limit is imposed by diffraction to the physical sharpness of the image.

The phenomena of the diffraction of light present exceedingly complicated mathematical problems that cannot be solved very exactly except in special cases. The so-called FRAUNHOFER diffraction phenomena illustrate what is meant here. These are the effects that are observed when light goes through a round aperture on the supposition that the source of light and the screen are both infinitely distant. The latter condition is satisfied by adjusting an optical system beyond the aperture so that its focal plane acts as the screen plane. Under these circumstances, a luminous point is reproduced as a bright disc surrounded by alternately bright and dark rings. The border of the central bright disc, whose brightness fades towards the edge, is at the first minimum of light as represented by the smallest dark ring. Let φ denote the angular distance of this minimum from the axis; then

$$\sin \varphi = 1.22 \frac{\lambda}{2R},$$

where λ denotes the wave-length of the light and R denotes the radius

of the circular aperture. The angle is so small that it is sufficiently accurate to substitute the angle itself in place of its sine. Hence, if φ is expressed in minutes, the formula becomes

$$\varphi = \frac{1.22}{R} \cdot \frac{\lambda}{0.00058}.$$

Therefore, for light of wave-length 0.00058 mm, the formula is:

$$\varphi = \frac{1.22'}{R},$$

where R is expressed in millimetres. According to the above, the angular diameter of the projection of the bright central disc on the infinitely distant object-plane is equal to 2φ. However, owing to the way the brightness fades out towards the edge, two points do not have to be at this angular distance from each other to be seen separately. Half of it is supposed to be enough for the purpose; and so the angle φ is the conventional measure of the *resolving power* of the instrument. No matter how much the refraction of the rays may be improved in the optical instrument on the other side of the aperture, and no matter how much the image in the focal plane may be magnified, the limit of efficiency is determined by the resolving power, entirely independently of all these devices.

This calculation is applicable at once to all optical systems that are focused for infinity, provided the stop is in front of the first refracting surface or coincides with its contour. It applies, therefore, to the reduced eye, in which the pupil coincides with the refracting surface. In the human eye, where the pupil is beyond the cornea, all we can get is an approximate value of the resolving power, by substituting in the formula the radius of the entrance-pupil. Of course, λ denotes the wave-length of the light in air; and the indices of refraction of the aqueous humor and vitreous humor are not involved in any way, because the value of φ is based on a projection in the object-space. The question is properly treated by SCHUSTER[1] and GLEICHEN,[2] but both DRUDE[3] and POCKELS[4] base the calculation on the wave-lengths in the aqueous and vitreous humors. POCKELS uses light of wave-length 0.00057 mm in his calculation, and finds $2\varphi = \frac{1}{R} \cdot 144''$. The formula given by HELMHOLTZ on page 196 applies to the reduced eye. By substituting in this formula

[1] ARTHUR SCHUSTER, *An introduction to the theory of optics*. London 1904.
[2] A. GLEICHEN, *Einführung in die medizinische Optik*. Leipzig 1904.
[3] PAUL DRUDE, *Lehrbuch der Optik*. Leipzig 1906.
[4] A. WINKELMANN, *Handbuch der Physik*. 2. Aufl. 6. Bd. Leipzig. 1905. S. 1075.

$$2\varphi, \lambda, 2R \quad \text{for} \quad \frac{n\delta}{r}, nl, d, \text{ respectively,}$$

the formula given above will be obtained. The reason why HELMHOLTZ's formula contains the index of refraction, is because it does not give the angular size of the apparent bright area in the object-space, but the actual size of it on the retina; which is obtained by using the wave-length of the light in the refracting medium. It is true that HELMHOLTZ calculated the size of the disc on the retina by putting $\lambda = l$ instead of $\lambda = nl$; but in calculating the angular size in the object-space, he put $2\varphi = \frac{\delta}{r}$ instead of $2\varphi = \frac{n\delta}{r}$, as can be seen from his numbers; so that the final result of the calculation is correct. Evidently, it was just an oversight in setting down the figures. The results found by DRUDE and POCKELS can be obtained by substituting in HELMHOLTZ's formula the wave-length of the light in the refracting medium, without taking into account the fact that for the projection of the retinal image of the bright disc in the object-space the index of refraction has to be used again.

The general formula, therefore, has to be used for the *resolving power* of the eye. For an entrance-pupil 2 mm in diameter and for yellow light of wave-length 0.00058 mm the value of this angle is found to be 1.22′; and for blue-green light of wave-length 0.0005 mm, 1.05′; while a distant luminous point has twice this apparent size. The angular measure of the resolving power is directly proportional to the wave-length and inversely proportional to the diameter of the entrance-pupil. Hence, for an entrance-pupil of 3 mm it is 0.82′ or 0.7′ for the wave-lengths selected above. Owing to the distribution of light in the solar spectrum and still more in the spectra of artificial sources of light, the former number is the one to be used for a comparison with the visual acuity of the eye. If in this comparison account is taken of the fact that, on account of the conventional definition of the resolving power, the angular measure was rather too small than too large, the result is that *the limit of the visual capacity of the eye as imposed by diffraction, as far as it can be calculated, is attained by the visual acuity of the normal eye with a pupil of the size corresponding to a good illumination.*

Thus, we see again that the complicated aberrations of higher order that are present in the eye and the astonishingly large positive aberration within the optical zone do not impair the visual acuity with a pupil of the size mentioned. However, the dioptrics of the crystalline lens tells us that the fomer aberration is due entirely, and the latter in large measure, to the fact that the lens is composed of a hetero-

geneous medium. The great advantage of such a medium is the elevation of the total index that takes place in accommodation, which denotes a change of optical focusing that is out of proportion to the change of form, and unattainable with homogeneous media *coeteris paribus*. The monochromatic aberrations are the necessary evil for obtaining this advantage; and even if the convergence of the rays is not so good as it might be, the clearness of the image in good illumination is still above the limit of the capacity of the eye as imposed by the laws of diffraction. Hence, the monochromatic aberrations are a witness for the perfection of the eye, if what is meant by the perfection of an optical instrument is good convergence of rays to the degree that is needed to obtain the greatest useful sharpness of image; anything in excess of this being sacrificed in order to gain some other end.

VI. Ophthalmoscopy[1]

Ophthalmoscopy in the widest meaning of the word includes all the dioptrical methods of investigating the various parts of the eye; but in its narrower sense as meaning the observation of the fundus of the eye, ophthalmoscopy has always been a subject of the greatest interest for physiology ever since HELMHOLTZ's invention of the ophthalmoscope. It deserves to be ranked highest of all the dioptrical methods as being the most beneficial and having the greatest practical importance.

The *theory of the illumination of the eye* was expressed by HELMHOLTZ in his famous Proposition II (p.231) as follows: "If the pupil of the patient's eye is to appear luminous, the image of the source of light on his retina must either wholly or partly overlap the image of the observer's pupil." Now what is meant here is an indistinct image composed of blur circles; and yet the statement is not absolutely correct, because the projection of the light-source or of the pupil of the observer in the pupil of the patient's eye, as it is produced in certain ophthalmoscopic methods, is, strictly speaking, not a question of an image on the retina at all or even of an indistinct image. By introducing the idea of the so-called region of radiation of an optical system, a statement can be formulated which will include all cases. The fundus of the eye must be illuminated in order to emit light. Consequently, there is always an *illumination system*, which extends from

[1] ¶This article on "Ophthalmoscopy" was not included in the original appendices in the first volume of the third edition of *The Treatise on Physiological Optics*. It is taken from Professor GULLSTRAND's book entitled *Einführung in die Methoden der Dioptrik des Auges des Menschen* (Leipzig, 1911), pp. 55-90. (J. P. C. S.)

the source of light to the fundus of the patient's eye, and for which the contour of the source of light is to be regarded as the aperture stop. On the other hand, the *ophthalmoscopic system* proper, or the *observation system*, extends from the fundus of the patient's eye to the entrance-pupil of the observer's eye. Now evidently, *the condition of illumination of the eye is that part of the fundus of the patient's eye shall be at the same time in the radiation region of the illumination system and in that of the observation system.*

Moreover, a necessary *condition of ophthalmoscopy* is that in the observation system an optical image of the fundus of the patient's eye shall be cast at a distance for which the observer's eye can be accommodated. Experience alone can decide whether, with a certain disposition of apparatus, the fulfilment of these two fundamental conditions is also sufficient to enable details of the fundus to be seen. For some requirements, either as to sharpness and magnification of the image or as to brightness and extent of field, the above conditions are not enough, and another condition has to be imposed whereby the harmful light in the patient's eye due to regular or diffused reflection must not be permitted to enter the observation system. The methods in which this latter condition is likewise satisfied will be called the methods of ophthalmoscopy without reflex (or reflex-free ophthalmoscopy) as distinguished from all the other methods of simple ophthalmoscopy.

For *simple ophthalmoscopy with erect image*, in case the eyes of both patient and observer are emmetropic, nothing more is needed in the way of instruments than an object-stand and a candle. The source of light is adjusted to one side of the patient's eye and moved downwards until no direct light any longer falls on the iris of this eye. The glass plate used for the mirror is held by the operator in a vertical plane in front of the eye, and as near it as possible, and turned until the light reflected from it falls on the pupil. Placing the pupil of his corresponding eye on the line joining the centre of the pupil of the patient and the middle of the virtual image of the flame in the mirror, without using his accommodation, the investigator beholds the details of the fundus of the eye as a small, brightly enough illuminated spot. By changing the adjustment of the mirror, with corresponding movements of his own head, or by varying the direction in which the patient looks, the observer can cause this spot to traverse the fundus of the eye to suit himself, although of course it takes practice to do it properly. The essential disadvantage connected with this simplest method of ophthalmoscopy *with transparent mirror* is the small field of the illumination system, whereas its feeble brightness is of secondary importance. In the dim illumination of the room by the weak source

of light the pupils of both patient and observer are dilated, and the small amount of light reflected into the eye produces very little contraction of the pupil of the patient. However, the intensity of illumination on the retina of his eye in the central part of the luminous area is proportional to the square of the diameter of the pupil, because, for emmetropic focusing, the useful exit-pupil is equal to the entire exit-pupil, provided the least width of the source of light is not less than the diameter of the exit-pupil of the eye. Moreover, supposing both eyes are emmetropically adjusted, the pupil of the observation system in the intervening medium between the two eyes is the smaller of the entrance-pupils, and the solid angle subtended by the useful exit-pupil at the fovea of the patient's eye is proportional to the square of its diameter. Hence, unless the pupil of the observer's eye is smaller than that of the patient, the intensity of illumination in the fovea of the observer is proportional to the fourth power of the diameter of the pupil of the patient's eye. Besides, the reflex from the cornea is so feeble that the blur circle which it makes in the observer's eye does not interfere with seeing, because the details of the fundus can be seen through it, so to speak. The result is that for physiological relations the brightness is generally completely sufficient, provided in investigating the fovea the image of the source of light does not fall on the centre of this region; which usually causes such a contraction of the patient's pupil that the apparent brightness of the fundus becomes too small in comparison with the brightness of the blur circle due to the corneal reflex. Hence, ordinarily, nothing is gained by increasing the specific brightness of the virtual image of the light-source in the mirror, because the brightness of the image reflected from the cornea increases in the same ratio. HELMHOLTZ advised for this purpose making the light fall more obliquely on the plate, or using several plates, but this method was not a practical success. With a large angle of incidence it is not so easy to protect the iris of the patient's eye from direct light, and light coming directly in this eye has an unfavourable effect on the size of the pupil. Besides, when the iris is illuminated directly, the brightness needed to recognize detail in the fundus is different from what it has to be otherwise, and this interferes with the observation. On the other hand, when more plates are used, the diffusely reflected light gets to be of more importance; for it is not an easy matter to keep the six glass surfaces of HELMHOLTZ's ophthalmoscope clean and free from dust. The field of the illumination system increases with the solid angle subtended by the source of light in the pupil of the patient's eye, and hence it is larger with a larger source of light, and when it is placed as near the mirror as possible. But it should be noted that this involves also an increase of brightness of the

blur circle in the observer's eye due to the reflex image in the cornea, and it will not be so easy to see through it. This is all the more true when an image of the source of light is projected in the patient's eye with a convex lens, as HELMHOLTZ proposed.

For the reasons above mentioned, the *opaque mirror with a peep-hole* is considered better for direct ophthalmoscopy by the method of the erect image. Since the central hole in the mirror has the same effect in the illumination system as a screen casting a shadow, whereas in the observation system it acts like a pupil or window, the connections here are rather complicated. However, on the one hand, the pupil of the observer's eye through which very little light enters, and which is enlarged by closing the other eye, is usually big enough not to be the pupil of the observation system; and, on the other hand, the window-effect of the aperture in the mirror may be disregarded, because, as the method is employed in practice, this effect is compensated by continual movements of mirror, head and eye. Practically, therefore, the hole in the mirror and the pupil of the patient's eye do not enter into the calculation except as apertures in the observation system. In order to calculate the field of the observation system, suppose that CD in Fig. 147 represents the diameter of the hole in the mirror, and AB the

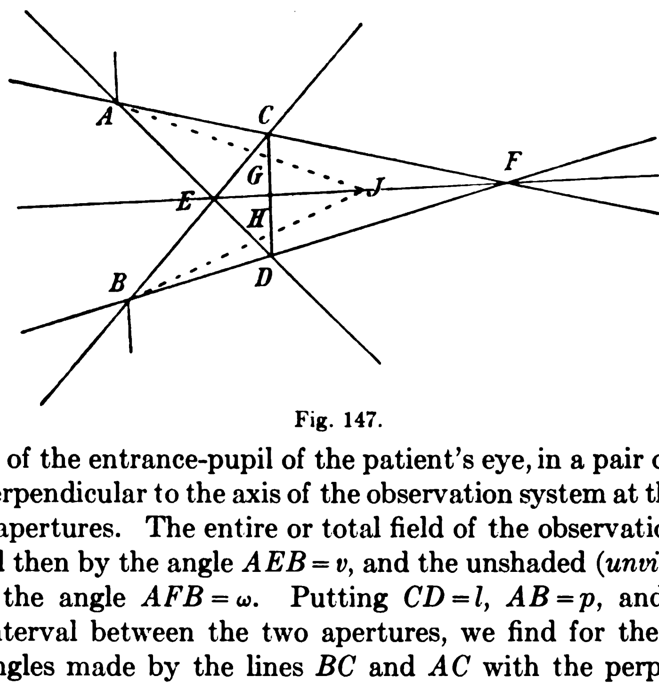

Fig. 147.

diameter of the entrance-pupil of the patient's eye, in a pair of parallel planes perpendicular to the axis of the observation system at the centres of these apertures. The entire or total field of the observation system is defined then by the angle $AEB = v$, and the unshaded (*unvignettierte*) field by the angle $AFB = \omega$. Putting $CD = l$, $AB = p$, and d equal to the interval between the two apertures, we find for the tangents of the angles made by the lines BC and AC with the perpendicular drawn from C to AB the following expressions:

$$\tan\frac{v}{2} = \frac{p+l}{2d}, \quad \tan\frac{\omega}{2} = \frac{p-l}{2d}.$$

However, these angles are so small that they can be substituted here in place of their tangents; consequently,

$$v=\frac{p+l}{d}, \qquad \omega=\frac{p-l}{d}.$$

With the degree of accuracy attainable by the laws of the first order, there would be no sense in using here a more exact schematic eye than the reduced eye; and hence AB is its own image after refraction in the dioptric system of the eye, and all we have to do is to determine the points conjugate to E and F. Suppose that the light proceeds from F towards AB, and let the distances of E and F from AB be denoted by e and f; so that

$$-e=\frac{pd}{p+l}, \qquad -f=\frac{pd}{p-l}.$$

Let D denote the refracting power of the optical system of the reduced eye whose principal points coincide at the middle of AB; consequently, the reduced distances of the points conjugate to E and F are found from the equations:

$$\frac{1}{e'}=D+\frac{1}{e}, \qquad \frac{1}{f'}=D+\frac{1}{f}.$$

Now the lines drawn from these points to the ends of AB determine the complete field on the retina and the unshaded (unvignetted) field. Hence, if the angular diameters of these fields are denoted by V and Ω, and if the reduced length of the eye is denoted by b, we may write the following expressions for the *extent of the field of the observation system*:

$$V=p\left(1-\frac{b}{e'}\right)=p(1-bD)+b.\frac{p+l}{d},$$

$$\Omega=p\left(1-\frac{b}{f'}\right)=p(1-bD)+b.\frac{p-l}{d}.$$

Like all similar formulae, these equations may be used with any system of units, provided the reciprocal of the refracting power is measured in terms of the same unit of length as the other linear magnitudes. Thus if D is given in dioptries, the unit of length is the metre.

The point F lies always in front of the anterior focal point of the eye or beyond the pupil, whereas the point E may be on either side of the focal point or at it. Thus, while f is invariably positive, e may be finite or infinite, positive or negative. Consequently, the formulae show that, as long as $f'>b$, the unshaded field is smaller when the axis of the eye is longer. When we investigate the illumination

system, we shall find that f is always greater than b Moreover, it appears that the complete field in all eyes is equal to the size of the pupil, provided the point E is at the anterior focal point of the eye. When E is nearer the eye than this, the complete field is larger, and *vice versa*. In either case the difference between the diameters of field and pupil is proportional to the length of the axis of the eye. If A denotes the refraction of the eye, then

$$1-bD=bA\,;$$

consequently, $$\frac{V}{b}=v+pA, \qquad \frac{\Omega}{b}=\omega+pA,$$

where the expressions on the left-hand sides of these equations are the principal point angles corresponding to the field-diameters. That is, they are the measures of the *angular diameters of the field.* Now, obviously from what has been stated, these angles are always positive, although for an hypermetropic eye it may happen that $l>p$. Therefore, other things being equal, the angular diameter of the field invariably increases with increase of refraction of the eye, and consequently is greater in hypermetropia than in myopia. The value of v or of ω for the emmetropic eye may be taken as the criterion, and will be designated therefore as the *characteristic angular diameter of the field.* It can be found exactly by using the formulae above containing the trigonometrical tangents. After all, however, the results of this investigation of the ophthalmoscopic field are to be regarded only as approximate, since they have been obtained by using simply the laws of the first order. Moreover, they have been derived on the assumption that the window-effect of the pupil of the observer's eye is compensated by movements of the head. By combination of these movements with displacements of the mirror and corresponding rotations of the observer's eye, a displacement of the field is produced on the fundus of the eye which amounts to a considerable addition to the region that can be seen. Now in order for these movements to have the most effect, the three apertures of the observation system should be as near together as possible. Moreover, not only the unshaded field but the entire field becomes larger when the interval between the mirror and the patient's eye is diminished, as appears from the formulae above. Hence, evidently, in the investigation with erect image, a good ophthalmoscope should be made in such a way as to enable the observer to get so close to the patient's eye that their foreheads come in contact during the investigation. For the same reason, as was pointed out above, the corresponding eyes of observer and patient should always be used in ophthalmoscopy with erect image, that is, either the right eyes of both individuals or the left eyes.

In order to produce an image of the fundus, for any arbitrary state of refraction of the patient's eye, at a place where it can be seen distinctly by the eye of any observer who desires to investigate it, it must be possible to insert conveniently lenses of many different powers behind the hole in the mirror, as they may be needed. But when the mirror is very close to the patient's eye it will be inclined to the axis of the observation system, and an oblique position of the lens usually involves astigmatism and a consequent impairment of the image. In order to avoid this, we must always count on a finite distance between the lens and the hole in the mirror, and therefore the lens-aperture must have a diameter big enough not to involve any so-called window-action. Suppose D_l denotes the refracting power of the interposed lens, which may be considered here as infinitely thin; and m denotes the reduced distance of the first principal point of the observer's eye from the optical centre of the lens (designated by M). The observer's eye is supposed to be accommodated for distinct vision at a certain point O on the axis of the observation system. The refracting power of his eye for this focusing may be denoted here by D. Let $a=\frac{1}{A}$ denote the reduced distance of the point O from the first principal point of the observer's eye; and, moreover, let $\frac{1}{A_m}$ denote the reduced distance of this same point O from the point M. Now at the same time the patient's eye is accommodated to see distinctly a point O', where O, O' designate, therefore, a pair of conjugate points on the axis of the interposed lens. The symbols m', D', A', A'_m, etc., have the same meanings with respect to the patient's eye as the symbols m, D, A, A_m, etc., respectively, have with respect to the observer's eye.

The refracting power of the lens is equal to $A_m+A'_m$. It may be assumed here with sufficient accuracy that the anterior focal point of the observer's eye coincides with the optical centre of the lens, and hence $A_m=L$, where L denotes the reduced focal point convergence of the bundle of rays incident on the observer's eye. Hence *the refracting power of the lens of the observation system* can be put equal to the sum (A'_m+L).

In order to study *the magnifying power in ophthalmoscopy by the method of the erect image*, we have the following general formula (as on p. 364):

$$-\frac{\omega_m}{a_0}=D_0(1+\Delta_m A_m),$$

where D_0 denotes the refracting power of the compound optical system composed of the lens and the patient's eye, and Δ_m denotes the

distance of the point M from the anterior focal point of this system. And if in this formula the values

$$D_0=(1-m_0'D_l), \qquad \Delta_m=\frac{m'\,D'}{D_0}-\frac{1}{D_0}=\frac{m_0'D'}{D_0}$$

are substituted for D_0 and Δ_m, we obtain:

$$-\frac{\omega_m}{a_0}=D_0+m_0'D'A_m=D'(1-m_0'A'_m).$$

Since for emmetropic focusing of the observer's eye the angle ω_m is equal to the principal point angle, this formula gives the *absolute magnifying power*, which, on the supposition of such focusing, is equal to D_0. Since m'_0 is always positive, the magnifying power is greater when the patient's eye is myopic than when it is hypermetropic; and, generally, the magnifying power diminishes as the refraction of the patient's eye becomes greater. Estimating the refracting power of the optical system of the eye roughly at 60 dptr, and assuming m'_0 equal to 1 cm, we may say, in the conventional mode of reckoning, that in emmetropia the magnifying power is 15-fold; and N% greater or smaller in myopia or hypermetropia, respectively, where N denotes the numerical value of A'_m. Thus, in the cases that occur in practice the magnification number varies between 13 and 18.

On the above assumption of the coincidence of the optical centre of the lens and the anterior focal point of the observer's eye, ω_m denotes the focal point angle. Hence, with relaxed accommodation the size of the retinal image in the observer's eye is independent of the axial length of the eye; and

$$-\frac{D'}{D}(1-m_0'A'_m)$$

denotes the *magnification-ratio* for the imagery of the retina of the patient's eye on the retina of the observer's eye.

Since

$$\omega_f'=a_0D', \qquad \frac{\omega_m'}{\omega_f'}=1-m_0'A'_m$$

the general expression for the magnifying power may be written in the form

$$\omega_m=-\omega_m';$$

where the negative sign is due to the fact that the two eyes are looking in opposite directions, and because no change of sign is to be made in the equations defining the angles ω.

The following two expressions for the individual magnifying power are obtained by using the principal point angle:

$$-\frac{\omega_h}{a_0}=D'(1-m_0'A'_m)(1+mA)=B'\cdot\frac{1+m\,A}{1+m'A'};$$

the latter of which shows in the simplest way the effect of the accommodation of the two eyes and the action of any astigmatism that may be present in the patient's eye. Evidently, the magnifying power is reduced by accommodation on the part of the observer and increased by accommodation on the part of the patient; although the variation of the refracting power of the lens in the observation system needed to compensate accommodation is in the same direction in both cases. Moreover, for investigating the fundus of an astigmatic eye the refracting power of the lens is so chosen that by using accommodation the imageable lines of both systems of imagery can be seen distinctly in succession. Hence, both A and A' must have different values for the two principal sections; for the condition that (A'_m+A_m) shall remain constant, is satisfied by accommodation on the part of the observer. Imposing this condition, and denoting the magnification-ratio $-\frac{\omega_h}{\omega'_h}$ by K_1 or by K_2 according as A' is equal to A'_1 or A'_2, respectively, a fairly complicated expression is obtained for the distortion due to astigmatism, as measured by the ratio $\frac{K_1}{K_2}$; in which the distortion depends not only on the astigmatism but also on the refraction states of the two eyes. However, in actual practice the numerical values of expressions of the type mA are so small as compared with unity that, where simply the imagery-laws of the first order are used, it is justifiable to neglect the higher powers of these products. Hence the following *approximate formula of the astigmatic distortion*, which is obtained in this way, is sufficiently accurate for practical purposes:

$$\frac{K_1}{K_2}=1-(m'+m)(A'_1-A'_2)\cdot$$

Since a smaller value of the refraction of the eye means a higher refracting power for it, the magnifying power is greater in that one of the principal sections, which is the more highly refracting. In case the observer has not the necessary power of accommodation, then instead of one of the images, or instead of both of them if he selects an intermediate focus, there will be an optical projection; for which the image of the centre of projection in the medium between the first principal point of the observer's eye and the lens must be substituted instead of this principal point. Accordingly, m here denotes the

distance of this image of the centre of projection from the optical centre of the lens used; and A denotes the convergence of the bundle of object-rays as measured at this same point. By the same process as before the same formula is obtained; and hence all that is necessary is to substitute the given value in place of m in the approximate formula above. However, it should be observed that in this approximation the difference between the distance of the centre of projection and that of its image from the optical centre of the lens is neglected in the product obtained by multiplying by the refraction. The hole in the mirror for all ordinary sizes that are used acts as the centre of projection; and so its distance from the lens, counted negatively, has to be substituted for m, and $m'+m=d$. The distortion is found to be less than in the ordinary investigation.

The astigmatic distortion thus found applies only to magnification of detail, with which the movement of the observer's eye has nothing to do. Usually, however, the distortion is estimated by the apparent form of the papilla, for which vision with the mobile eye is required. In this case the centre of rotation of the eye is to be taken as centre of projection; which amounts roughly to increasing the magnitude $(m'+m)$ in the approximate formula by about 1 cm, in order to obtain an expression for the apparent distortion.

The intensity of the light in the observation system is measured by the solid angle subtended by the exit-pupil of this system; the intensity of illumination on the retina of the observer's eye being equal to this intensity multiplied by the specific intensity of the light of the illuminated retina of the patient's eye. Having constructed the image of the hole in the mirror as cast by the optical system of the patient's eye, from the extremities of a diameter of this image draw a pair of lines to the opposite ends of the parallel diameter of the pupil, intersecting on the axis at the point defined by f' on page 447. Now since $f'>b$, this point lies behind the retina, and hence the image of the hole in the mirror subtends at the retina a smaller solid angle than the pupil. Moreover, as in ophthalmoscopy the retina of the observer's eye is imaged on that of the patient's eye, the image of the hole in the mirror is the entrance-pupil.

Using the reduced eye as being accurate enough for this investigation, we find for the solid angle subtended by the exit-pupil the following value:

$$\frac{\pi p_n^2 B^2}{4n^2},$$

where n denotes the index of refraction of the vitreous humor, and p_n is *the diameter of the useful pupil of the eye*, the latter magnitude being

defined as follows: The useful pupil is the projection on the pupil of the image of the hole in the mirror, as formed by the lens of the observation system, from the point, on which the observer's eye is focused, as the centre of projection; and its diameter is given by the equation:

$$p_n : \frac{1}{A} = \frac{l}{1-(m'-d)D_l} : \left(\frac{1}{A_m} + \frac{m'-d}{1-(m'-d)D_l}\right) = l : \frac{1-(m'-d)A'_m}{A_m}.$$

It follows that

$$p_n = l \cdot \frac{1-mA_m}{1-(m'-d)A'_m};$$

and hence the useful pupil of the eye is greatest when the observer is hypermetropic and the patient myopic. Substituting the value

$$B = D \cdot \frac{1-m_0A_m}{1-mA_m}$$

in the expression for the solid angle subtended by the exit-pupil, we may write this expression as follows:

$$\frac{\pi l^2 D^2}{4n^2}\left\{\frac{1-m_0A_m}{1-(m'-d)A'_m}\right\}^2.$$

Accordingly, in case the optical centre of the lens is at the anterior focal point of the observer's eye, the intensity of illumination on the retina of this eye depends simply on the refraction of the patient's eye.

In *the illumination system* let us try, first, to estimate the mutual influence on each other of the hole in the mirror and the pupil of the patient's eye, apart from the source of light. As was shown above, the complete shadow between the points conjugate to F and E in Fig. 147 due to the hole in the mirror extends into the medium beyond the optical system of the patient's eye. Hence, the first point must necessarily be beyond the retina, if the central part of the field is to be illuminated. It is true this is not absolutely necessary, because it is possible that the observation system is not centered. Since the dark spot on the fundus of the eye made by the shadow is always central around the axis of the observation system, it is extremely disturbing in its effect. Therefore, perhaps not so much in the actual practice of ophthalmoscopy, but certainly for ophthalmoscopy with a centered observation system, the condition $f' > b$, as formulated above, has to be satisfied. Now this is equivalent to saying that the hole in the mirror shall be the pupil of the observation system; and therefore it defines also the greatest diameter that this opening can have for a centered observation system. In order to obtain a practically useful expression, all that is necessary is to project the hole on the pupil from the point

on which the eye is focused. Thus, for *the condition as to the size of the hole in the mirror*, the following formula is obtained:

$$l<p(1+dA).$$

If this condition is not satisfied, then, as stated above, the observation system will have to be decentered. This is accomplished by decentering the mirror until it no longer acts as a perforated mirror, but merely transmits to the eye the light reflected from one side of the hole. This method of using the perforated mirror comes therefore under the head of ophthalmoscopy with an unperforated mirror, and will be considered when we come to treat of that subject.

Supposing, however, the above condition is satisfied, the values of ω and Ω for *the extent of the field as seen through the hole in the mirror and the pupil without being shaded off*, are obtained in the same way as in the observation system. But beyond these limits, except for such shading off of the field as is due to the position and size of the actual source of light, the shading off of the field occurs in the reverse sense, with brightness gradually increasing out towards the periphery.

There does not have to be any other shading than this in the field of the illumination system. Any window-action of the source of light itself can be obviated by casting an image of the source right on the fundus of the eye. In this case the shading off of the field in the illumination system and in the observation system being in opposite sense, the brightness of the ophthalmoscopic image is very uniform, provided the specific brightness of the source of light is itself uniform. Theoretically, by projecting an image of the source of light in the pupil or near by it, a larger field can be obtained without any shading besides. But, practically, nothing is gained in this way in simple ophthalmoscopy, because the investigation is hampered by greatly augmented corneal reflex, apart from the complication of the instrumental arrangements. In order to enlarge the field, practically, therefore, there is no other choice except to produce an image of the light at some other place; which must be done at the cost of further shading off of the field. It would lead too far to attempt here a complete investigation of this matter; for since the contour of the source of light is to be considered as a stop, there are two apertures and a central screen that are involved, and for such a case the general formulae are rather complicated. Here, therefore, it will be sufficient to examine merely the questions that are of practical importance. Let the diameter of the image of the source of light in the mirror be denoted by q, and let the distance of this image from the mirror be denoted by c, this latter magnitude being reckoned as positive when the mirror is between the pupil of the eye and the image in question.

In the first place, with regard to the mutual action on each other of the image of the source of light and the hole in the mirror, the condition will be imposed, as before, that there is no complete shadow on the fundus of the eye. This condition amounts to saying that the angle subtended by the image of the source of light at the point for which the eye is focused is greater than that subtended at this same point by the hole in the mirror. Let the symbol $|K|$ be employed to denote the numerical value of any magnitude denoted by K; then we may write:

$$q : \left|\frac{1}{A}+c+d\right| > l : \left|\frac{1}{A}+d\right|,$$

that is,

$$q > l \cdot \left|\frac{1+(c+d)A}{1+dA}\right| .$$

Neglecting powers of dA higher than the first and magnitudes of similar order of smallness as compared with unity, we derive the following approximate formula:

$$q > l \cdot |1+cA| ;$$

which gives, therefore, accurately enough *the condition as to the size of the image of the source of light.* Unless this condition is satisfied, there will be again a dark spot in the centre of the ophthalmoscopic field. But it is different from that described above in this way, namely, that with rotations of the mirror around an axis passing through the centre of the hole, without decentering the observation system, this spot undergoes displacements in the field of this system, and consequently is not so disturbing as the other spot was. However, since it is liable to cause confusion, it is best to satisfy the condition above.

The characteristic angular size of the field that is not shaded off by the mutual action of the hole in the mirror and the source of light is found by the method used above, and is equal to $\dfrac{q-l}{c}$. According as the numerical value of this magnitude does or does not exceed ω, the extent of the central unshaded field in the emmetropic eye is measured by the angle ω or by the other angle, respectively.

The entire *angular measure of the field of the illumination system* is obtained in the same way as that of the observation system. If the characteristic angular size of the field is denoted by w, and if the linear dimension of the field on the retina is denoted by W, then

$$w = \frac{p+q}{c+d}, \qquad BW = w + pA.$$

Using corresponding symbols λ, Λ to denote the extent of the outer

borders of the field that is not shaded off by the mutual action of the source of light and the pupil, we obtain corresponding formulæ, namely:

$$\lambda = \frac{p-q}{c+d}, \qquad B\Lambda = \lambda + pA.$$

The *intensity of the light of the illumination system* depends on the area P_n of the useful pupil, the intensity of illumination on the retina being equal to

$$eP_nB^2,$$

where e denotes the specific intensity of the source of light. In order to find the useful pupil, the image of the source of light and the hole in the mirror must be projected on the pupil from the point of fixation of the eye as centre of projection. According as the projection of the image of the source of light is smaller or greater than the pupil, the area of the projection of the hole in the mirror must be subtracted from the area of the former or from that of the latter, respectively. Accordingly, P_n is the same as the smaller of the two values given by the following expressions;

$$P'_n = \frac{\pi}{4}\left\{p^2 - \frac{l^2}{(1+dA)^2}\right\}, \qquad P''_n = \frac{\pi}{4}\left\{\frac{q^2}{[1+(c+d)A]^2} - \frac{l^2}{(1+dA)^2}\right\}.$$

By auxiliary optical appliances the position and size of the image of the source of light in the mirror can be arbitrarily modified. But as one of the prime factors in the practical employment of the simple ophthalmoscope is the unhampered movement of the mirror, any combination of it with lenses, which (entirely aside from limitations of space) requires complicated manipulation for changing the direction of the incident light, is ruled out in advance. It is only in certain electrical forms of ophthalmoscope when the lamp and mirror are rigidly connected that such combinations as are referred to here are really practical. Generally speaking, therefore, *the form of the mirror* is the only optical method of influencing the image of the source of light. If it is a question of getting the largest possible field with the best possible brightness, evidently the lamp and mirror should be as near together as possible. Now the ratio $\frac{q}{c} = K'$ depends merely on the size of the source of light and its distance, and not on the form of the mirror, since, in both the tangential and the sagittal imagery, the principal points coincide at the surface of the mirror. Hence, starting with the value of K' that is practicable, we must see how the field and the intensity of the light are altered when c is varied. The simplest method to use for this purpose is that of differentiation; although the

process will be illustrated without this by the aid of Fig. 147. Treating K' as constant, and hence $q \,.\, dc = c \,.\, dq$, we find by differentiation:

$$\frac{dw}{dc} = \frac{K'd - p}{(c+d)^2}, \qquad \frac{d\lambda}{dc} = -\frac{K'd + p}{(c+d)^2};$$

where the value of K' is negative when c is negative, because the sign of q is always regarded as positive. As to the size of the entire field, c and w always have the same sign, as the image of the light source must never be between mirror and pupil on account of the prevalence of light reflected from the cornea. Hence, according as the value of the differential coefficient is positive or negative, the numerical values of c and w either both increase or decrease together, or as one increases the other decreases, respectively. But when c is negative, and therefore K' is negative too, the differential coefficient is negative; and when c and K' are positive, the sign of the differential coefficient will be the same as that of the magnitude $(K'd - p)$. Since $|K'|$ and $\frac{p}{d}$ denote the angular diameters of the source of light and the pupil, respectively, as measured at the centre of the hole in the mirror, evidently, starting with a very concave mirror which is supposed to get flatter and flatter until it is plane and then more and more convex, the entire field of illumination will continually diminish, provided the angle subtended by the source of light at the centre of the hole in the mirror is smaller than the angle subtended by the pupil; whereas in the opposite case the field is least when c becomes infinite, and begins to increase again when the image of the source of light is virtual and comes nearer to the mirror. With regard to the part of the field that is not shaded off by the source of light and the pupil, the only cases of any practical importance are those for which $(q-p)$ is positive and Λ and λ have the same sign. Under these circumstances, the signs of λ and c are opposite, so that, according as the differential coefficient is positive or negative, the numerical values of λ and c vary in opposite directions or in the same direction, respectively. When c is positive, the differential coefficient is negative; and when c is negative, the sign of the differential coefficient is opposite to that of $(K'd+p)$. The result is, that, provided the source of light subtends a greater angle at the centre of the hole in the mirror than the pupil, the part of the field that is not shaded off by the source of light and the pupil gets smaller and smaller when the form of the mirror is varied as described above; whereas in the opposite case, in the region $q > p$, $\lambda\Lambda > 0$, the field increases when the image of the source of light is real, reaches its maximum value when c is infinite, and thereafter diminishes when the image becomes virtual. Both when the image of the source of light is real and when it is virtual, there is a

certain point, when the image approaches the mirror, for which the entire field is shaded by the mutual action of the source of light and the pupil on each other. Thus, putting $\Lambda = 0$, we obtain:

$$c = \frac{p(1+dA)}{K'-pA},$$

in which the value of K' is negative when c is negative. For emmetropia these points coincide with the points where the image of the source of light has been reduced, by bringing it nearer to the mirror, until it has the same size as the pupil; but for cases of decided ametropia there may be considerable departures from this rule, as the formula indicates.

The effect of the form of the mirror on the field of illumination may be illustrated by the aid of Fig. 147, as was mentioned above. For instance, suppose AB represents a real image of the source of light, and CD represents the pupil; the light proceeding from right to left, so that the point J in the diagram may be regarded as the centre of the hole in the mirror. Then by varying the form of the mirror, AB may be displaced farther to the left, without, however, altering the angle AJB. In this case it is plain that the angle $AEB = w$ decreases, and the angle $AFB = \lambda$ increases; whereas if the point F were on the other side of J, the angle λ would decrease also when the interval between the mirror and the image increased. In the latter case, $-K' > \frac{p}{d}$; but the opposite case is the one shown in the figure. These conclusions apply to the case when the image is real. When the image of the source of light is virtual, the point J will be between AB and CD, and everything happens in this case in similar fashion.

The conclusions can be summarized as follows: For $|K'| > \frac{p}{d}$, a real image of the source of light, for every reason, is better for the size of the field than an infinitely distant image, and the latter is better than a virtual image. On the other hand, for $|K'| < \frac{p}{d}$, both a real image and a virtual image are better for the total size of the field, but worse for the portion that is not shaded off by the pupil and the source of light, than an infinitely distant image. Since in the practice of simple ophthalmoscopy by the method of the erect image the technical difficulties are augmented by reduction of the size of the pupil, and since there is no trouble about realizing the condition $|K'| > \frac{p}{d}$ with small pupil, usually *the concave mirror is more advantageous* when the

size of the field of the illumination system is kept in view. However, the curvature of the mirror must not be increased too much, because the disturbing effect of the corneal reflex will interfere in this case. The more of the light coming from the mirror that is intercepted by the pupil, the greater will be the portion of the flux of light, as determined by the dimensions of the source and of the mirror, that succeeds in entering the eye. Hence the quantity of light penetrating the eye increases with increase of curvature of the concave mirror, and this causes the pupil to contract; all the more because in the larger field the macula region is more exposed to the illumination. Now since the total amount of light falling on the retina passes through the pupil, the proportion of the intensity of illumination on the fundus of the eye to the density of the light on the cornea and lens is never favourable when the curvature of the mirror is increased; that is, the diffusely reflected light coming from these media is always a disturbance. The result is that for general use the radius of the mirror should not be less than about 15 cm. Of course, this limit is merely a fact of experience in agreement with the size of the pupil. If the pupil is dilated—as is usually the case in ophthalmological experiments in physiological laboratories and clinics—a more curved mirror may be used without disadvantage. But in practice the curvature mentioned above is found to be rather too high. It should be remarked too, that, owing to the astigmatism produced by reflection at oblique incidence, the size of the field has to be calculated separately in tangential and sagittal directions.

In working with a mirror of this kind, with the ordinary sources of light, usually the field obtained is bright enough and of sufficient size for finding pathological variations, as long as the macula region is not under investigation. The corneal reflex, which for the other parts of the fundus does not hinder seeing, being more central here, is essentially more disturbing; because also the pupil is more contracted by reason of the stronger illumination of the centre of the retina. In many instances, therefore, it is not possible to investigate this region with this mirror without artificial dilatation of the pupil. Frequently the purpose is achieved by not attempting to have a big field and using a *plane mirror* and a small source of light.

The intensity of illumination in the central part of the field, as shown by the formulae given above, does not depend on the form of the mirror, provided the source of light is not so small that $P''_n < P'_n$. But if the latter is the case, the intensity for an emmetropic eye has precisely the same value for either a convex or a concave mirror of same radius. Hence, the common notion that a convex mirror involves weak intensity of light is true only with respect to the quantity of light that enters the eye.

The size of the mirror should be such that the field of the illumination system can be brought by a single rotation around an axis through the hole in the mirror to the farthest edge of the field of the observation system, without the mirror's intercepting the flow of light. For an emmetropic eye this means that the surface of the mirror should extend above and below the hole for a distance equal to the diameter of the pupil; rather more than this towards the temporal side, and somewhat less on the nasal side. With a round mirror therefore a diameter of 20 mm is ample, and should never be exceeded. A smaller diameter would be better, because less superfluous light would fall on the iris. *The size of the hole* with reference to the condition derived above should be made as small as possible with respect to the practical applicability of the mirror, and under no circumstances should be more than 2 mm in diameter. The chief thing is to reduce as much as possible the detrimental space between the illumination system and the observation system. Hence the perforated hole in the glass mirror should be as thin as possible and cylindrical. A dull black surface on the inside of the hole is a matter of very great importance. Even in the best ophthalmoscopes a little reflex image of the source of light originates here, whose blur circle tends to obscure the ohpthalmoscopic image. When the black surface is poor, the light from the blur circle may be strong enough to hinder the observation in case of a small pupil. They have tried to avoid this trouble by scratching off the silver from the mirror instead of boring a hole through the glass; and in fact this method reduces the detrimental space to a minimum. But a mirror of this sort has to be kept scrupulously clean on both sides, which takes a lot of time to use it for one thing; and, besides, there are technical difficulties as to its rear side. Quite recently PRIESTLEY SMITH[1] has proposed cementing another glass on the rear side, which makes it easier to keep clean. How such a mirror works with the method of erect image, has not yet been ascertained by practical experience.

Simple ophthalmoscopy with unperforated mirror is of no value as an independent method. Thus since the mirror must be at one side, its edge past which the observer has to look and the visible portion of the edge of the pupil act like windows, by which there is a one-sided shading off of the field. Besides, it amounts to having to look always eccentrically through the pupil where the convergence of the rays is not so good as at the centre. But this method is used all the time in cases where in the ordinary investigation the pupil is so contracted that the condition as to the size of the hole in the mirror is not satisfied. Hence, as has been stated above, the mirror must be decentered and the per-

[1] A new simple ophthalmoscope. *Ophthalmic Rev.*, XXIX. 1910. p. 33.

forated mirror employed, therefore, as if it were not perforated. One advantage of it is that the edge of the mirror may be conveniently adjusted in any direction of the line of sight. This method is very frequently used for investigating the macula region with contracted pupil. We can tell when we are using it, because the movements of the mirror that are needed for it are automatic, on account of the fact that the macula region can be seen in an eccentric part of the blur circle of the hole in the mirror, but not in the central part.

Simple Ophthalmoscopy by the Method of the Inverted Image

If an image of the pupil of the observer is projected by a convex lens in the pupil of the patient's eye, an inverted image of the fundus of the latter eye will be formed between the lens and the observer's eye; and if the fundus is illuminated, the observer, by using his accommodation or with the aid of a suitable lens, can see the image distinctly. The advantage which this method has over that of the erect image is in the possibility of getting a larger field. As this means that more light must be sent in the eye, the unsilvered mirror used with ordinary source of light is not to be considered. For practical reasons, we may likewise leave out of consideration the case of illumination with unperforated, opaque mirror, in which both mirror and light are placed to one side. On the other hand, both of these methods of illumination may be advantageously used in reflex-free ophthalmoscopy, and therefore they will be discussed presently. With illumination by *perforated mirror* there are four apertures in the *observation system*, namely, the lens of the ophthalmoscope, the hole in the mirror and the two pupils. The window-action of the pupil of the observer's eye is to all intents and purposes practically neutralized by the continuous movements of his eye and of the mirror. Hence, this aperture may be left out of account in determining the field, and the hole in the mirror is to be regarded therefore as being the entrance-pupil of the observer's eye. Now if an image of this latter aperture is projected by the lens of the ophthalmoscope in the entrance-pupil of the patient's eye, evidently a part of this pupil will not be covered by the optical image. For as no light leaves the hole in the mirror, its optical image is that of an opaque screen. Therefore, either the image of this hole must be smaller than the pupil of the patient or else it must be decentered with respect to it. The result is that the hole in the mirror must act as a pupil of the observation system. In those ophthalmoscopic methods in which an image of the entrance-pupil of the observer's eye is formed in that of

the patient's eye, the symbol K was used to denote the magnification-ratio for this imagery; and, therefore, let the same notation be used here for the imagery of the hole in the mirror. Since the lens of the ophthalmoscope produces an inverted image of this hole, K is negative, and the diameter of the image is $-Kl$, where as above l denotes the diameter of the hole. Moreover, let D_0 denote the refracting power of the lens of the ophthalmoscope and L_0 its diameter; then since the distance between the lens and the entrance-pupil of the patient's eye is equal to $\frac{1-K}{D_0}$, the *angular dimensions of the characteristic field* will be given as follows:

$$\tan\frac{v}{2}=\frac{D_0(L_0-Kl)}{2(1-K)}, \qquad \tan\frac{\omega}{2}=\frac{D_0(L_0+Kl)}{2(1-K)}.$$

However, Kl being very small as compared with L_0, the shading off of the field as represented by the difference in the values of these two angles is practically of no importance at all; and hence the *aperture angle* ω_0 for the observation system may be defined by the equation:

$$\tan\frac{\omega_0}{2}=\frac{D_0L_0}{2(1-K)};$$

the aperture itself being given by the number

$$\frac{D_0L_0}{1-K}.$$

In ophthalmoscopy by the method of the inverted image, the mirror does not have to be inclined to the axis of the correction lens inserted in front of the observer's eye, but the lens is placed close to the back of the mirror. Hence, in calculating the *magnifying power* (p. 362) in the observation system it is sufficiently accurate to consider the hole as being where the lens is. In practical work the entire field cannot always be utilized. Similarly, the hole in the mirror is not invariably imaged in the pupil of the patient's eye; so that the magnifying power will be calculated here for any adjustment whatever. The point in the observer's eye or in the patient's eye that is designated by M is to be regarded as the centre of the hole in the mirror or of the image of this hole made by the lens, respectively. Accordingly,

$$\omega_m=-K\omega_m';$$

and hence substituting the principal point angle, just as was done in case of the erect image, we find:

$$-\frac{\omega_h}{a_0}=KB'.\frac{1+mA}{1+m'A'}=KD'(1-m_0'A'_m)(1+mA).$$

The *absolute magnifying power*, therefore, for $m' = 0$, as is approximately realized in the ordinary way of working, is equal to KB'; being inversely proportional to the axial length of the patient's eye. Generally, the magnifying power in this case is independent of the accommodation of the observer's eye, provided the change of focusing in the observation system which it causes is not compensated by change of the state of accommodation of the patient's eye, but by a change of the correction lens. Varying the distance m' on the same assumption, the distance of the lens from the observer's eye being kept constant, we find that when the lens is so close to the observer's eye that m' is negative, the magnifying power is greater when the patient's eye is axially hypermetropic than when it is emmetropic, but less when his eye is axially myopic. When the distance between lens and eye is increased, the magnifying power is unaltered for an emmetropic eye, but increases for a myopic eye, and decreases for an hypermetropic eye. When $m'_0 = 0$, the image of the hole in the mirror is at the anterior focal point of the patient's eye, and then the magnifying power is independent of the axial length of his eye, its absolute value being equal to KD'. But if the distance between lens and eye is increased still more, the magnifying power will be greatest in case of a myopic eye, and least in case of an hypermetropic eye.

The reason why these results are different from the prevalent opinion on the subject is that most people are thinking chiefly of the size of the image made by the ophthalmoscopic lens, whereas the real criterion is the angle subtended by this image.

If, when the interval between the lens and the patient's eye is increased, the change of optical focus of the observation system is compensated by change of accommodation of the observer's eye, the variations of magnifying power are qualitatively the same as before; but the point for which the magnifying power is independent of the axial length of the patient's eye has a different position. According to the general law of optical imagery, the condition for seeing the fundus of the eye distinctly is

$$\frac{A_m}{K^2} = \frac{D_{00}}{K} - A'_m,$$

where D_{00} denotes the refracting power of the system composed of the lens of the ophthalmoscope and the correction glass. By means of this equation and the relations formerly given between A and A_m and between A' and A'_m, A can be eliminated from the factor $(1 + mA)$. The resulting expression is fairly complicated; but if, just as in the case of the erect image, the higher powers of the products of the type

mA are neglected, the following *approximate formula* will be obtained by development in series:

$$-\frac{\omega_h}{a_0}=KB'\left\{1+KmD_{00}-(m'+K^2m)A'\right\}$$
$$=KD'\left\{1+KmD_{00}-(m_0'+K^2m)A'\right\}$$

Accordingly, when one and the same correction lens is used, the magnifying power is the same for different axially ametropic eyes, provided the image of the hole in the mirror is situated at a place between the patient's eye and its anterior focal point and at a distance from the latter equal to K^2m. Practically, therefore, this place is not far from the focal point, because K^2 is seldom more than 1/9.

Suppose the patient's eye is astigmatic, the refractions in the first and second principal sections being denoted by A'_1 and A'_2, respectively; and let K_1 and K_2 be the values of the ratio $\frac{\omega_h}{\omega'_h}$ when the patient's eye is focused for the imageable lines of the primary and secondary systems of imagery, respectively. The following *approximate formula for the astigmatic distortion* is obtained by series-development:

$$\frac{K_1}{K_2}=1-(m'+K^2m)(A_1'-A_2').$$

If, in place of either imagery, or in place of both of them, there is an optical projection, the same formulae as in the case of the erect image are similarly obtained; but now *m* denotes the distance of the image of the centre of projection from the optical centre of the correction lens. But in ophthalmoscopy with inverted image the hole in the mirror is too big to act as centre of projection, and the magnifying power is so slight that even the form of the papilla is recognized with the immobile eye. Since, therefore, the pupil is the centre of projection, in making the calculation with the reduced eye, *m* has the same value, whether it is a case of an optical image or of an optical projection. It is evident from the approximate formula that the astigmatic distortion when the lens of the ophthalmoscope is far from the patient's eye proceeds in the same way as with the erect image, but when the lens is near the eye, the reverse relation occurs. Practically, the focusing for which there is no astigmatic distortion cannot be distinguished from the adjustment for which the image of the hole in the mirror lies in the pupil of the patient's eye. From the fact that, for $m'=0$, the absolute magnifying power is independent of the refracting power of the optical system of the patient's eye, it might be inferred

that with this focusing there was no astigmatic distortion. But this is not mathematically true, as can be seen by considering what is meant by absolute magnifying power which supposes an emmetropic adjustment of the observer's eye. Hence, for $m'=0$, the astigmatic distortion would not be abolished, unless the observer used a different correction glass in passing from one imagery to the other, so as to keep the adjustment emmetropic.

In the same way as with erect image and on the same assumption, namely, that the optical centre of the correction lens is at the anterior focal point of the observer's eye, *the magnification-ratio* for the image of the retina of the patient's eye on that of the observer is given by the expression

$$-\frac{KD'}{D}(1-m_0'A'_m).$$

The intensity of the light in the observation system is obtained from the same formulae as for ophthalmoscopy with erect image, by equating to zero the distance $(m'-d)$ of the hole in the mirror from the correction lens. If the value found for p_n is not less than the diameter of the entrance pupil, the latter must be used instead of p_n.

In *the illumination system* the *condition* $-Kl<p$ *for the size of the hole* has been formulated already. If it is not satisfied, a decentration is necessary, whereby the pupil of the patient's eye is covered only by a part of the image of the hole in the mirror. The only case that will be considered here is that of the centered system for which the illumination system produces at the centre of the pupil of the eye the image of a screen that casts a shadow; so that, as a matter of fact, the aperture under these circumstances is an annular one. The other apertures that have to be taken into account are the lens of the ophthalmoscope and the contour of the source of light; the mirror itself being of such size that it has no influence on the radiation region. As in the investigation of the erect image, let q denote the diameter of the image of the source of light as made by the lens of the ophthalmoscope in the medium in front of the eye; and let c denote its distance from the image of the hole in the mirror, the sign of this distance being reckoned in the same way here as in the previous investigation. Moreover, let K' denote the angular diameter of the source of light as measured at the centre of the hole in the mirror, and put $\frac{q}{c}=K''$. By means of the angular magnification-ratio we get then $K''=-\frac{K'}{K}$. Provided the effect of the aperture of the lens of the ophthalmoscope is not involved,

the same formulae can be used here as in case of the erect image; m' being substituted for d and K'' for K', and the value $-Kl$ being used instead of l. *The condition as to the image of the source of light*, namely,

$$q > -Kl \mid 1+cA \mid ,$$

without which a complete shadow of the hole in the mirror is visible on the ophthalmoscopic image, is in this case absolutely valid for $m'=0$. When this condition is satisfied, the area of the *useful pupil* P_n is equal to the smaller of the following two expressions:

$$P_n' = \frac{\pi}{4}(p^2 - K^2l^2), \qquad P_n'' = \frac{\pi}{4}\left\{\frac{q^2}{(1+cA)^2} - K^2l^2\right\};$$

and the *angular magnitudes of the characteristic field* are

$$w = \frac{p+q}{c}, \qquad \lambda = \frac{p+q}{c}.$$

If D_s denotes the reflecting power of the mirror and A_s the reduced convergence of the bundle of rays that is incident on it, all that is necessary for determining the best *form of the mirror* for the case when $m'=0$ is to use the formula:

$$-\frac{K^2}{c} = KD_0 + D_s + A_s,$$

where the desired numerical value of c is substituted first with a positive sign and then with a negative sign, since the value of q is the same in both cases. In this way two values D_{s1} and D_{s2} will be obtained which depend on each other according to the relation:

$$D_{s1} + D_{s2} = -2(KD_0 + A_s).$$

Therefore, supposing that in actual practice the image of the hole in the mirror is always in the pupil of the patient's eye, there would be two equally advantageous forms of mirror to be selected. But since, for reasons to be explained presently, the value of m' is very often negative, the form of mirror to be taken is that one of the two that is most advantageous in this case. In precisely the same way as in ophthalmoscopy with erect image, it follows that, according as m' is positive or negative, a negative or positive value of c, respectively, will be more advantageous for the field of the illumination system. Thus, concave mirrors of such power, that the image of the source of light formed by them falls between the mirror and the focus of the ophthalmoscopic lens, are excluded from being chosen, because $(-KD_0+A)$ gives the maximum reflecting power of the mirror. With a mirror having this reflecting power, the image of the source of light is thrown

on the retina of the emmetropic eye, and there is no shading off of any part of the whole field. However, this latter advantage is usually offset by the fact that none of the available sources of light are of sufficient uniformity of brightness. It is better, therefore, to choose a somewhat larger focal length for the mirror, which will reduce the shaded part of the field, but increase the unshaded part. If in this way the decrease of c were allowed to go on until $q = p$, the unshaded field in the emmetropic eye would be contracted until it vanished entirely. Now since ophthalmoscopy with inverted image is not so much for the purpose of investigating detail as for the purpose of general survey of the image, this behaviour would manifest itself as a decrease of the intensity of light. In order to avoid this dimming of light, the best thing to do is not to reduce the curvature of the mirror any more than is just enough not to detect in the image any lack of uniformity of the source of light, even in the highest degrees of myopia. What this limit is depends on the peculiarity of the source of light and can be ascertained by experiment only. Besides, different investigators prefer different sources of light and different distances; and therefore it is easy to see that radii of curvature between about 60 and 30 cm can be used. Other things being equal, a longer focal length gives a bigger field, but a smaller portion that is unshaded; which gives the impression of less intensity of light. This is clearer still when a plane mirror is used.

By bringing the source of light nearer the mirror and at the same time protecting the patient's eye from the light, it is easy to illuminate the unshaded part of the field of the observation system. However, in practical experience, without artificial dilatation of the pupil, this causes the pupil to contract, and the corneal reflex becomes so annoying that it is better to give up this big field, and to place the source of light near the patient's eye, as in the method with erect image. But then every part of the field of observation can be illuminated by slight movements of the mirror.

The diameter of the hole in the mirror is usually controlled by the magnifying power to be used, and should be so chosen that the condition as to the size of the hole is satisfied in most cases, but it must not be made smaller than this. Since in general it is not advisable to use a stronger magnifying power than $K = -\frac{1}{3}$, unless the pupil is artificially dilated, the hole should ordinarily be 4 mm in diameter. This affords the advantage of more intensity of light in the observation system, and usually produces a big enough pupil, when the other eye is closed, for a hole of this size to be used. On the other hand, particularly if a

smaller magnifying power in the observation system is sufficient, the condition as to the size of the hole may be satisfied even with very small pupil. But, in general, if a higher magnification is desired, the hole in the mirror must be smaller. The diameter of the mirror should be so great that the image of the pupil of the patient's eye is completely covered by the reflecting surface, and consequently it must be 30 mm at least. Unless this condition is satisfied, part of the light in the illumination system is lost.

In *performing the investigation* we always begin with a shorter distance between the lens and the patient's eye, in order to extend the field of the observation system by increasing this distance. If in this way we have contrived so that none of the edge of the iris is visible, the focusing has practically been found for which the image of the hole in the mirror lies in the pupil of the patient's eye.

However, when the pupil is small this focusing means a steady hand on the part of the observer, together also with a certain docility on the part of the patient, because slight movements of the patient's eye are enough to darken the field entirely, and then the right focus has to be found all over again in the same way as before. For these reasons, beginners especially, and more experienced practitioners too, when the patient is not an easy subject, frequently prefer the less delicate focusing with the lens not so far away, and choose also the more favourable form of mirror mentioned above for which the value of m' is negative. Along with these variations of the distance of the lens of the ophthalmoscope, the distance of the image of the fundus of the patient's eye is likewise altered. This involves a change of focus of the eye to get sufficient magnification of the ophthalmoscopic image. Now in order that this change of focus may not have to be too great, *the refracting power of the lens of the ophthalmoscope* must not be too great. On the other hand, it should not be so small that the hand holding it cannot be supported against the patient's brow. Perhaps, the lower limit demanded by this consideration, which is about 14 dptr, is the best value to use. The lens may have a diameter of 50 mm with advantage.

In numerous instances it is impossible to investigate the macula region by simple ophthalmoscopy. Indeed when the pupil has a minimum size, the method is hardly possible at all. We always get the impression that the real trouble is with the corneal reflex. Before passing to the consideration of ophthalmoscopy without reflexes, let us therefore investigate first the possibility of *abolishing the corneal reflex* only. Obviously, it is only when we have disposed of methods that fulfil this condition that we can decide whether this necessary requirement in the case of ophthalmoscopy without reflexes is likewise sufficient.

The optical system to be studied now is that in which the rays are reflected at the anterior surface of the cornea, and which extends from the medium where the source of light is situated to the entrance-pupil of the observer's eye—the so-called *first catadioptric system*. Supposing it is the case of the ordinary method with *erect* image, the simplest way is to investigate the form of the region of radiation in the medium where the light travels after having been reflected at the mirror and before being reflected at the cornea. For this purpose only two apertures need to be considered, namely, that of the mirror itself and that of the hole in the mirror. The first of these belongs to the medium in question, and the image of the latter in this medium is the image of the hole that is reflected in the cornea. By drawing, therefore, all the straight lines that pass through the aperture of the mirror and through the corneal reflex of the hole, the maximum region of radiation will be constructed for the first catadioptric system in the medium in question. If some other opening, as, for example, the rim of the correction lens or the pupil of the observer's eye should act here as a window, all the effect it would have would be to restrict somewhat this region of radiation. Now it is clear that, in order to abolish the corneal reflex, all that is necessary is to shift the image of the source of light in the mirror outside this region of radiation; for if none of the lines in this region passes through the image of the source of light, no ray lying therein can go through the system and through the hole in the mirror into the observer's eye. When a transparent unperforated mirror is used, the corneal reflex of the pupil of the observer's eye or of the rim of the correction glass (if present) takes the place of the corneal reflex of the hole in the mirror. But there is also another way of getting rid of the corneal reflex with a hole in the mirror. Since this hole casts a shadow, all that we have to do is to adjust the source of light so that the full shadow of the hole covers its reflected image in the cornea. The apex of the umbra is found from the equation

$$\frac{x}{l}=\frac{x+c}{q}.$$

If here x is put equal to the distance of the centre of curvature of the cornea from the hole in the mirror, then for a given value of the value is found which q cannot exceed if the image of the hole reflected in the cornea is to be inside the umbra. A practical way of making this adjustment is by moving the source of light farther away. Whether the umbra has been extended far enough by this means, can be told by the disappearance of the corneal reflex when the cornea is expressly illuminated for this purpose. We may therefore either see past the corneal reflex or cause it to vanish when it lies right in the way. Al-

though both the field and the intensity are very small, in many cases it is possible in this way to see the small reflex from the fovea, when it is not possible to do so in the ordinary investigation. However, a sufficient intensity can be obtained with the NERNST *slit lamp*, which the writer designed originally for use in ophthalmometry, and which is consequently known also as the ophthalmometric NERNST lamp. It consists of a closed tube, at one end of which the lamp is inserted; a lens-system being used to form an image of the little incandescent filament in a slit at the other end of the tube, so that this illuminated slit is to be regarded as the source of light. With a vertical slit a bright enough image is obtained by this arrangement, and in most cases the macula region can be investigated without dilating the pupil. Of course, the field becomes smaller as the pupil is more contracted; but even when a pupil is contracted by treatment with eserin, it is possible by this method to see a small field in the fovea. When the distance between slit and mirror is too great, the condition as to the size of the image of the source of light is not satisfied; but a central dark spot will be seen in the illuminated field, which moves in conformity with the movements of the mirror. However, by using a concave mirror with a radius of about 15 cm, a slit from 12 to 15 cm away from it, and an angle of incidence of approximately 45°, this spot will usually disappear. The reason why this is so with a source of light almost linear in form is due to the fact that the bundle of rays after being reflected is not only astigmatic, but has a fairly high asymmetry-value. With this adjustment, after refraction of the light into the optical system of the eye, the second focal point falls on the retina or very near it, provided the refraction of the eye is very anomalous; and owing to the asymmetry the width of the bundle of rays at the first focal point is finite. If this investigation is performed with dilated pupil, it constitutes a diagnostical method which is more delicate than any previous method of this sort. The illuminated field here is very bright and large enough to be investigated conveniently. Besides, owing to the abolition of the corneal reflex, the investigator is enabled to use the central part of the pupil of the patient's eye in the observation system; which would otherwise be impossible, because the corneal reflex here compels the observer to make a movement to one side. Hence, in the ordinary investigation of the macula region by the method of erect image, the observer is always forced to use the more unfavourable part of the pupil for the optical imagery in the observation system; whereas with this method of *simple central ophthalmoscopy* the resolving power of the optical system of the eye is essentially greater, thereby permitting much finer details to be perceived.

The only disadvantage of this method of investigation with undilated pupil is the small field. In order to get rid of the corneal reflex when it is desired to obtain a larger field with a smaller pupil, an image of the source of light has to be formed near the narrowest part of the region of radiation of the catadioptric system, that is, near the image of the hole reflected in the cornea, or near the pupil of the observer. This is accomplished by combining a suitable aplanatic convex lens with a plane parallel plate of glass and using the NERNST slit lamp as source of light. If the distance of the lens from the plate of glass used as a mirror is correctly measured, an image of the slit may be produced in the pupillary plane of the patient's eye. It is found also that the corneal reflex may be abolished without difficulty by small movements of this ophthalmoscope, and that the size of the illuminated field is quite large; but very little can be seen in this field due to the hindrance of foreign light. It is not hard to understand that the cause of this obscurity is light that is reflected diffusely in the cornea and in the crystalline lens. All that is necessary in order to see this is to separate the eye and mirror a little. This experiment shows that, when more is required with respect to the extent of the field of the illumination system, it is not enough merely to abolish the corneal reflex, but that *the light diffusely reflected in the cornea and in the crystalline lens must be warded off also.*

However, provided we are content to have a smaller field, the following is a very useful method also: A lens of refracting power about 30 dptr and not more than 10 mm in diameter, combined with a plane mirror perforated by a hole 2 mm in diameter, makes a very suitable instrument for investigating the macula without dilating the pupil. With a horizontal slit a concave mirror may also be used, either by itself or in combination with a convex lens, depending on the curvature of the mirror.

Moreover, in the method of *the inverted image*, the corneal reflex may be abolished without difficulty, provided a suitable aplanatic ophthalmoscope-lens is available, such as can be had nowadays. In order to construct the region of radiation of the first catadioptric system in the medium where the light travels before it is reflected at the cornea, the lens of the ophthalmoscope and the image reflected in the cornea must be considered as the apertures; because, after being reflected from the cornea of the patient's eye, the light must pass through the lens of the ophthalmoscope before entering the pupil of the observer's eye as a corneal reflex. The maximum extent of the radiation space includes all the straight lines that can be drawn through these two apertures. Now if the image of the pupil of the patient's eye made by the lens of the ophthalmoscope falls outside this region of

radiation, obviously, this maximum region of radiation will be thereby reduced to zero; because, although the light reflected from the cornea may get through the lens of the ophthalmoscope, it cannot enter the pupil of the observer's eye; and therefore not a single ray of light can traverse the system from the first medium to the last. In the practical employment of simple ophthalmoscopy by the method of the inverted image, where an aplanatic ophthalmoscopic lens is used to get rid of the corneal reflex, the observer must be careful to look towards a point near the edge of the lens, so as to prevent the image of the lens as reflected in the cornea from covering the entire pupil. Under these circumstances, provided the lens is held at the right distance from the patient's eye, by simply decentering it slightly towards the line of fixation of the patient's eye, the corneal reflex will be seen to disappear. Now when this experiment is tried with a very small pupil, it is found that the image of the fundus of the eye is obscured again by foreign light; which goes to prove that for considerable extent of the illuminated field the light diffusely reflected in the cornea and crystalline lens has to be excluded when the pupil is small. As the concentration of light in a beam is inversely proportional to the cross section thereof, in general, the diffusely reflected light gets to be more and more important in proportion as the illuminated area of the retina is larger, as compared with the part of the pupil used in the illumination system.

The necessary and sufficient condition for excluding the light diffusely reflected in the cornea and crystalline lens may be most simply formulated by saying, that *no part of the cornea or crystalline lens of the patient's eye shall be at once in the region of radiation of the illumination system and also in that of the observation system.*

If the exclusion of all the light reflected from the patient's eye, including not only that which is regularly reflected from the surfaces of separation of the ocular media but also that which is diffusely reflected in the cornea and crystalline lens, is the absolute requirement for what is called here *ophthalmoscopy without reflex*, this condition is equivalent to that which has just been stated. For if no part of the anterior surface of the cornea on which the light falls is in the region of radiation of the observation system, no ray of light proceeding from the illuminated area of this surface can come into the pupil of the observer's eye. The same thing is true with respect to the other reflecting surfaces, so that as a matter of fact there is no region of radiation in all three catadioptric systems. But if a small portion of the cornea lies in both regions of radiation, all that this may involve is that a narrow peripheral border of the ophthalmoscopic field will appear dim; whereas, when the corneal reflex is not excluded, the whole

image may be invisible. Consequently, another condition must be that, when there is a region of radiation, the source of light must be outside it at least in the first catadioptric system, and preferably in all three of these systems. In accordance therewith the writer has formulated the *condition of ophthalmoscopy without reflex*, as follows:

While a part of the fundus of the patient's eye must be situated in the region of radiation of the illumination system and in that of the observation system at the same time, this must not be the case with any portion of the cornea or crystalline lens; and, moreover, in the three catadioptric systems, the source of light must be outside the region of radiation, if there is such a region.

In this statement it is to be noted that the illumination system contains the source of light as one of the apertures, whereas the three catadioptric systems extend in this direction only to the medium of the source of light.

The most advantageous way of realizing this condition in the case of *central ophthalmoscopy without reflex* is shown in the diagram (Fig. 148),

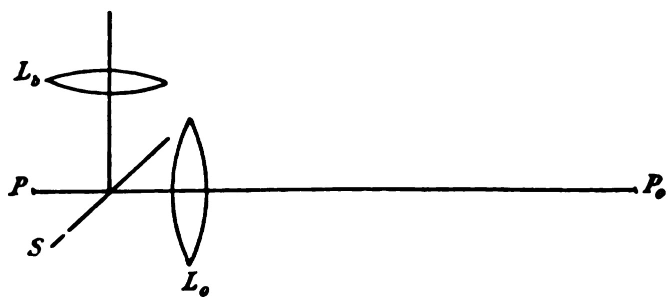

Fig. 148.

where P and P_0 designate the centres of the entrance-pupils of the eyes of patient and observer, respectively, L_0 represents the lens of the ophthalmoscope, and L_b a lens belonging to the illumination system, and S indicates a thin parallel plate of glass inclined at an angle of 45°. The two points P and P_0 are a pair of conjugate axial points with respect to the lens L_0. First, it must be shown how this arrangement can satisfy the conditions of absence of reflex. To begin with the first catadioptric system, disregarding the pupil of the patient's eye, suppose that the maximum region of radiation of this system is constructed in the medium of the source of light. As there are only two apertures in this region of radiation, namely, the openings of the two lenses, and since the opening of lens L_b already lies in the medium of the source of light, all that remains to be done is to find the image of the opening of the lens L_0 for which the light goes first through the transparent mirror S and is reflected then from the cornea and from

this mirror in succession, so as to pass finally through the lens L_b. The image of the lens L_0 as thus found will represent the smallest cross section of the region of radiation of the first catadioptric system. It is situated in the vicinity of the image of the pupil of the patient's eye which is made by the lens L_b. If this pupil is dilated as much as possible, there will be place enough here for a source of light, which will be therefore inside the maximum region of radiation of the illumination system as determined by the pupil of the patient's eye and the lens L_b, but which will lie outside the region of radiation of the first catadioptric system. The region of radiation of the third catadioptric system is found in the same way; whereas for investigating the second catadioptric system it is better to take into account the pupil of the patient's eye, from the start. Since its image is to be formed at the centre of the pupil of the patient's eye, the same is true also with respect to the images of the two pupils made by the lens L_b in the medium of the source of light; the image first mentioned being situated on the anterior surface of the lens and consequently coinciding with the image reflected in this surface. Actual calculation[1] based on the exact schematic eye gives the following result:—Take the refracting power of each of the two lenses to be 14 dptr, and the diameter of each to be 50 mm. Consider the mirror S as being at first infinitely thin, and suppose that the lenses are combined with it in such fashion that one of the lenses is the image of the other as reflected in the mirror. Moreover, assume that the diameters of the entrance-pupils of the eyes of both patient and observer are 6 mm, and that an image of the observer's pupil, reduced one-third in size, is formed in the patient's pupil. Then provided a central portion in the plane of the entrance-pupil of the patient's eye about 2.4 mm in diameter is shielded from the light, *no light regularly reflected from the three reflecting surfaces of the eye* will enter the pupil of the observer's eye. In the region of the radiation of the first and third catadioptric systems the pupil of the observer's eye was not taken into account; hence, in general, light regularly reflected from the cornea and from the posterior surface of the crystalline lens will not arrive at the lens L_0.

But in order also to keep *the light that is diffusely reflected* by the cornea and the crystalline lens far from the image in the ophthalmoscope, the place must be essentially restricted where the image of the source of light is formed in the entrance-pupil of the patient's eye. For this purpose the regions of radiation of the observation system and the illumination system, both of which are contracted in the patient's eye, need to be separated so far apart from each other, that

[1] Die reflexlose Ophthalmoskopie. *Arch. f. Augenheilk.* LXVIII, S. 101, 1911.

a point either on the posterior surface of the crystalline lens or on the anterior surface of the cornea will be in the free space between the two regions of radiation. The best way to see this is by constructing these regions of radiation in the medium lying between the mirror S and the cornea. If the patient's eye is emmetropic, and the illuminated portion of the fundus is entirely within the visible field of the ophthalmoscope, the boundary lines of the section of these regions of radiation made by a meridian plane, four for each region, are parallel to each other, as shown in Fig. 149. P'_0 is the image of the pupil of

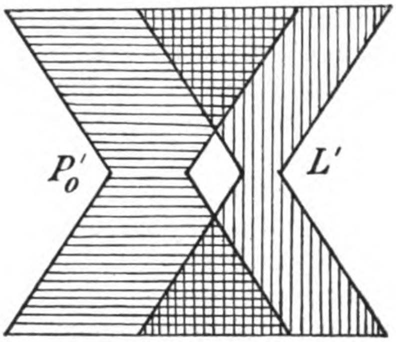

Fig. 149.

the observer's eye, and its centre is at the centre of the entrance-pupil of the patient's eye; while L' represents a small image of the source of light lying in the outer part of this pupil. The radiation regions of the observation system and the illumination system are indicated by horizontal and vertical lines, respectively. It is immediately obvious that the condition stated above, namely, that no portion of the cornea or of the crystalline lens may be at the same time within the regions of radiation of both systems, amounts to saying that both the anterior surface of the cornea and the aerial image of the posterior surface of the crystalline lens must intersect the free rhombic section. Now the distance of the vertex of the cornea is about 3 mm, and that of the aerial image of the posterior vertex of the crystalline lens about 4 mm. from the aerial image of the anterior vertex of the crystalline lens, In calculating the region of radiation of the catadioptric system, supposing also that the entrance-pupil is considered as being at the aerial image of the anterior pole of the crystalline lens, it may easily be shown, on the other hand, that a displacement backwards amounting to 0.5 mm, which is not far from what actually occurs with dilated pupil, will have no perceptible effect on the region of radiation. If the angle at the vertex of the unshaded rhombic figure in the diagram is denoted by ω, and if the width (in mm) of the open space between the two regions P'_0 and L' is denoted by a, all that is required in order to keep

the light that is diffusely reflected in the ocular media far away from the image in the ophthalmoscope is to satisfy the condition

$$\tan\frac{\omega}{2}=\frac{a}{7}\cdot$$

In central ophthalmoscopy without reflex it is always possible to use stops to cut out the portions of the fields of the illumination system and observation system that are shaded (*vignettierten*) and to make the two fields overlap each other. Consequently, with suitable instruments the angle ω as determined by the equation above is the same as the angle previously denoted by this symbol, which gives the characteristic angular dimensions of the unshaded field in the observation system, and which therefore may be called the *aperture angle* of an instrument of this kind.

If the width of the space available for the image of the source of light is supposed to be not more than 1 mm, the value of a for a wide open pupil is 2 mm, which means that $\omega=31°50'$. Thus, under favourable circumstances, in central ophthalmscopy without reflex, we can get an unshaded field, free from obscurity, of from 5 to 6 times the diameter of the papilla, although the diameters of the two lenses would have to be a little larger than they were in the calculation mentioned above.

Since the intensity of illumination on the fundus of the patient's eye is directly proportional to the area of the image of the source of light and to the specific intensity of the light, besides sunlight and electric arclight, the light of the NERNST slit lamp is the only light that can be employed for the small space that is necessary for the image of the source of light.

For the purpose of obtaining higher magnification in the observation system, a stop acting as entrance-pupil of an instrument on the order of an astronomical telescope and having its image in the entrance-pupil of the observer's eye at the point P_0 may take the place of the latter. In this case, according to the definition previously given, K will denote the magnification-ratio with respect to the entrance-pupil of the observer's eye and its image lying in the plane of the entrance-pupil of the patient's eye; and $\omega_h=-K\omega_h'$, where K is positive in sign, meaning therefore that the image in the ophthalmoscope itself is erect. In order to have sufficient brightness with more magnification, the field, which anyhow should not be excessive, may be curtailed, so as to get more space for the image of the source of light and for the image of the entrance-pupil of the telescopic lens in the entrance-pupil of the patient's eye. The diameter of this latter lens may be as

much as 10 mm, and, by proper choice of the refracting power and position of the lens L_b, the magnification of the image of the source of light may be increased also at the same time.

An indispensable requirement in all cases is that the images of the stop of the observation system and source of light in the entrance-pupil of the patient's eye shall be sufficiently free from aberrations. This has been achieved in the new *aplanatic ophthalmoscope lenses* as calculated from the writer's data by Dr. v. Rohr and manufactured by Zeiss; the aplanatism being obtained by using an aspherical surface.

It would take too much space to describe the construction in detail and to explain how it is necessary to prevent any bad effect from light that goes through the mirror S without being reflected. Experience shows that with an aperture-angle of 20°, giving therefore a field 3 or 4 times the diameter of the papilla, a magnification can be used of $K=2$; so that with the larger image correspondingly more detail is visible, whereas previously it was impossible to gain anything with a magnification higher than that of the ordinary erect image. Apparently this is not the limit of the available magnification, but it is probably not more than $K=3$.

The writer is confident, from certain special experiments, that by using more extended sources of light there is use for the *perforated opaque mirror* in central ophthalmoscopy. But owing to the unavoidable shading off of the field this method is essentially inferior to that with the unsilvered mirror; and hence it will not be discussed more particularly.

Eccentrical (or "acentral") *ophthalmoscopy without reflex*, where there is no longer any attempt to produce an image of the pupil of the observer's eye or of the special stop of the observation system at the centre of the entrance-pupil of the patient's eye, may be easily realized by increasing the distance a; as follows from the foregoing investigation. Thus, in general, a field is obtained that is relatively larger in comparison with the size of the pupil; and this advantage is useful in two ways, namely, either for obtaining *large ophthalmoscopic images for comprehensive survey*, or as affording a method of *ophthalmoscopy without reflex for case of small pupil*. However, a simpler method, to be described below, can be used with advantage for both of these purposes.

If an *opaque* mirror is used for eccentrical ophthalmoscopy without reflex, it is best to adjust it to one side, and not to perforate it. With this arrangement it is easy to fulfil the condition of freedom from reflex, because the mirror, which acts in the illumination system as a stop and in the observation system as a screen, to a certain extent automatically separates the two regions of radiation. This method has been

worked out with much success by DIMMER[1] in *photographing the fundus of the eye;* and is used likewise in THORNER's[2] stationary ophthalmoscope. In both methods an image of a stop in the observation system is formed in the entrance-pupil of the patient's eye. But whereas in the first method an image of the source of light is formed in the other part of this pupil, in the latter method there is at this place an image of a stop that belongs to the illumination system. In DIMMER's method the distance *a* is large enough to shut off also the diffusely reflected light, but this does not seem to be the case with THORNER's ophthalmoscope, so far as the crystalline lens is concerned. However, here this light is not so harmful, because the part of the pupil affected by the illumination system is comparatively large. In WOLFF's[3] method of ophthalmoscopy without reflex there is no image of a stop in the observation system in the pupil of the patient's eye, and the observation system is separated from the illumination system by the screening of the mirror. By their methods also both THORNER and WOLFF have made photographs of the fundus of the eye. WOLFF's electrical ophthalmoscope is intended for clinical use and is employed for investigating with the ordinary erect image, but it has the disadvantage of requiring the pupil to have a certain diameter.

All methods of ophthalmoscopy without reflex in which an opaque mirror is adjusted to one side give a one-sided shading off of the field, and are disadvantageous in this way.

In THORNER's[4] latest method of ophthalmoscopy without reflex, he employs a means of simple ophthalmoscopy which is due orginally to SCHULTÉN,[5] and in which the lens of the ophthalmoscope is replaced by a concave mirror belonging to both the observation system and the illumination system. Here only a small source of light is needed, with its image in the mirror formed near the entrance-pupil of the eye or of the telescopic lens, in order to fulfil the conditions of ophthalmoscopy without reflex. Besides some inconveniences of a technical nature, the disadvantages of the method, such as astigmatism and lack of symmetry in the bundle of rays used in the imagery, are due to the unavoidable obliquity of the mirror. A glass mirror silvered on the back gives double images, the fainter one perhaps being comparatively

[1] FR. DIMMER, *Die Photographie des Augenhintergrundes.* Wiesbaden 1907.

[2] W. THORNER, *Die Theorie des Augenspiegels und die Photographie des Augenhintergrundes.* Berlin 1903.

[3] H. WOLFF, Zur Photographie des menschlichen Augenhintergrundes. *Arch. f. Augenheilkunde.* LIX. S. 115. 1908.

[4] W. THORNER, Ein reflexloser Handaugenspiegel. *Zft. f. Augenheilkunde.* XXVI. S. 1. 1910.

[5] SCHULTÉN, Beobachtungen des Augenhintergrundes bei hochgradiger Vergrösserung. *Arch. f. Anat. u. Physiol.* 1883. S. 285.

harmless usually, and yet for certain effects necessarily in the way; and other mirrors are too delicate to be used in practical ophthalmoscopy.

Stereoscopic ophthalmoscopy without reflex occupies an intermediate place between central and eccentrical (or acentral) ophthalmoscopy; because in order to obtain the maximum stereoscopic effect, the images of the two stops of the observation systems must be close to the ends of a diameter of the entrance-pupil of the patient's eye; which means that the axis of symmetry of the two systems shall be centrally situated. If these stops are the entrance-pupils of two telescopic lenses, and if their images are formed in the entrance-pupils of the two eyes of the observer, the condition of correct stereoscopic effect is, that the magnification-ratio for this imagery of the distance between the eyes shall have the same sign, as for the imagery of the pupils. Hence, the astronomical telescope cannot be used, but the telescopic lenses must be made on the order of the terrestrial or prism telescope, in which the image of the fundus as presented to the eyes is inverted. The image of the slit may be vertical and midway between the images of the two stops, or horizontal and above or below these images. Assuming that the magnification-ratio for the images of the entrance-pupils of the telescopic lenses in the entrance-pupil of the patient's eye is $-1/3$, it is advisable to take the distance between the centres of the stops at 16 mm, the diameter of each stop being 6 mm. The distance might perhaps be made greater, but then the instrument could be used only for a pupil of maximum size. On the other hand, there is not much advantage in getting a greater stereoscopic effect than can be obtained in this way. The magnification $K = -1$ may be used with advantage.

THORNER's stationary ophthalmoscope mentioned above was used also for stereoscopic ophthalmoscopy, by halving again the half of the pupil intended for the observation system. Aside from other drawbacks of this method, the maximum stereoscopic effect is not obtained in this arrangement.

The apparatus for ophthalmoscopy without reflex is fairly elaborate, due principally to the separation of the observation system from the illumination system, which is necessary in order to avoid the reflex images in the lens of the ophthalmoscope. But if these reflex images can be tolerated, the apparatus can be essentially simplified. As a matter of fact, with a small source of light, these images are not very much in the way. Besides, as distinguished from the images reflected at the surfaces of the ocular media, they can be made to fall in front of any part of the fundus of the patient's eye. Thus, provided the magnification is not excessive, the advantages of ophthalmoscopy without reflex can be obtained in this way. Consequently, those

methods in which the condition of ophthalmoscopy without reflex as formulated above is fulfilled, although reflections of light are tolerated in the lens of the ophthalmoscope, may be called the methods of *simplified ophthalmoscopy without reflex*. If it is simply a question of magnification of the ordinary inverted image, nothing more is needed in a method of this kind than the aplanatic lens of the ophthalmoscope and the author's *electric hand-ophthalmoscope*, in which the image of the filament of an incandescent lamp is produced in a slit near the ophthalmoscope hole containing the necessary correction lens and at a variable distance from it; the illumination tube with the source of light being far enough away to protect the observer's vision from being injured by the heat. In using this mirror with the technique of the ordinary inverted image, it is best to start with undilated pupil and a distance of from 4 to 5 mm between slit and hole, looking a little to the nasal side in order to see macula and papilla at the same time. When, with pupil undilated, the macula is in the field of the ophthalmoscope, the hole should always be between the visual axis of the patient's eye and the slit; for which purpose the illumination tube can be rotated about the optical axis of the correction lens. The lens of the ophthalmoscope with its most curved surface towards the observer is held at first right in front of the patient's eye, in which case reddish light is visible on the temporal edge of the pupil. If this should not be so, either the distance between the lens of the ophthalmoscope and the eye of the observer is to be increased, or the distance between hole and slit diminished. The reddish light being kept steadily in view, and pains being taken to see that the two small images reflected in the lens are as nearly as possible in line with each other, the distance between the lens and the patient's eye is then increased. At the right distance the field is fully illuminated and free from reflex, provided the lens is moved in the nasal direction as far as possible with fully illuminated field. If any obscurity should appear, it is a sign that the distance *a* has been made too short. If owing to the smallness of the pupil it cannot be increased, often the unobscured part of the field can be used without further adjustment. Obviously, the light that obscures the macula region of the image originates in the cornea, while the region of the papilla gets the light that is diffusely reflected in the crystalline lens. Now the distance of the lens of the ophthalmoscope can be varied in both directions, within certain limits, without danger of introducing regularly reflected light. Hence, unless the pupil is abnormally small, the region of the macula can be seen free from obscurity by increasing the distance a little between the lens and the patient's eye; and the region of the papilla in the same way by slightly diminishing this distance. But with the smallest pupil the field is necessarily limited.

However, since the source of light is to one side, this contraction of the field does not have to take place concentrically; but a rectangular stop can be used in conjunction with the lens of the ophthalmoscope, which contracts the field, about one-third on the average, in the direction at right angles to the linear extent of the source of light, without producing any contraction parallel to it. In this way, with a pupillary diameter as small as 2 mm, and with a magnification $K = -1/3$, an aperture angle of 30° may be utilized; giving therefore a visible field free from obscurity that is at least 5 times the diameter of the papilla. And if the aperture angle is reduced by the rectangular stop to one-third the size in one direction, the macula can still be investigated; in fact even when the pupil is contracted by treatment with eserin, although not without difficulty. However, as a rule, one should never try to see the macula in the centre of the field, with undilated pupil. For the condition of cutting off the light that is reflected regularly at the cornea is that the image of the source of light shall be inside the entrance-pupil of the patient's eye, but outside the image of the lens of the ophthalmoscope as reflected in the cornea; which cannot be achieved with small pupil and central direction of sight.

In order with dilated pupil to obtain a larger image for inspection by this method, all that is necessary is to increase the diameter of the lens of the ophthalmoscope without altering its refracting power. How far we can go in this way cannot be determined in advance, owing to technical considerations. With such lenses as have been manufactured already, a field about 7 times the diameter of the papilla has been obtained.

The simplified method of ophthalmoscopy without reflex is well adapted for *demonstration ophthalmoscopes*. All that is necessary for this purpose is to fasten the ophthalmoscope and the lens on a stand, the field being adjusted to another stand, which is so arranged that the image of the slit is formed in the proper part of the pupil, according to the rules given above. But when a stand is used, the advantages of this arrangement should be utilized, by substituting in place of the hole in the mirror, which acts always as a window-opening (*Luke*), the entrance-pupil of a telescopic lens whose exit-pupil is in the entrance-pupil of the observer's eye. Moreover, in order to be able to increase the magnification, it is a good plan to substitute the NERNST lamp for the little incandescent lamp; either the slit being adjusted in similar fashion to the arrangement in the electrical hand ophthalmoscope, or a virtual image of it near the entrance-pupil of the telescopic lens being produced with a transparent mirror. When the NERNST lamp is used, particularly in the first way, a magnification of $K = 1$ is a convenient value, with no risk of getting any disturbing effects from reflexes

in the lens. Whether in central ophthalmoscopy by this method even higher magnifications may not be preferable, cannot be stated in advance; as the reflexes are more troublesome with increase of magnification. On the other hand, this method requires less expensive apparatus than the one described above.

Stereoscopic ophthalmoscopy may also be advantageously used in the simplified method without reflex; and with a stationary instrument the magnification may be as much as $K=1$. With somewhat lower magnification a suitable binocular telescopic lens can be used without a stand, in combination with the illumination tube of the author's electrical ophthalmoscope and the aplanatic lens. By suitable alteration GIRAUD-TEULON'S[1] binocular ophthalmoscope, which in its original form was hardly adapted for stereoscopy, may also be converted into a practically very useful instrument with illumination tube and the aplanatic lens of the ophthalmoscope. FRAENKEL'S[2] modification of it lacks only the illumination tube and the proper lens. However, the principle employed by GIRAUD-TEULON, of bringing the two visual axes close to each other by double reflection, can never be free from window-action, and therefore a binocular telescopic lens is always preferable.

[1] Described in Vol. III.

[2] FR. FRAENKEL, Demonstration eines binokularen Augenspiegels. *Ber. über d. 36. Vers. d. Ophth. Ges. Heildelberg* 1910. S. 314.